

DP IB Maths: AA HL



Your notes

1.2 Exponentials & Logs

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1.2.1 Introduction to Logarithms

Introduction to Logarithms

What are logarithms?

- A logarithm is the inverse of an exponent
 - If $a^x = b$ then $\log_a(b) = x$ where $a > 0, b > 0, a \neq 1$
 - This is in the formula booklet
 - The number a is called the **base** of the logarithm
 - Your GDC will be able to use this function to solve equations involving exponents
- Try to get used to 'reading' logarithm statements to yourself
 - $\log_a(b) = x$ would be read as "the power that you raise a to, to get b , is x "
 - So $\log_5 125 = 3$ would be read as "the power that you raise 5 to, to get 125, is 3"
- Two important cases are:
 - $\ln x = \log_e(x)$
 - Where e is the mathematical constant 2.718...
 - This is called the **natural logarithm** and will have its own button on your GDC
 - $\log x = \log_{10}(x)$
 - Logarithms of **base 10** are used often and so abbreviated to **log x**

Why use logarithms?

- Logarithms allow us to solve equations where the exponent is the unknown value
 - We can solve some of these by inspection
 - For example, for the equation $2^x = 8$ we know that x must be 3
 - Logarithms allow use to solve more complicated problems
 - For example, the equation $2^x = 10$ does not have a clear answer
 - Instead, we can use our GDCs to find the value of $\log_2 10$

Examiner Tip

- Before going into the exam, make sure you are completely familiar with your GDC and know how to use its logarithm functions



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 **Worked example**

Solve the following equations:

i) $x = \log_3 27,$

$$x = \log_3 27 \iff 3^x = 27$$

We can see from inspection:

$$3^3 = 27 \iff x = 3$$

$$x = 3$$

OR: use GDC to find answer directly.

ii) $2^x = 21.4,$ giving your answer to 3 s.f.

$2^x = 21.4$ This cannot be seen
from inspection:

$$2^x = 21.4 \iff x = \log_2 21.4$$

use GDC to find answer directly.

$$\log_2 21.4 = 4.4195\dots$$

$$x = 4.42 \text{ (3 s.f.)}$$



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1.2.2 Laws of Logarithms

Laws of Logarithms

What are the laws of logarithms?

- Laws of logarithms allow you to simplify and manipulate expressions involving logarithms
 - The laws of logarithms are equivalent to the **laws of indices**
- The laws you need to know are, given $a, x, y > 0$:
 - $\log_a xy = \log_a x + \log_a y$
 - This relates to $a^x \times a^y = a^{x+y}$
 - $\log_a \frac{x}{y} = \log_a x - \log_a y$
 - This relates to $a^x \div a^y = a^{x-y}$
 - $\log_a x^m = m \log_a x$
 - This relates to $(a^x)^y = a^{xy}$
- These laws are **in the formula booklet** so you do not need to remember them
 - You must make sure you know how to use them

$$\log_a xy = \log_a x + \log_a y$$

$$\text{RELATES TO } a^x \times a^y = a^{x+y}$$

$$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\text{RELATES TO } \frac{a^x}{a^y} = a^{x-y}$$

$$\log_a x^k = k \log_a x$$

$$\text{RELATES TO } (a^x)^y = a^{xy}$$

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Useful results from the laws of logarithms

- Given $a > 0, a \neq 1$
 - $\log_a 1 = 0$
 - This is equivalent to $a^0 = 1$
- If we substitute b for a into the given identity in the formula booklet



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- $a^x = b \Leftrightarrow \log_a b = x$ where $a > 0, b > 0, a \neq 1$
- $a^x = a \Leftrightarrow \log_a a = x$ gives $a^1 = a \Leftrightarrow \log_a a = 1$
 - This is an important and useful result
- Substituting this into the third law gives the result
 - $\log_a a^k = k$
- Taking the inverse of its operation gives the result
 - $a^{\log_a x} = x$
- From the third law we can also conclude that
 - $\log_a \frac{1}{x} = -\log_a x$

$$\log_a a = 1$$

"THE POWER YOU RAISE
a TO, TO GET a, IS 1"

$$\log_a a^x = x$$

$$\log_a a^x = x \log_a a$$

$$= x$$

$$a^{\log_a x} = x$$

AN OPERATION AND
ITS INVERSE

$$\log_a 1 = 0$$

$$a^0 = 1$$

$$\log_a \frac{1}{x} = -\log_a x$$

$$\log_a \frac{1}{x} = \log_a x^{-1}$$

$$= -\log_a x$$

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- These useful results are **not in the formula booklet** but can be deduced from the laws that are
- Beware...
 - ... $\log_a (x + y) \neq \log_a x + \log_a y$
- These results apply to $\ln x$ ($\log_e x$) too
 - Two particularly useful results are
 - $\ln e^x = x$

- $e^{\ln x} = x$
- Laws of logarithms can be used to ...
 - simplify expressions
 - solve logarithmic equations
 - solve exponential equations

 **Examiner Tip**

- Remember to check whether your solutions are valid
 - $\log(x+k)$ is only defined if $x > -k$
 - You will lose marks if you forget to reject invalid solutions



Your notes



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 **Worked example**

- a) Write the expression $2 \log 4 - \log 2$ in the form $\log k$, where $k \in \mathbb{Z}$.

Using the law $\log_a x^m = m \log_a x$

$$2 \log 4 = \log 4^2 = \log 16$$

$$\begin{aligned} 2 \log 4 - \log 2 &= \log 4^2 - \log 2 \\ &= \log 16 - \log 2 \end{aligned}$$

Using the law $\log_a \frac{x}{y} = \log_a x - \log_a y$

$$\log 16 - \log 2 = \log \frac{16}{2} = \log 8$$

$$\boxed{2 \log 4 - \log 2 = \log 8}$$

- b) Hence, or otherwise, solve $2 \log 4 - \log 2 = -\log \frac{1}{x}$.



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To solve $2\log 4 - \log 2 = \log \frac{1}{x}$ rewrite as

$$\begin{array}{l} \text{from} \\ \text{part (a)} \end{array} \log 8 = -\log \frac{1}{x}$$

Use the index law $\frac{1}{x} = x^{-1}$

$$\log 8 = -\log x^{-1}$$

$$\log 8 = \log x \quad \leftarrow \log_a x^m = m \log_a x$$

$$8 = x$$

$$x = 8$$



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Change of Base

Why change the base of a logarithm?

- The laws of logarithms can only be used if the logs have the same **base**
 - If a problem involves logarithms with different bases, you can change the base of the logarithm and then apply the laws of logarithms
- **Changing the base** of a logarithm can be particularly useful if you need to evaluate a log problem **without a calculator**
 - Choose the base such that you would know how to solve the problem from the equivalent exponent

How do I change the base of a logarithm?

- The formula for changing the base of a logarithm is

$$\log_a x = \frac{\log_b x}{\log_b a}$$

- This is **in the formula booklet**
- The value you choose for b does not matter, however if you do not have a calculator, you can choose b such that the problem will be possible to solve

Examiner Tip

- Changing the base is a key skill which can help you with many different types of questions, make sure you are confident with it
 - It is a particularly useful skill for examinations where a GDC is not permitted



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Worked example

By choosing a suitable value for b , use the change of base law to find the value of $\log_8 32$ without using a calculator.

Change of base law: $\log_a x = \frac{\log_b x}{\log_b a}$

$$\log_8 32$$

$2^5 = 32$ (pointing to 32)
 $2^3 = 8$ (pointing to 8)

Choose $b = 2$ to allow for a solution by inspection

$$\log_8 32 = \frac{\log_2 32}{\log_2 8} = \frac{5}{3}$$

$$\log_8 32 = \frac{5}{3}$$



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1.2.3 Solving Exponential Equations

Solving Exponential Equations

What are exponential equations?

- An exponential equation is an equation where the unknown is a power
 - In simple cases the solution can be spotted without the use of a calculator
 - For example,

$$\begin{aligned} 5^{2x} &= 125 \\ 2x &= 3 \\ x &= \frac{3}{2} \end{aligned}$$

- In more complicated cases the laws of logarithms should be used to solve exponential equations
- The **change of base** law can be used to solve some exponential equations without a calculator
 - For example,

$$\begin{aligned} 27^x &= 9 \\ x &= \log_{27} 9 \\ &= \frac{\log_3 9}{\log_3 27} \\ &= \frac{2}{3} \end{aligned}$$

How do we use logarithms to solve exponential equations?

- An exponential equation can be solved by taking logarithms of both sides
- The **laws of indices** may be needed to rewrite the equation first
- The **laws of logarithms** can then be used to solve the equation
 - ln (log_e)** is often used
 - The answer is often written in terms of ln
- A question may ask you to give your answer in a particular form
- Follow these steps to solve exponential equations
 - STEP 1: Take logarithms of both sides
 - STEP 2: Use the laws of logarithms to remove the powers
 - STEP 3: Rearrange to isolate x
 - STEP 4: Use logarithms to solve for x

What about hidden quadratics?

- Look for hidden squared terms that could be changed to form a quadratic
 - In particular look out for terms such as
 - $4^x = (2^2)^x = 2^{2x} = (2^x)^2$
 - $e^{2x} = (e^2)^x = (e^x)^2$



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Examiner Tip

- Always check which form the question asks you to give your answer in, this can help you decide how to solve it
- If the question requires an exact value you may need to leave your answer as a logarithm



Your notes

Worked example

Solve the equation $4^x - 3(2^{x+1}) + 9 = 0$. Give your answer correct to three significant figures.

Spot the hidden quadratic: $4^x = (2^2)^x = (2^x)^2$

By the laws of indices $2^{x+1} = 2^x \times 2^1$

$$(2^x)^2 - 3(2^{x+1}) + 9 = 0$$

$$= 2^x \times 2^1$$

$$(2^x)^2 - 3 \times 2 \times 2^x + 9 = 0$$

$$(2^x)^2 - 6 \times 2^x + 9 = 0$$

Let $u = 2^x$ $u^2 - 6u + 9 = 0$

$$(u - 3)(u - 3) = 0$$

$$u = 3 \quad \therefore 2^x = 3$$

Solve the exponential equation $2^x = 3$

Step 1: Take logarithms of both sides: $\ln(2^x) = \ln(3)$

Step 2: Use the law $\log_a x^m = m \log_a x$ $x \ln 2 = \ln 3$

Step 3: Rearrange to isolate x $x = \frac{\ln 3}{\ln 2}$

Step 4: Solve

$$x = \frac{\ln 3}{\ln 2} = 1.584\dots$$

$$x = 1.58 \text{ (3s.f.)}$$