DP IB Maths: AA SL

Your notes

5.2 Further Differentiation

Contents

5.2.1 Differentiating Special Functions

Your notes

Differentiating Trig Functions

How do I differentiate sin and cos?

- The derivative of $y = \sin x$ is $\frac{dy}{dx} = \cos x$
- The derivative of $y = \cos x$ is $\frac{dy}{dx} = -\sin x$
- For the **linear** function $ax + b$, where a and b are constants,
  - the derivative of $y = \sin(ax + b)$ is $\frac{dy}{dx} = a\cos(ax + b)$
  - the derivative of $y = \cos(ax + b)$ is $\frac{dy}{dx} = -a\sin(ax + b)$
- For the **general** function $f(x)$,
  - the derivative of $y = \sin(f(x))$ is $\frac{dy}{dx} = f'(x)\cos(f(x))$
  - the derivative of $y = \cos(f(x))$ is $\frac{dy}{dx} = -f'(x)\sin(f(x))$
- These last two sets of results can be derived using the **chain rule**
- For calculus with trigonometric functions angles must be measured in **radians**
  - Ensure you know how to change the angle mode on your GDC

Examiner Tip

- As soon as you see a question involving differentiation and trigonometry put your GDC into radians mode

Your notes

Worked example

a) Find $f'(x)$ for the functions

i. $f(x) = \sin x$
ii. $f(x) = \cos 2x$
iii. $f(x) = 3\sin 4x - \cos(2x - 3)$

i. $\boxed{f'(x) = \cos x}$

ii. $\boxed{f'(x) = -2\sin 2x}$

iii. Differentiate 'term by term'

$f'(x) = 3(4\cos 4x) - (-2\sin(2x-3))$

$\boxed{f'(x) = 12\cos 4x + 2\sin(2x-3)}$

b) Find the gradient of the tangent to the curve $y = \sin\left(2x + \frac{\pi}{6}\right)$ at the point where $x = \frac{\pi}{8}$.

Give your answer as an exact value.

Gradient of tangent is equal to gradient of curve

$\frac{dy}{dx} = 2\cos\left(2x + \pi/6\right)$

At $x = \frac{\pi}{8}$, $\frac{dy}{dx} = 2\cos\left(2\left(\frac{\pi}{8}\right) + \pi/6\right)$ ← Ensure your GDC is in RADIANS mode

$\boxed{\therefore \text{Gradient of tangent, at } x = \pi/8, \text{ is } \tfrac{1}{2}(\sqrt{6} - \sqrt{2})}$

Your notes

Differentiating e^x & lnx

How do I differentiate exponentials and logarithms?

- The derivative of $y = e^x$ is $\frac{dy}{dx} = e^x$ where $x \in \mathbb{R}$
- The derivative of $y = \ln x$ is $\frac{dy}{dx} = \frac{1}{x}$ where $x > 0$
- For the **linear** function $ax + b$, where a and b are constants,
  - the derivative of $y = e^{(ax+b)}$ is $\frac{dy}{dx} = ae^{(ax+b)}$
  - the derivative of $y = \ln(ax+b)$ is $\frac{dy}{dx} = \frac{a}{(ax+b)}$
    - in the special case $b = 0$, $\frac{dy}{dx} = \frac{1}{x}$ (a's cancel)
- For the **general** function $f(x)$,
  - the derivative of $y = e^{f(x)}$ is $\frac{dy}{dx} = f'(x)e^{f(x)}$
  - the derivative of $y = \ln(f(x))$ is $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
- The last two sets of results can be derived using the **chain rule**

Examiner Tip

- Remember to avoid the common mistakes:
  - the derivative of $\ln kx$ with respect to x is $\frac{1}{x}$, NOT $\frac{k}{x}$!
  - the derivative of e^{kx} with respect to x is ke^{kx}, NOT kxe^{kx-1}!

Your notes

Worked example

A curve has the equation $y = e^{-3x+1} + 2\ln 5x$.

Find the gradient of the curve at the point where $x = 2$ gving your answer in the form $y = a + be^c$, where a, b and c are integers to be found.

$y = e^{-3x+1} + 2(\ln 5x)$

$\therefore \frac{dy}{dx} = -3e^{-3x+1} + 2\left(\frac{1}{x}\right)$

"$y = e^{ax+b}$, $\frac{dy}{dx} = ae^{ax+b}$"

"$y = \ln(ax+b)$, special case $b = 0$, $\frac{dy}{dx} = \frac{1}{x}$"

At $x = 2$, $\frac{dy}{dx} = -3e^{-3(2)+1} + \frac{2}{2} = -3e^{-5} + 1$

Your GDC may be able to find gradients but probably not in the exact form required. It is still helpful to check approximate answers though.

$\therefore$ Gradient at $x = 2$ is $1 - 3e^{-5}$

i.e. $a = 1$, $b = -3$, $c = -5$

5.2.2 Techniques of Differentiation

Your notes

Chain Rule

What is the chain rule?

- The **chain rule** states if y is a function of u and u is a function of x then

$$y = f(u(x))$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

  - This is given in the **formula booklet**
- In **function notation** this could be written

$$y = f(g(x))$$

$$\frac{dy}{dx} = f'(g(x))g'(x)$$

How do I know when to use the chain rule?

- The chain rule is used when we are trying to differentiate **composite functions**
  - "function of a function"
  - these can be identified as the variable (usually x) does not 'appear alone'
    - $\sin x$ – **not** a composite function, x 'appears alone'
    - $\sin(3x + 2)$ **is a composite function;** x is tripled and has 2 added to it before the sine function is applied

How do I use the chain rule?

STEP 1
Identify the two functions
Rewrite y as a function of u; $y = f(u)$
Write u as a function of x; $u = g(x)$

STEP 2
Differentiate y with respect to u to get $\frac{dy}{du}$
Differentiate u with respect to x to get $\frac{du}{dx}$

STEP 3

Your notes

Obtain $\frac{\mathrm{d}y}{\mathrm{d}x}$ by applying the formula $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$ and substitute u back in for $g(x)$

- In trickier problems **chain rule** may have to be applied **more than once**

Are there any standard results for using chain rule?

- There are **five** general results that can be useful
  - If $y = (f(x))^n$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = nf'(x)\mathrm{f}(x)^{n-1}$
  - If $y = \mathrm{e}^{f(x)}$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)e^{f(x)}$
  - If $y = \ln(f(x))$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{f'(x)}{f(x)}$
  - If $y = \sin(f(x))$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)\cos(f(x))$
  - If $y = \cos(f(x))$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = -f'(x)\sin(f(x))$

Examiner Tip

- You should aim to be able to spot and carry out the chain rule mentally (rather than use substitution)
  - every time you use it, say it to yourself in your head
    "differentiate the first function ignoring the second, then multiply by the derivative of the second function"

Your notes

Worked example

a) Find the derivative of $y = (x^2 - 5x + 7)^7$.

STEP 1 Identify the two functions and rewrite

$y = u^7$ i.e. $f(u) = u^7$

$u = x^2 - 5x + 7$ i.e. $g(x) = x^2 - 5x + 7$

STEP 2 Find $\frac{dy}{du}$ and $\frac{du}{dx}$.

$\frac{dy}{du} = 7u^6$ $\quad \frac{du}{dx} = 2x - 5$

STEP 3 Apply chain rule, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Chain rule is in the formula booklet

$\frac{dy}{dx} = 7u^6(2x - 5)$

and substitute u back for g(x)

$$\boxed{\frac{dy}{dx} = 7(2x-5)(x^2-5x+7)^6}$$

b) Find the derivative of $y = \sin(e^{2x})$.

Your notes

$y = \sin(e^{2x})$ "... differentiate $\sin \square$, ignore e^{2x}..."

$\frac{dy}{dx} = \cos(e^{2x}) \times 2e^{2x}$ "... multiply by derivative of e^{2x}..."

"$y = e^{ax+b}$, $\frac{dy}{dx} = ae^{ax+b}$"
or by applying chain rule again

$$\boxed{\therefore \frac{dy}{dx} = 2e^{2x}\cos(e^{2x})}$$

Your notes

Product Rule

What is the product rule?

- The **product rule** states if y is the product of two functions $u(x)$ and $v(x)$ then

$$y = uv$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

  - This is given in the **formula booklet**
- In **function notation** this could be written as

$$y = f(x)g(x)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g'(x) + g(x)f'(x)$$

- **'Dash notation'** may be used as a **shorter** way of writing the rule

$$y = uv$$

$$y' = uv' + vu'$$

- Final answers should match the notation used throughout the question

How do I know when to use the product rule?

- The **product rule** is used when we are trying to **differentiate** the **product** of **two functions**
  - these can easily be confused with composite functions (see **chain rule)**
    - $\sin(\cos x)$ is a composite function, "sin of cos of x"
    - $\sin x \cos x$ is a product, "sin *x* times cos x"

How do I use the product rule?

- Make it clear what u, v, u' and v' are
  - arranging them in a square can help
    - opposite diagonals match up

STEP 1

Identify the two functions, u and v

Differentiate both u and v with respect to x to find u' and v'

STEP 2

Obtain $\frac{\mathrm{d}y}{\mathrm{d}x}$ by applying the product rule formula $\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$

Simplify the answer if straightforward to do so or if the question requires a particular form

- In trickier problems **chain rule** may have to be used when finding u' and v'

Examiner Tip

- Use u, v, u' and v' for the elements of product rule
  - lay them out in a 'square' (imagine a 2×2 grid)
  - those that are paired together are then on opposite diagonals (u and v', v and u')
- For trickier functions chain rule may be required inside product rule
  - i.e. chain rule may be needed to differentiate u and v

Your notes

Your notes

Worked example

a) Find the derivative of $y = e^x \sin x$.

$y = e^x \sin x$

STEP 1 Identify functions and differentiate

$u = e^x \qquad v = \sin x$

$u' = e^x \qquad v' = \cos x$

arranging u, v, u', v' in a square makes product rule 'diagonal pairs'

STEP 2 Apply product rule: '$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$'

(As it is given in the formula booklet)

$y' = e^x \cos x + e^x \sin x$

$$\boxed{\therefore \frac{dy}{dx} = e^x(\cos x + \sin x)}$$

It is straightforward to take a factor of e^x out

b) Find the derivative of $y = 5x^2 \cos 3x^2$.

$y = 5x^2 \cos 3x^2$

STEP 1 $\quad u = 5x^2 \qquad v = \cos 3x^2$

$u' = 10x \qquad v' = -\sin 3x^2 \times 6x$ chain rule

$v' = -6x \sin 3x^2$

STEP 2 $\quad y' = -30x^3 \sin 3x^2 + 10x \cos 3x^2$

$$\boxed{\therefore \frac{dy}{dx} = 10x(\cos 3x^2 - 3x^2 \sin 3x^2)}$$

Your notes

Quotient Rule

What is the quotient rule?

- The **quotient rule** states if y is the quotient $\frac{u(x)}{v(x)}$ then

$$y = \frac{u}{v}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

  - This is given in the **formula booklet**
- In **function notation** this could be written

$$y = \frac{f(x)}{g(x)}$$

$$\frac{dy}{dx} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

- As with product rule, **'dash notation'** may be used

$$y = \frac{u}{v}$$

$$y' = \frac{vu' - uv'}{v^2}$$

- Final answers should match the notation used throughout the question

How do I know when to use the quotient rule?

- The **quotient rule** is used when trying to differentiate a fraction where **both** the **numerator** and **denominator** are **functions** of x
  - if the **numerator** is a **constant**, **negative powers** can be used
  - if the **denominator** is a **constant**, treat it as a **factor** of the expression

How do I use the quotient rule?

- Make it clear what u, v, u' and v' are
  - arranging them in a square can help
    - opposite diagonals match up (like they do for product rule)

STEP 1

Identify the two functions, u and v

Differentiate both u and v with respect to x to find u' and v'

STEP 2

Obtain $\frac{\mathrm{d}y}{\mathrm{d}x}$ by applying the quotient rule formula $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

Be careful using the formula – because of the **minus sign** in the **numerator**, the **order** of the functions is important

Simplify the answer if straightforward or if the question requires a particular form

- In trickier problems **chain rule** may have to be used when finding u' and v',

Examiner Tip

- Use u, v, u' and v' for the elements of quotient rule
  - lay them out in a 'square' (imagine a 2×2 grid)
  - those that are paired together are then on opposite diagonals (v and u', u and v')
- Look out for functions of the form $y = f(x)(g(x))^{-1}$
  - These can be differentiated using a combination of **chain rule** and **product rule** (it would be good practice to try!)
  - ... but it can also be seen as a quotient rule question in disguise
  - ... and vice versa!
    - A quotient could be seen as a product by rewriting the denominator as $(g(x))^{-1}$

Your notes

Worked example

Differentiate $f(x) = \dfrac{\cos 2x}{3x+2}$ with respect to x.

STEP 1 Identify u and v, differentiate

$u = \cos 2x \qquad v = 3x+2$

$u' = -2\sin 2x \qquad v' = 3$

chain rule

opposite diagonals match up

STEP 2 Apply quotient rule: '$\dfrac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$'

(As it is given in the formula booklet)

$$f'(x) = \frac{(3x+2)(-2\sin 2x) - (\cos 2x)(3)}{(3x+2)^2}$$

$$\boxed{\therefore\ f'(x) = \frac{-2(3x+2)\sin 2x - 3\cos 2x}{(3x+2)^2}}$$

(Nothing obvious/easy to simplify and question does not specify a particular form)

5.2.3 Second Order Derivatives

Your notes

Second Order Derivatives

What is the second order derivative of a function?

- If you **differentiate** the **derivative** of a **function** (i.e. differentiate the function a second time) you get the **second order derivative** of the function
- There are two forms of **notation** for the **second order derivative**
  - $y = f(x)$
  - $\frac{dy}{dx} = f'(x)$ (First order derivative)
  - $\frac{d^2y}{dx^2} = f''(x)$ (Second order derivative)
- Note the position of the superscript 2's
  - **d**ifferentiating twice (so $\mathbf{d^2}$) with respect to x twice (so x^2)
- The **second order derivative** can be referred to simply as the **second derivative**
  - Similarly, the **first order derivative** can be just the **first derivative**
- A **first order derivative** is the **rate** of **change** of a function
  - a **second order derivative** is the **rate** of **change** of the **rate** of **change** of a function
    - i.e. the **rate** of **change** of the function's **gradient**
- **Second order derivatives** can be used to
  - test for local minimum and maximum points
  - help determine the nature of stationary points
  - determine the concavity of a function
  - graph derivatives

How do I find a second order derivative of a function?

- By **differentiating twice!**
- This may involve
  - rewriting **fractions, roots,** etc as **negative** and/or **fractional powers**
  - differentiating **trigonometric** functions, **exponentials** and **logarithms**
  - using **chain rule**
  - using **product** or **quotient** rule

Examiner Tip

- Negative and/or fractional powers can cause problems when finding second derivatives so work carefully through each term

Your notes

Worked example

Given that $f(x) = 4 - \sqrt{x} + \dfrac{3}{\sqrt{x}}$

a) Find $f'(x)$ and $f''(x)$.

a) $f(x) = 4 - x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}$

REWRITE AS POWERS OF x

$f'(x) = -\left(\frac{1}{2}\right)x^{\frac{1}{2}-1} + 3\left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1}$

$f'(x) = -\frac{1}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}}$

DIFFERENTIATE ONCE TO FIND $f'(x)$

$f''(x) = -\frac{1}{2}\left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1} - \frac{3}{2}\left(-\frac{3}{2}\right)x^{-\frac{3}{2}-1}$

$f''(x) = \frac{1}{4}x^{-\frac{3}{2}} + \frac{9}{4}x^{-\frac{5}{2}}$

DIFFERENTIATE A SECOND TIME TO FIND $f''(x)$

b) Evaluate $f''(3)$.

Give your answer in the form $a\sqrt{b}$, where b is an integer and a is a rational number.

b) $f''(x) = \dfrac{1}{4x\sqrt{x}} + \dfrac{9}{4x^2\sqrt{x}}$

$x^{\frac{3}{2}} = x\sqrt{x}$ $\quad x^{\frac{5}{2}} = x^2\sqrt{x}$

$f''(3) = \dfrac{1}{12\sqrt{3}} + \dfrac{9}{36\sqrt{3}}$

$= \dfrac{12}{36\sqrt{3}} = \dfrac{1}{3\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{\sqrt{3}}{9}$

RATIONALISE DENOMINATOR

$f''(3) = \frac{1}{9}\sqrt{3}$

5.2.4 Further Applications of Differentiation

Your notes

Stationary Points & Turning Points

What is the difference between a stationary point and a turning point?

- A **stationary point** is a point at which the **gradient function** is equal to zero
  - The **tangent** to the **curve** of the **function** is **horizontal**
- A **turning point** is a type of stationary point, but in addition the **function changes** from **increasing** to **decreasing**, or **vice versa**
  - The curve **'turns'** from **'going upwards'** to **'going downwards'** or **vice versa**
  - **Turning points** will either be **(local) minimum** or **maximum** points
- A **point** of **inflection** ***could*** also be a **stationary point** but is **not** a turning point

How do I find stationary points and turning points?

- For the function $y = f(x)$, **stationary points** can be found using the following process

  STEP 1

  Find the **gradient function**, $\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$

  STEP 2

  Solve the equation $f'(x) = 0$ to find the x-coordiante(s) of any stationary points

  STEP 3

  If the y-coordinates of the stationary points are also required then substitute the x-coordinate(s) into $f(x)$

- A GDC will solve $f'(x) = 0$ and most will find the coordinates of turning points (minimum and maximum points) in graphing mode

Your notes

Testing for Local Minimum & Maximum Points

What are local minimum and maximum points?

- Local **minimum** and **maximum** points are two types of **stationary** point
  - The **gradient function** (derivative) at such points equals zero
  - i.e. $f'(x) = 0$
- A **local minimum** point, $(x,\ f(x))$ will be the lowest value of $f(x)$ in the **local** vicinity of the value of x
  - The function may reach a **lower** value further afield
- Similarly, a **local maximum** point, $(x,\ f(x))$ will be the highest value of $f(x)$ in the **local** vicinity of the value of x
  - The function may reach a **greater** value further afield
- The graphs of many functions **tend** to **infinity** for **large** values of x (and/or **minus infinity** for **large negative** values of x)
- The **nature** of a stationary point refers to whether it is a **local minimum** point, a **local maximum** point or a **point of inflection**
- A **global** minimum point would represent the **lowest** value of $f(x)$ for **all values** of x
  - similar for a **global** maximum point

How do I find local minimum & maximum points?

- The **nature** of a **stationary point** can be determined using the **first derivative** but it is *usually* quicker and easier to use the **second derivative**
  - only in cases when the second derivative is **zero** is the first derivative method needed
- For the function $f(x)$ ...

  STEP 1

  Find $f'(x)$ and solve $f'(x) = 0$ to find the x-coordinates of any stationary points

  STEP 2 (Second derivative)

  Find $f''(x)$ and evaluate it at each of the stationary points found in **STEP 1**

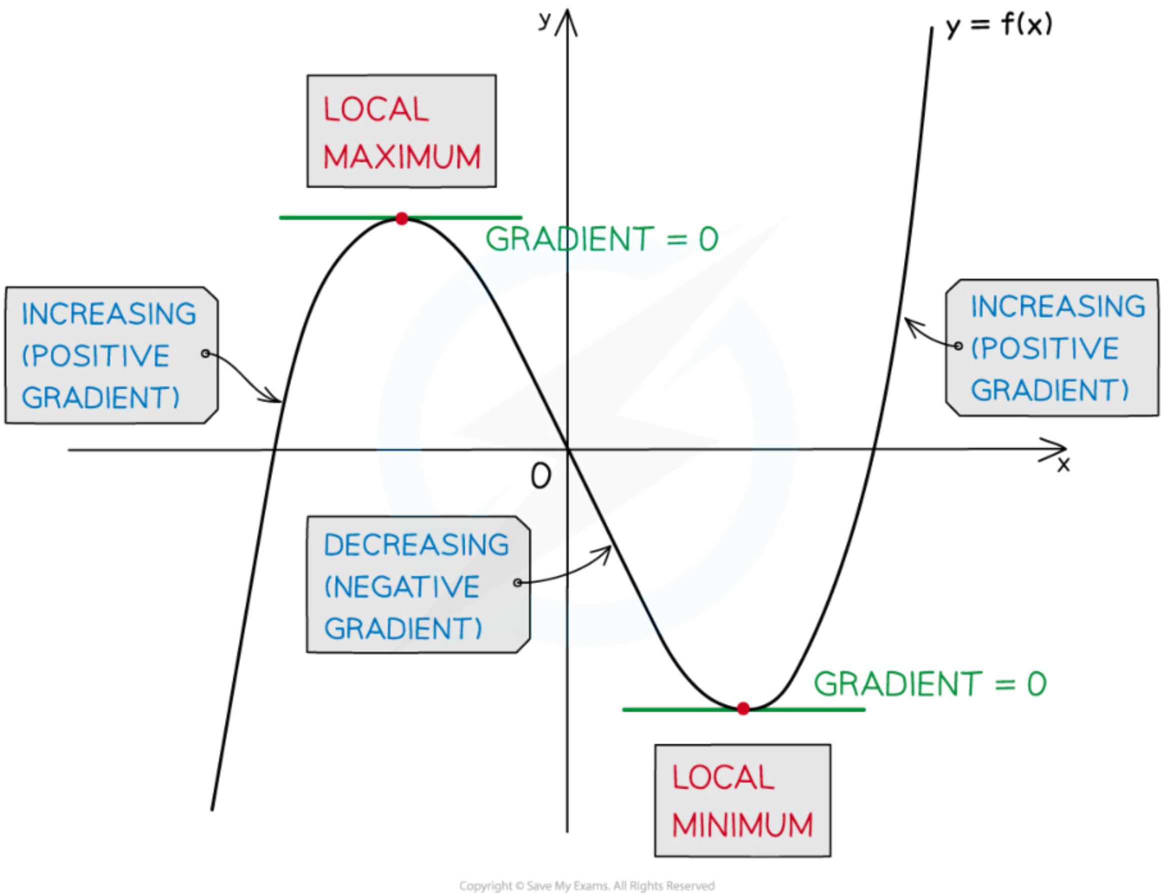
  STEP 3 (Second derivative)

  - If $f''(x) = 0$ then the nature of the stationary point **cannot** be determined; use the **first derivative** method (**STEP 4**)
  - If $f''(x) > 0$ then the curve of the graph of $y = f(x)$ is **concave up** and the stationary point is a **local minimum** point
  - If $f''(x) < 0$ then the curve of the graph of $y = f(x)$ is **concave down** and the stationary point is a **local maximum** point

  STEP 4 (First derivative)

  Find the sign of the first derivative just either side of the stationary point; i.e. evaluate $f'(x-h)$ and $f'(x+h)$ for small h

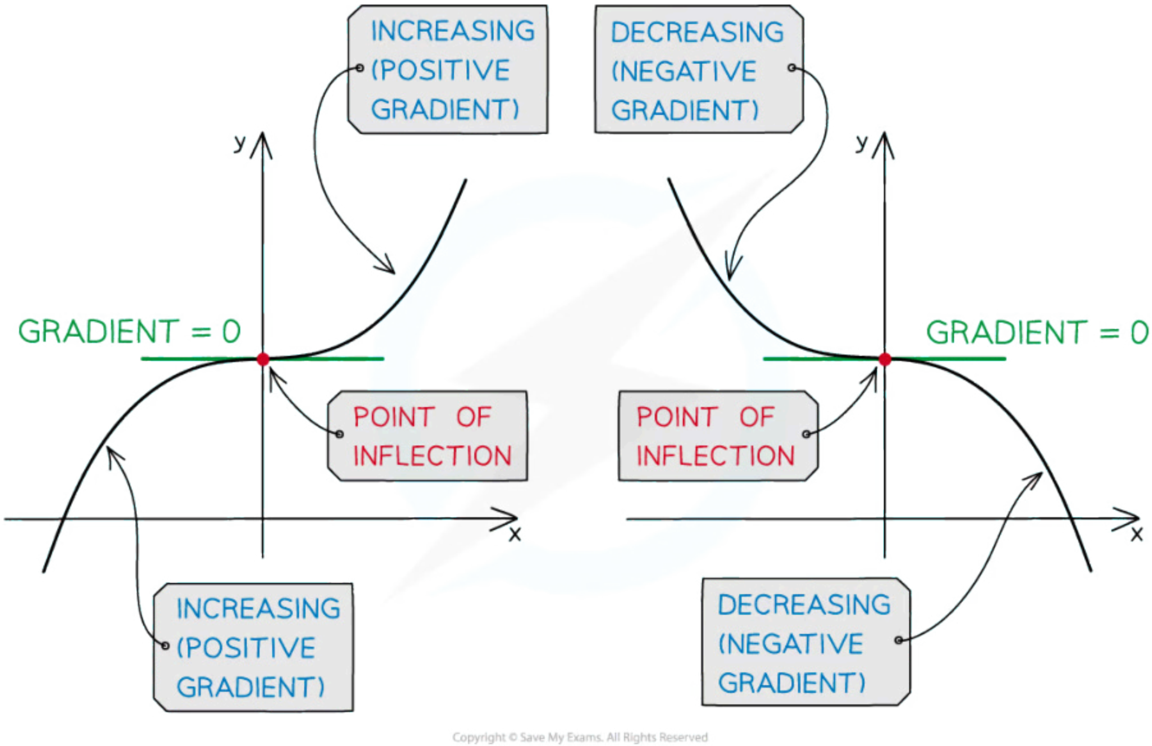
- A **local minimum point** changes the function from **decreasing** to **increasing**
  - the **gradient** changes from **negative** to **positive**
  - $f'(x-h)<0,\ f'(x)=0,\ f'(x+h)>0$
- A **local maximum point** changes the **function** from **increasing** to **decreasing**
  - the **gradient** changes from **positive** to **negative**
  - $f'(x-h)>0,\ f'(x)=0,\ f'(x+h)<0$



- A **stationary point** of **inflection** results from the function **either increasing or decreasing** on **both sides** of the stationary point
  - the **gradient** does **not change** sign
  - $f'(x-h)>0,\ f'(x+h)>0$ or $f'(x-h)<0,\ f'(x+h)<0$
  - a **point** of **inflection** does **not** necessarily have $f'(x)=0$
    - this method will only find those that do - and are often called **horizontal points** of **inflection**

Your notes

Your notes



Examiner Tip

- Exam questions may use the phrase "classify turning points" instead of "find the nature of turning points"
- Using your GDC to sketch the curve is a valid test for the nature of a stationary point in an exam unless the question says "show that..." or asks for an algebraic method
- Even if required to show a full algebraic solution you can still use your GDC to tell you what you're aiming for and to check your work

Your notes

Worked example

Find the coordinates and the nature of any stationary points on the graph of $y = f(x)$ where $f(x) = 2x^3 - 3x^2 - 36x + 25$.

Your notes

At stationary points, $f'(x) = 0$

$$f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6)$$

$$6(x^2 - x - 6) = 0$$

$$(x-3)(x+2) = 0$$

$x = 3$, $\quad y = f(3) = 2(3)^3 - 3(3)^2 - 36(3) + 25 = -56$

$x = -2$, $\quad y = f(-2) = 2(-2)^3 - 3(-2)^2 - 36(-2) + 25 = 69$

Using the second derivative to determine their nature

$$f''(x) = 12x - 6 = 6(2x - 1)$$

$$f''(3) = 6(2 \times 3 - 1) = 30 > 0$$

$\therefore$ $x = 3$ is a local minimum point

$$f''(-2) = 6(2 \times -2 - 1) = -30 < 0$$

$\therefore$ $x = -2$ is a local maximum point

(Note: In this case, both stationary points are turning points)

Turning points are:
$(3, -56)$ local minimum point
$(-2, 69)$ local maximum point

Use a GDC to graph $y = f(x)$ and the max/min solving feature to check the answers.

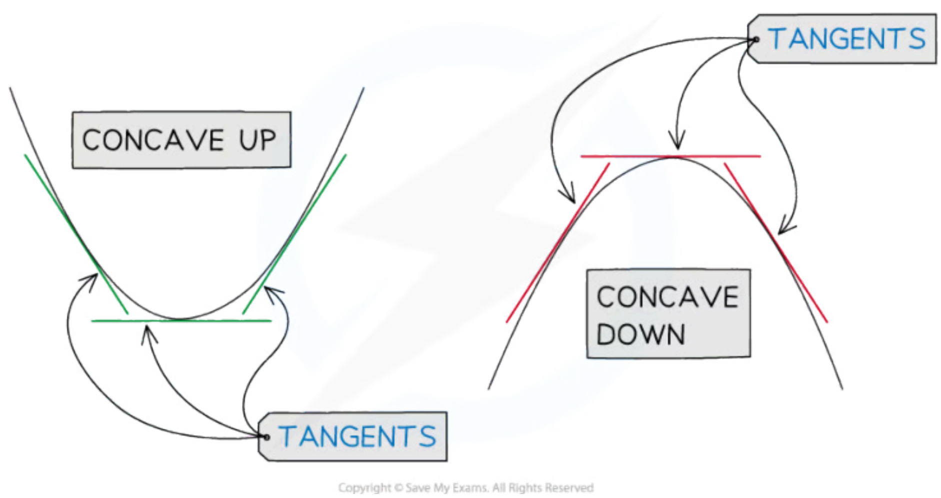
Your notes

5.2.5 Concavity & Points of Inflection

Concavity of a Function

What is concavity?

- **Concavity** is the way in which a **curve** (or surface) **bends**
- Mathematically,
  - a curve is **CONCAVE DOWN** if $f''(x) < 0$ for all values of x in an interval
  - a curve is **CONCAVE UP** if $f''(x) > 0$ for all values of x in an interval



Examiner Tip

- In an exam an easy way to remember the difference is:
  - Concave **down** is the shape of (the mouth of) a sad smiley ☹
  - **Concave up** is the shape of (the mouth of) a happy smiley ☺

Your notes

Worked example

The function $f(x)$ is given by $f(x) = x^3 - 3x + 2$.

a) Determine whether the curve of the graph of $y = f(x)$ is concave down or concave up at the points where $x = -2$ and $x = 2$.

$f(x) = x^3 - 3x + 2$
$f'(x) = 3x^2 - 3$
$f''(x) = 6x$

$f''(-2) = 6 \times -2 = -12 < 0$ (concave down)

$f''(2) = 6 \times 2 = 12 > 0$ (concave up)

$$\boxed{\begin{array}{l}\text{At } x = -2, \ y = f(x) \text{ is concave down} \\ \text{At } x = 2, \ y = f(x) \text{ is concave up}\end{array}}$$

Use your GDC to plot the graph of $y = f(x)$ and to help see if your answers are sensible

b) Find the values of x for which the curve of the graph $y = f(x)$ of is concave up.

$f''(x) = 6x$ from part (a)
Concave up is $f''(x) > 0$
$6x > 0$ when $x > 0$

$$\boxed{\therefore \ y = f(x) \text{ is concave up for } x > 0}$$

Use your GDC to check your answer

Points of Inflection

Your notes

What is a point of inflection?

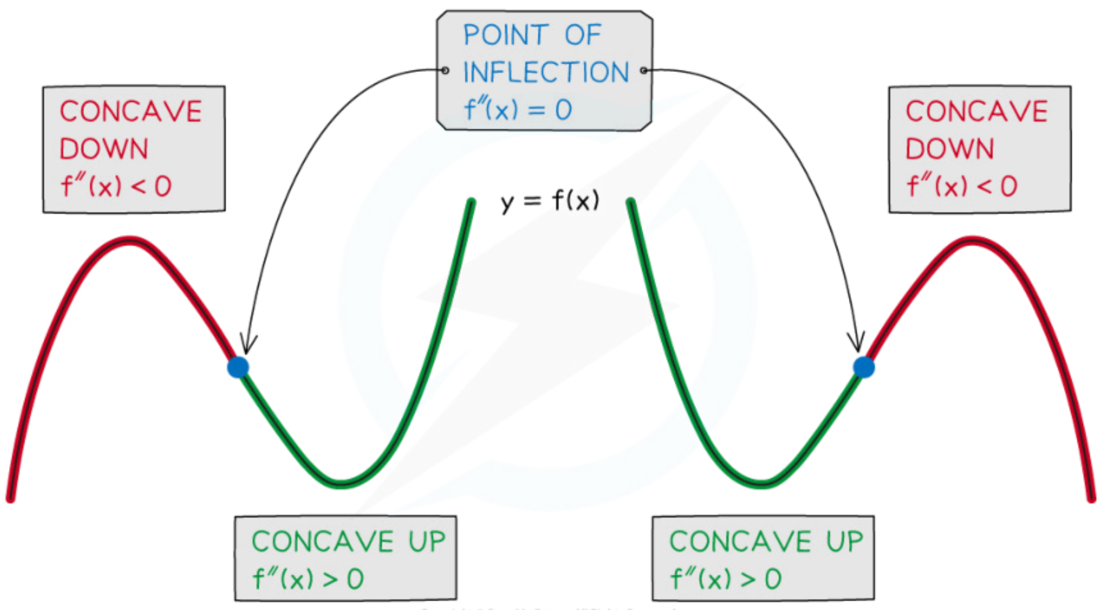
- A point at which the curve of the graph of $y = f(x)$ changes **concavity** is a **point** of **inflection**
- The alternative spelling, **inflexion,** may sometimes be used

What are the conditions for a point of inflection?

- A point of inflection requires **BOTH** of the following two conditions to hold
  - the **second derivative** is zero
    - $f''(x) = 0$

  AND

  - the graph of $y = f(x)$ changes **concavity**
    - $f''(x)$ changes **sign** through a **point** of **inflection**



- It is important to understand that the first condition is **not** sufficient on its own to locate a point of inflection
  - points where $f''(x) = 0$ could be **local minimum** or **maximum** points
    - the **first derivative** test would be needed
  - However, if it is already known $f(x)$ has a point of inflection at $x = a$, say, then $f''(a) = 0$

What about the first derivative, like with turning points?

Your notes

- A **point of inflection**, unlike a turning point, does not necessarily have to have a first derivative value of 0 ($f'(x) = 0$)
  - If it does, it is also a **stationary point** and is often called a **horizontal point of inflection**
    - the tangent to the curve at this point would be horizontal
  - The **normal distribution** is an example of a commonly used function that has a graph with two non-stationary points of inflection

How do I find the coordinates of a point of inflection?

- For the function $f(x)$

  STEP 1
  Differentiate $f(x)$ **twice** to find $f''(x)$ and solve $f''(x) = 0$ to find the x-coordinates of possible points of inflection

  STEP 2
  Use the **second derivative** to **test** the **concavity** of $f(x)$ either side of $x = a$

  - If $f''(x) < 0$ then $f(x)$ is concave down
  - If $f''(x) > 0$ then $f(x)$ is concave up

  If concavity changes, $x = a$ is a **point of inflection**

  STEP 3
  If required, the y-coordinate of a point of inflection can be found by substituting the x-coordinate into $f(x)$

Examiner Tip

- You can find the *x*-coordinates of the point of inflections of $y = f(x)$ by drawing the graph $y = f'(x)$ and finding the *x*-coordinates of any local maximum or local minimum points
- Another way is to draw the graph $y = f''(x)$ and find the *x*-coordinates of the points where the graph crosses (not just touches) the *x*-axis

Your notes

Worked example

Find the coordinates of the point of inflection on the graph of $y = 2x^3 - 18x^2 + 24x + 5$.
Fully justify that your answer is a point of inflection.

STEP 1: Differentiate twice, solve $f''(x) = 0$

$f(x) = 2x^3 - 18x^2 + 24x + 5$
$f'(x) = 6x^2 - 36x + 24$
$f''(x) = 12x - 36$
$12x - 36 = 0$ when $x = 3$

STEP 2: Use the second derivative to test concavity

$f''(3) = 0$
$f''(2.9) < 0$ (concave down)
$f''(3.1) > 0$ (concave up)
$\therefore$ concavity changes through $x = 3$

STEP 3: The y-coordinate is required

$f(3) = 2(3)^3 - 18(3)^2 + 24(3) + 5 = -31$

Since $f''(3) = 0$ AND the graph of $y = f(x)$ changes concavity through $x = 3$, the point $(3, -31)$ is a point of inflection.

Use your GDC to plot the graph of $y = f(x)$ and to help see if your answer is sensible

Your notes

5.2.6 Derivatives & Graphs

Derivatives & Graphs

How are derivatives and graphs connected?

- If the graph of a function $y = f(x)$ is known, or can be sketched, then it is also possible to sketch the graphs of the derivatives $y = f'(x)$ and $y = f''(x)$
- The key properties of a graph include
  - the $\boldsymbol{y}$**-axis intercept**
  - the $\boldsymbol{x}$**-axis intercepts** – the **roots** of the function; where $f(x) = 0$
  - **stationary points**; where $f'(x) = 0$
    - turning points – (local) **minimum** and **maximum** points
    - (horizontal) **points of inflection**
  - (non-stationary, $f'(x) \neq 0$) points of inflection
  - **asymptotes** – **vertical** and **horizontal**
  - intervals where the graph is **increasing** and **decreasing**
  - intervals where the graph is **concave down** and **concave up**
- **Not** all graphs have all of these properties and **not** all can be determined without knowing the expression of the function
- However questions will provide enough information to sketch
  - the **shape** of the graph
  - **some** of the **key properties** such as **roots** or **turning points**

How do I sketch the graph of *y* = f'(*x*) from the graph of *y* = f(*x*)?

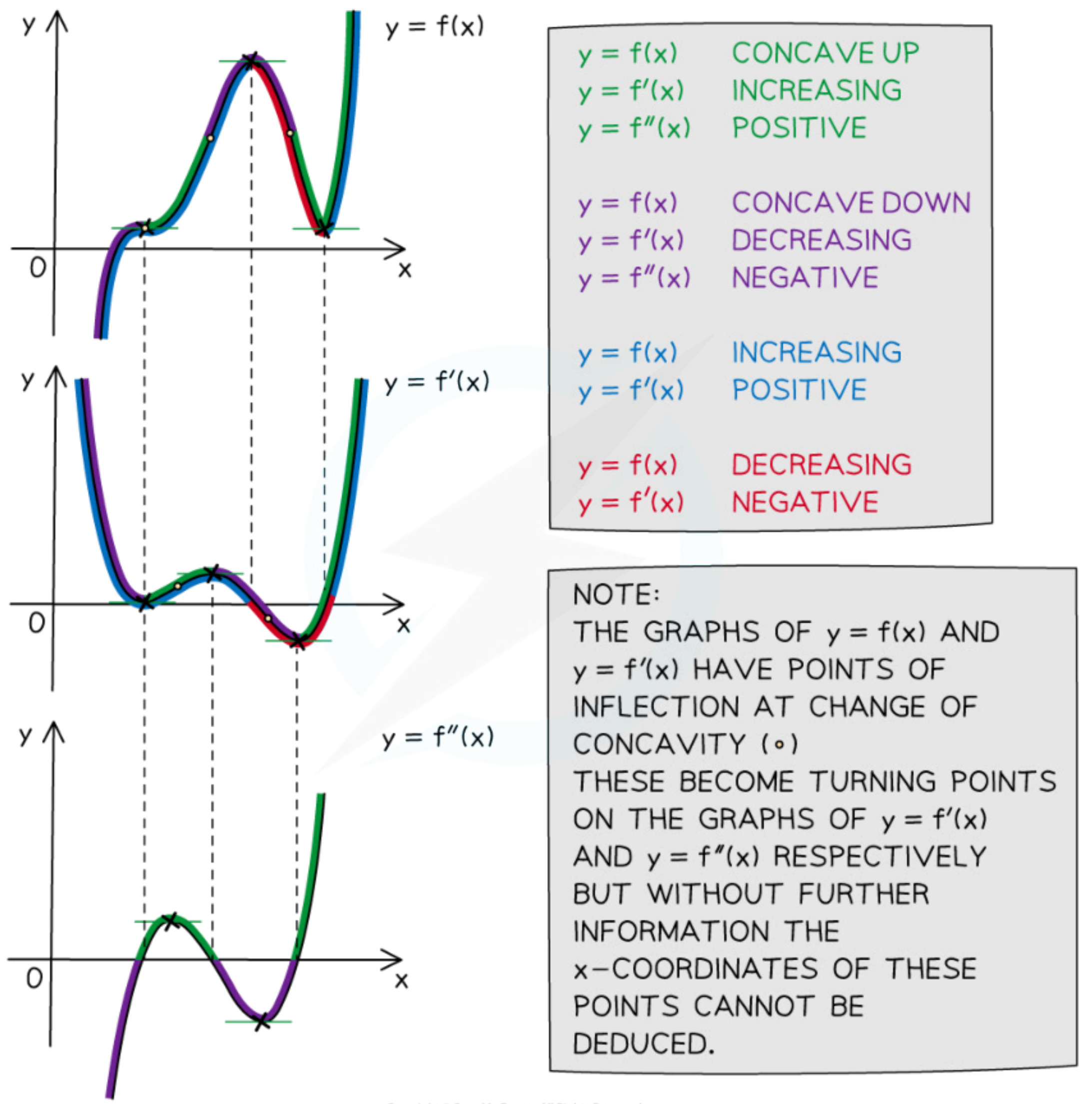
- The graph of $y = f'(x)$ will have its
  - x-axis **intercepts** at the x-coordinates of the **stationary points** of $y = f(x)$
  - **turning points** at the x-coordinates of the **points of inflection** of $y = f(x)$
- For intervals where $y = f(x)$ is concave up, $y = f'(x)$ will be increasing
- For intervals where $y = f(x)$ is concave down, $y = f'(x)$ will be decreasing
- For intervals where $y = f(x)$ is increasing, $y = f'(x)$ will be positive
- For intervals where $y = f(x)$ is decreasing, $y = f'(x)$ will be negative

How do I sketch the graph of *y* = f''(*x*) from the graph of *y* = f(*x*)?

- First sketch the graph of $y = f'(x)$ from $y = f(x)$, as per the above process
- Then, using the same process, sketch the graph of $y = f''(x)$ from the graph of $y = f'(x)$
- There are a couple of things you can deduce about the graph of $y = f''(x)$ directly from the graph of $y = f(x)$

Your notes

- The graph of $y = f''(x)$ will have its x-axis **intercepts** at the x-coordinates of the **points of inflection** of $y = f(x)$
- For intervals where $y = f(x)$ is concave up, $y = f''(x)$ will be positive
- For intervals where $y = f(x)$ is concave down, $y = f''(x)$ will be negative



Is it possible to sketch the graph **of y = f(x) from the graph of a derivative?**

- It is possible to **sketch** a graph of $y = f(x)$ by considering the reverse of the above
  - For intervals where $y = f'(x)$ is **positive**, $y = f(x)$ will be **increasing** but is **not** necessarily positive

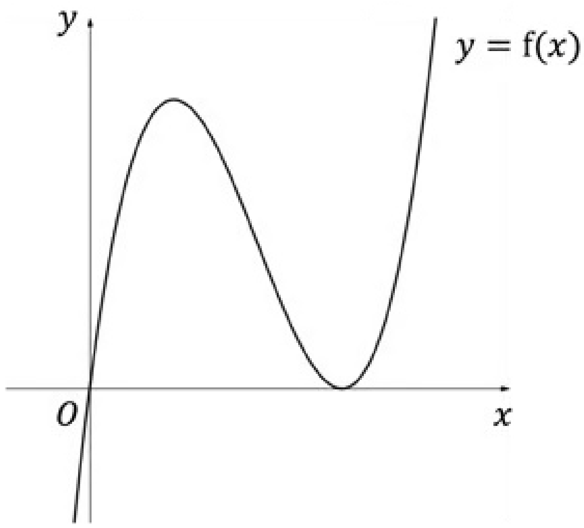
Your notes

- For intervals where $y = f'(x)$ is **negative**, $y = f(x)$ will be **decreasing** but is **not** necessarily negative
- **Roots** of $y = f'(x)$ give the x-coordinates of the **stationary points** of $y = f(x)$
- There are some properties of the graph of $y = f(x)$ that **cannot** be determined from the graph of $y = f'(x)$
  - the $\boldsymbol{y}$**-axis intercept**
  - the **intervals** for which $y = f(x)$ is **positive** and **negative**
  - the roots of $y = f(x)$
- Unless a **specific** point the **curve passes through** is **known**, the **constant** of **integration cannot** be determined
  - the *exact* location of the curve will remain unknown
  - but it will still be possible to **sketch** its **shape**
- If starting from the graph of the **second derivative**, $y = f''(x)$, it is easier to sketch the graph of $y = f'(x)$ first, then sketch $y = f(x)$

Your notes

Worked example

The graph of $y = f(x)$ is shown in the diagram below.



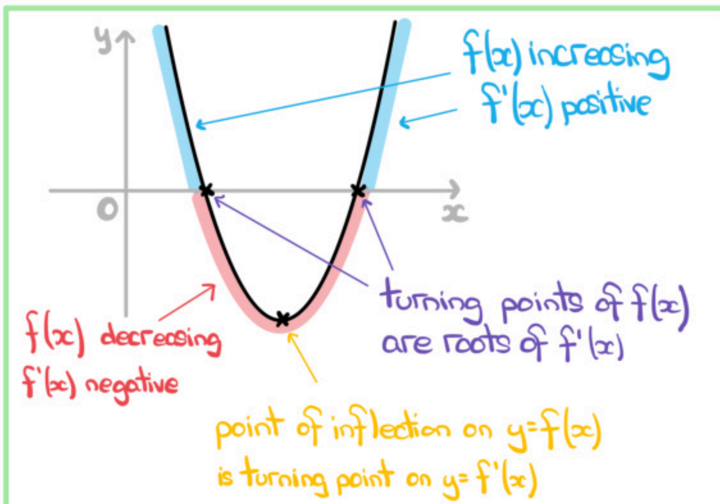
On separate diagrams sketch the graphs of $y = f'(x)$ and $y = f''(x)$, labelling any roots and turning points.

Your notes

Key features from graph of $y = f(x)$ are:

- local maximum and local minimum
- $f(x)$ is increasing 'at either end'
- $f(x)$ is decreasing between max. and min.
- point of inflexion (non-stationary), graph changes from concave down to concave up

Graph of $y = f'(x)$:



Graph of $y = f''(x)$: