DP IB Maths: AI SL

Your notes

4.6 Normal Distribution

Contents

4.6.1 The Normal Distribution

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Properties of Normal Distribution

The binomial distribution is an example of a discrete probability distribution. The normal distribution is an example of a **continuous** probability distribution.

What is a continuous random variable?

- A continuous random variable (often abbreviated to CRV) is a random variable that can take **any value** within a range of infinite values
  - Continuous random variables **usually measure** something
  - For example, height, weight, time, etc

What is a continuous probability distribution?

- A continuous probability distribution is a probability distribution in which the random variable X is continuous
- The probability of X being a **particular value is always zero**
  - $\mathrm{P}(X = k) = 0$ for any value *k*
  - Instead we define the **probability density function** $\mathrm{f}(x)$ for a specific value
    - This is a function that describes the **relative likelihood** that the random variable would be close to that value
  - We talk about the **probability** of X being within a **certain range**
- A continuous probability distribution can be represented by a continuous graph (the values for X along the horizontal axis and probability **density** on the vertical axis)
- The **area under the graph** between the points $x = a$ and $x = b$ is equal to $\mathrm{P}(a \le X \le b)$
  - The **total area under the graph equals 1**
- As $\mathrm{P}(X = k) = 0$ for any value *k*, it does not matter if we use strict or weak inequalities
  - $\mathrm{P}(X \le k) = \mathrm{P}(X < k)$ for any value *k* when *X* is a **continuous random variable**

What is a normal distribution?

- A normal distribution is a **continuous probability distribution**
- The **continuous random variable** X can follow a normal distribution if:
  - The distribution is **symmetrical**
  - The distribution is **bell-shaped**
- If X follows a normal distribution then it is denoted $X \sim \mathrm{N}(\mu,\ \sigma^2)$
  - μ is the **mean**
  - σ^2 is the **variance**
  - σ is the **standard deviation**
- If the **mean** changes then the graph is **translated horizontally**

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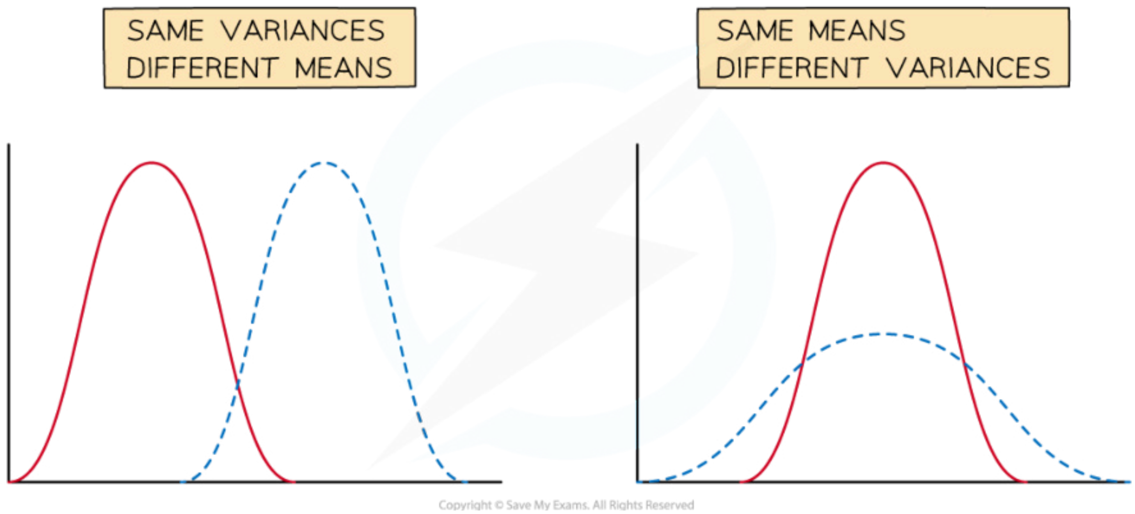
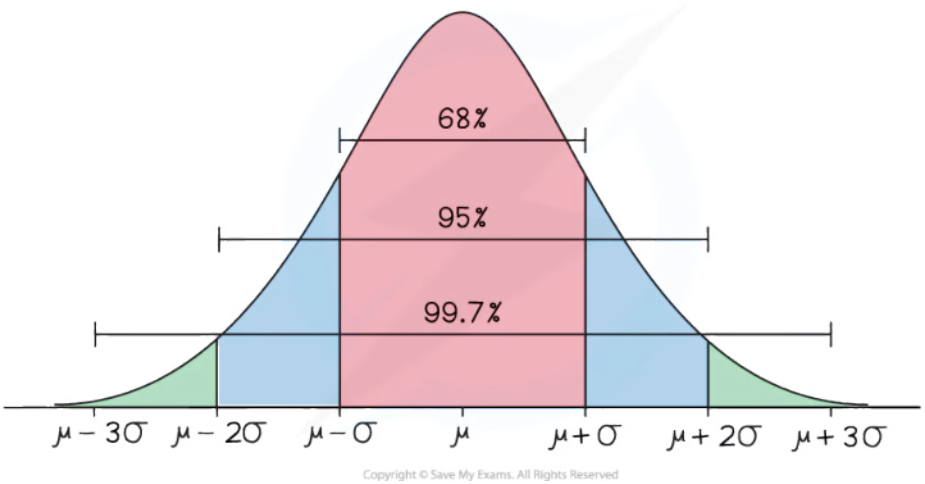
- If the **variance** increases then the graph is **widened horizontally** and **made shorter vertically** to maintain the same area
  - A **small variance** leads to a **tall** curve with a **narrow** centre
  - A **large variance** leads to a **short** curve with a **wide** centre



What are the important properties of a normal distribution?

- The **mean** is μ
- The **variance** is σ^2
  - If you need the **standard deviation** remember to square root this
- The normal distribution is symmetrical about $x = \mu$
  - Mean = Median = Mode = μ
- There are the results:
  - Approximately **two-thirds (68%)** of the data lies within **one standard deviation** of the mean ($\mu \pm \sigma$)
  - Approximately **95%** of the data lies within **two standard deviations** of the mean ($\mu \pm 2\sigma$)
  - Nearly **all of the data (99.7%)** lies within **three standard deviations** of the mean ($\mu \pm 3\sigma$)


68%
95%
99.7%
$\mu-3\sigma$
$\mu-2\sigma$
$\mu-\sigma$
μ
$\mu+\sigma$
$\mu+2\sigma$
$\mu+3\sigma$
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Modelling with Normal Distribution

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What can be modelled using a normal distribution?

- A lot of real-life continuous variables can be modelled by a normal distribution provided that the population is large enough and that the variable is **symmetrical** with **one mode**
- For a normal distribution X can take any real value, however values far from the mean (more than 4 standard deviations away from the mean) have a probability density of **practically zero**
  - This fact allows us to model variables that are not defined for all real values such as height and weight

What can not be modelled using a normal distribution?

- Variables which have **more than one mode** or **no mode**
  - For example: the number given by a random number generator
- Variables which are **not symmetrical**
  - For example: how long a human lives for

Examiner Tip

- An exam question might involve different types of distributions so make it clear which distribution is being used for each variable

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Worked example

The random variable S represents the speeds (mph) of a certain species of cheetahs when they run. The variable is modelled using $\mathrm{N}(40,\ 100)$.

a) Write down the mean and standard deviation of the running speeds of cheetahs.

$\mu = 40$ and $\sigma^2 = 100$

↑ Square root to get standard deviation

Mean $\mu = 40$
Standard deviation $\sigma = 10$

b) State two assumptions that have been made in order to use this model.

We assume that the distribution of the speeds is
- symmetrical
- bell-shaped

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4.6.2 Calculations with Normal Distribution

Calculating Normal Probabilities

Throughout this section we will use the random variable $X \sim \mathrm{N}(\mu,\ \sigma^2)$. For *X* distributed normally, *X* can take any real number. Therefore any values mentioned in this section will be assumed to be real numbers.

How do I find probabilities using a normal distribution?

- The **area under a normal curve** between the points $x = a$ and $x = b$ is equal to the **probability** $\mathrm{P}(a < X < b)$
  - Remember for a normal distribution you do not need to worry about whether the inequality is strict (< or >) or weak (≤ or ≥)
    - $\mathrm{P}(a < X < b) = \mathrm{P}(a \le X \le b)$
- You will be **expected to use** distribution functions on your **GDC** to find the probabilities when working with a normal distribution

How do I calculate P(*X* = *x*): the probability of a single value for a normal distribution?

- The probability of a **single value** is **always zero** for a normal distribution
  - You can picture this as the area of a single line is zero
- $\mathrm{P}(X = x) = 0$
- Your GDC is likely to have a "**Normal Probability Density**" function
  - This is sometimes shortened to NPD, Normal PD or Normal Pdf
  - **IGNORE THIS FUNCTION** for this course!
  - This calculates the **probability density function** at a point **NOT the probability**

How do I calculate P(*a* < *X* < *b*): the probability of a range of values for a normal distribution?

- You need a **GDC** that can calculate **cumulative normal probabilities**
- You want to use the "**Normal Cumulative Distribution**" function
  - This is sometimes shortened to NCD, Normal CD or Normal Cdf
- You will need to enter:
  - The 'lower bound' - this is the value *a*
  - The 'upper bound' - this is the value *b*
  - The 'μ' value - this is the mean
  - The 'σ' value - this is the standard deviation
- **Check the order carefully** as some calculators ask for standard deviation before mean
  - Remember it is the standard deviation
    - so if you have the **variance** then **square root it**
- **Always sketch** a quick diagram to visualise which area you are looking for

How do I calculate P(*X* > *a*) or P(*X* < *b*) for a normal distribution?

- You will still use the "**Normal Cumulative Distribution"** function
- $\mathrm{P}(X > a)$ can be estimated using an **upper bound that is sufficiently bigger** than the **mean**
  - Using a value that is more than 4 standard deviations **bigger than the mean** is quite accurate
  - Or an easier option is just to input lots of 9's for the upper bound (**99999999...** or **10^{99}**)
- $\mathrm{P}(X < b)$ can be estimated using a **lower bound that is sufficiently smaller** than the **mean**
  - Using a value that is more than 4 standard deviations **smaller than the mean** is quite accurate
  - Or an easier option is just to input lots of 9's for the lower bound with a negative sign (**-99999999...** or **-10^{99}**)

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Are there any useful identities?

- $\mathrm{P}(X < \mu) = \mathrm{P}(X > \mu) = 0.5$
- As $\mathrm{P}(X = a) = 0$ you can use:
  - $\mathrm{P}(X < a) + \mathrm{P}(X > a) = 1$
  - $\mathrm{P}(X > a) = 1 - \mathrm{P}(X < a)$
  - $\mathrm{P}(a < X < b) = \mathrm{P}(X < b) - \mathrm{P}(X < a)$
- These are useful when:
  - The mean and/or standard deviation are unknown
  - You only have a diagram
  - You are working with the **inverse distribution**

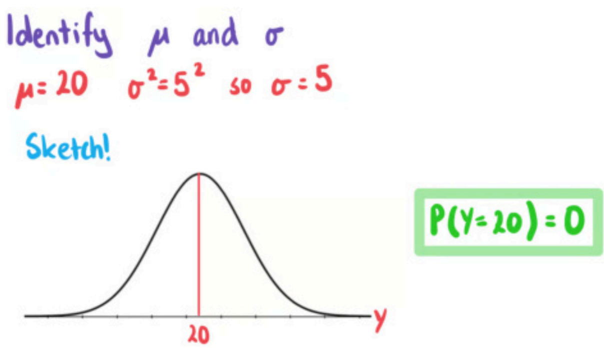
Examiner Tip

- Check carefully whether you have entered the standard deviation or variance into your GDC

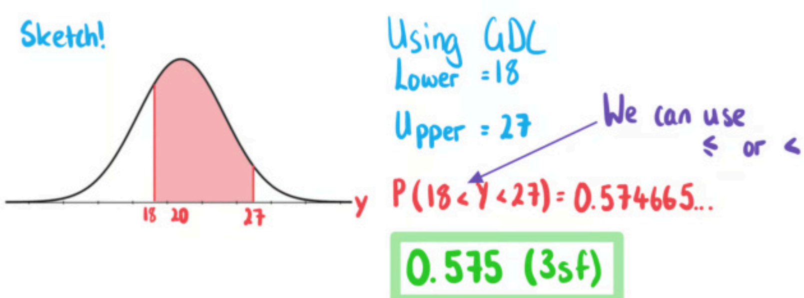
Worked example

The random variable $Y \sim \mathrm{N}(20, 5^2)$. Calculate:

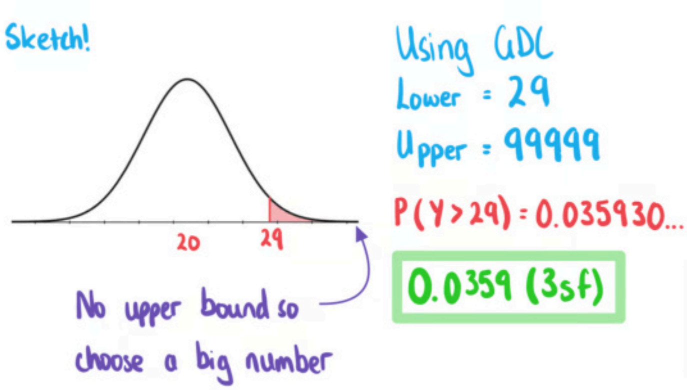
i) $\mathrm{P}(Y = 20)$.



ii) $\mathrm{P}(18 \le Y < 27)$.



iii) $\mathrm{P}(Y > 29)$



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Inverse Normal Distribution

Given the value of P(*X* < *a*) how do I find the value of *a*?

- Your **GDC** will have a function called "**Inverse Normal Distribution**"
  - Some calculators call this InvN
- Given that $\mathrm{P}(X < a) = p$ you will need to enter:
  - The 'area' - this is the value *p*
    - Some calculators might ask for the 'tail' - this is the left tail as you know the area to the left of *a*
  - The 'μ' value - this is the mean
  - The 'σ' value - this is the standard deviation

Given the value of P(*X* > *a*) how do I find the value of *a*?

- If your calculator **does** have the **tail option** (left, right or centre) then you can use the "Inverse Normal Distribution" function straightaway by:
  - Selecting 'right' for the tail
  - Entering the area as '*p*'
- If your calculator **does not** have the **tail option** (left, right or centre) then:
  - Given $\mathrm{P}(X > a) = p$
  - Use $\mathrm{P}(X < a) = 1 - \mathrm{P}(X > a)$ to rewrite this as
    - $\mathrm{P}(X < a) = 1 - p$
  - Then use the **method for P(*X* < a)** to find *a*

Examiner Tip

- Always check your **answer makes sense**
  - If P(*X* < *a*) is **less than 0.5** then *a* should be **smaller than the mean**
  - If P(*X* < *a*) is **more than 0.5** then *a* should be **bigger than the mean**
  - A sketch will help you see this

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Worked example

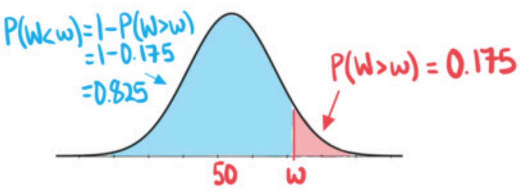
The random variable $W \sim \mathrm{N}(50, 36)$.

Find the value of w such that $\mathrm{P}(W > w) = 0.175$.

Identify μ and σ

$\mu = 50$ $\sigma^2 = 36$ so $\sigma = 6$

Sketch!



P(W > w) is less than 0.5 so w is bigger than the mean

Area from left is 0.825

Use Inverse Normal Distribution function on GDC

$w = 55.6075...$

$w = 55.6$ (3sf)