DP IB Maths: AA HL

Your notes

5.11 MacLaurin Series

Contents

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5.11.1 Maclaurin Series

Maclaurin Series of Standard Functions

What is a Maclaurin Series?

- A Maclaurin series is a way of representing a function as an infinite sum of increasing integer powers of x (x^1, x^2, x^3, etc.)
  - If all of the infinite number of terms are included, then the Maclaurin series is exactly equal to the original function
  - If we **truncate** (i.e., shorten) the Maclaurin series by stopping at some particular power of x, then the Maclaurin series is only an approximation of the original function
- A truncated Maclaurin series will always be exactly equal to the original function for $x = 0$
- In general, the approximation from a truncated Maclaurin series becomes less accurate as the value of x moves further away from zero
- The accuracy of a truncated Maclaurin series approximation can be improved by including more terms from the complete infinite series
  - So, for example, a series truncated at the x^7 term will give a more accurate approximation than a series truncated at the x^3 term

How do I find the Maclaurin series of a function 'from first principles'?

- Use the **general Maclaurin series formula**

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots$$

- This formula is in your exam formula booklet
- STEP 1: Find the values of $f(0)$, $f'(0)$, $f''(0)$, etc. for the function
  - An exam question will specify how many terms of the series you need to calculate (for example, "up to and including the term in x^4")
  - You may be able to use your GDC to find these values directly without actually having to find all the necessary derivatives of the function first
- STEP 2: Put the values from Step 1 into the general Maclaurin series formula
- STEP 3: Simplify the coefficients as far as possible for each of the powers of x

Is there an easier way to find the Maclaurin series for standard functions?

- Yes there is!
- The following Maclaurin series expansions of standard functions are contained in your exam formula booklet:

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots$$

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$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$$

- Unless a question specifically asks you to derive a Maclaurin series using the general Maclaurin series formula, you can use those standard formulae from the exam formula booklet in your working

Is there a connection Maclaurin series expansions and binomial theorem series expansions?

- Yes there is!
- For a function like $(1+x)^n$ the binomial theorem series expansion is **exactly the same** as the Maclaurin series expansion for the same function
  - So unless a question specifically tells you to use the general Maclaurin series formula, you can use the binomial theorem to find the Maclaurin series for functions of that type
  - Or if you've forgotten the binomial series expansion formula for $(1+x)^n$ where n is not a positive integer, you can find the binomial theorem expansion by using the general Maclaurin series formula to find the Maclaurin series expansion

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Worked example

a) Use the Maclaurin series formula to find the Maclaurin series for $f(x) = \sqrt{1+2x}$ up to and including the term in x^4.

$$f(x) = \sqrt{1+2x} = (1+2x)^{\frac{1}{2}}$$

STEP 1: $f(0) = 1 \quad f'(0) = 1 \quad f''(0) = -1$

$f'''(0) = 3 \quad f^{(4)}(0) = -15$

STEP 2: $f(x) = 1 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(3) + \frac{x^4}{4!}(-15) + \ldots$

STEP 3: Up to the x^4 term,

$$\boxed{\sqrt{1+2x} = 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{8}x^4}$$

Note: This is the same as the binomial theorem expansion of $(1+2x)^{\frac{1}{2}}$

b) Use your answer from part (a) to find an approximation for the value of $\sqrt{1.02}$, and compare the approximation found to the actual value of the square root.

Your notes

Up to the x^4 term,

$$\boxed{\sqrt{1+2x} = 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{8}x^4}$$ from part (a)

Let $x = 0.01$. Then $\sqrt{1+2x} = \sqrt{1+2(0.01)} = \sqrt{1.02}$.

So

$$\sqrt{1.02} \approx 1 + (0.01) - \frac{1}{2}(0.01)^2 + \frac{1}{2}(0.01)^3 - \frac{5}{8}(0.01)^4$$

$$\boxed{\sqrt{1.02} \approx 1.00995049375}$$

The exact value of the square root is

$$\sqrt{1.02} = 1.00995049383 6\ldots$$

The approximation is accurate to 10 d.p. or 11 s.f.

Your notes

Maclaurin Series of Composites & Products

How can I find the Maclaurin series for a composite function?

- A **composite function** is a 'function of a function' or a 'function within a function'
  - For example sin(2*x*) is a composite function, with 2*x* as the 'inside function' which has been put into the simpler 'outside function' sin *x*
  - Similarly e^{x^2} is a composite function, with x^2 as the 'inside function' and e^x as the 'outside function'
- To find the Maclaurin series for a composite function:
  - STEP 1: Start with the Maclaurin series for the basic 'outside function'
    - Usually this will be one of the 'standard functions' whose Maclaurin series are given in the exam formula booklet
  - STEP 2: Substitute the 'inside function' every place that *x* appears in the Maclaurin series for the 'outside function'
    - So for sin(2*x*), for example, you would substitute 2*x* everywhere that *x* appears in the Maclaurin series for sin *x*
  - STEP 3: Expand the brackets and simplify the coefficients for the powers of *x* in the resultant Maclaurin series
- This method can theoretically be used for quite complicated 'inside' and 'outside' functions
  - On your exam, however, the 'inside function' will usually not be more complicated than something like kx (for some constant k) or x^n (for some constant power n)

How can I find the Maclaurin series for a product of two functions?

- To find the Maclaurin series for a product of two functions:
  - STEP 1: Start with the Maclaurin series of the individual functions
    - For each of these Maclaurin series you should only use terms up to an appropriately chosen power of *x* (see the worked example below to see how this is done!)
  - STEP 2: Put each of the series into brackets and multiply them together
    - Only keep terms in powers of *x* up to the power you are interested in
  - STEP 3: Collect terms and simplify coefficients for the powers of *x* in the resultant Maclaurin series

Worked example

a) Find the Maclaurin series for the function $f(x) = \ln(1+3x)$, up to and including the term in x^4.

Maclaurin series for special functions	$e^x = 1 + x + \frac{x^2}{2!} + \ldots$	$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$
	$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$	$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$
	$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$	

from exam formula booklet

STEP 1: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

STEP 2: $\ln(1+3x) = 3x - \frac{(3x)^2}{2} + \frac{(3x)^3}{3} - \frac{(3x)^4}{4} + \ldots$

STEP 3: $\boxed{\ln(1+3x) = 3x - \frac{9}{2}x^2 + 9x^3 - \frac{81}{4}x^4 + \ldots}$

b) Find the Maclaurin series for the function $g(x) = e^x \sin x$, up to and including the term in x^4.

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Maclaurin series for special functions		
	$e^x = 1 + x + \frac{x^2}{2!} + \ldots$	$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$
	$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$	$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$
	$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$	

from exam formula booklet

STEP 1: $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$

Higher powers of x here will give powers higher than 4 when multiplied by the sin x series.

$\sin x = x - \frac{x^3}{6} + \ldots$

Don't need terms in powers of x higher than 4

STEP 2: $e^x \sin x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right)\left(x - \frac{x^3}{6} + \ldots\right)$

$$= x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} - \frac{x^3}{6} - \frac{x^4}{6} - \underbrace{\cancel{\frac{x^5}{12}} - \cancel{\frac{x^6}{36}}}_{\text{Discard terms for powers higher than 4}}$$

Note that the x^4 terms cancel out

STEP 3: $\boxed{e^x \sin x = x + x^2 + \frac{1}{3}x^3 + \ldots}$

Differentiating & Integrating Maclaurin Series

Your notes

How can I use differentiation to find Maclaurin Series?

- If you differentiate the Maclaurin series for a function $f(x)$ term by term, you get the Maclaurin series for the function's derivative $f'(x)$
- You can use this to find new Maclaurin series from existing ones
  - For example, the derivative of $\sin x$ is $\cos x$
  - So if you differentiate the Maclaurin series for $\sin x$ term by term you will get the Maclaurin series for $\cos x$

How can I use integration to find Maclaurin series?

- If you integrate the Maclaurin series for a derivative $f'(x)$, you get the Maclaurin series for the function $f(x)$
  - Be careful however, as you will have a constant of integration to deal with
  - The value of the constant of integration will have to be chosen so that the series produces the correct value for $f(0)$
- You can use this to find new Maclaurin series from existing ones
  - For example, the derivative of $\sin x$ is $\cos x$
  - So if you integrate the Maclaurin series for $\cos x$ (and correctly deal with the constant of integration) you will get the Maclaurin series for $\sin x$

Your notes

Worked example

a) (i) Write down the derivative of $\arctan x$.

(ii) Hence use the Maclaurin series for $\arctan x$ to derive the Maclaurin series for $\dfrac{1}{1+x^2}$.

Standard derivatives	
$\arctan x$	$f(x)=\arctan x \Rightarrow f'(x)=\dfrac{1}{1+x^2}$

Maclaurin series for special functions	$e^x = 1+x+\dfrac{x^2}{2!}+\ldots$	$\ln(1+x) = x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\ldots$
	$\sin x = x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\ldots$	$\cos x = 1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\ldots$
	$\arctan x = x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\ldots$	

from exam formula booklet

(i) $$\boxed{\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}}$$

(ii) $$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

$$\Rightarrow \frac{1}{1+x^2} = \frac{d}{dx}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\right)$$

$$\boxed{\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots}$$

Note : This is the same as the binomial theorem expansion of $(1+x^2)^{-1}$

b) (i) Write down the derivative of $-\sin x$.

(ii) Hence derive the Maclaurin series for $\cos x$, being sure to justify your method.

Your notes

Maclaurin series for special functions	
	$e^x = 1 + x + \frac{x^2}{2!} + \dots$ $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$
	$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$
	$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$

from exam formula booklet

(i) $\boxed{-\sin x = -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \dots}$

(ii) $-\sin x$ is the derivative of $\cos x$, so we can integrate the Maclaurin series for $-\sin x$ to find the Maclaurin series for $\cos x$.

$$\cos x = \int \left(-x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \dots\right) dx$$

constant of integration

$$= c - \frac{1}{2}x^2 + \frac{1}{4}\cdot\frac{x^4}{3!} - \frac{1}{6}\cdot\frac{x^6}{5!} + \frac{1}{8}\cdot\frac{x^8}{7!} - \dots$$

$$= c - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

And $\cos(0) = 1$, so $c - \frac{0^2}{2!} + \frac{0^4}{4!} - \frac{0^6}{6!} + \frac{0^8}{8!} - \dots = 1 \Rightarrow c = 1$

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots}$$

Your notes

5.11.2 Maclaurin Series from Differential Equations

Maclaurin Series for Differential Equations

Can I apply Maclaurin Series to solving differential equations?

- If you have a differential equation of the form $\frac{dy}{dx} = g(x, y)$ along with the value of $y(0)$ it is possible to build up the Maclaurin series of the solution $y = f(x)$ term by term
  - This does not necessarily tell you the explicit function of x that corresponds to the Maclaurin series you are finding
  - But the Maclaurin series you find is the exact Maclaurin series for the solution to the differential equation
- The Maclaurin series can be used to approximate the value of the solution *y* = *f*(*x*) for different values of x
  - You can increase the accuracy of this approximation by calculating additional terms of the Maclaurin series for higher powers of x

How can I find the Maclaurin Series for the solution to a differential equation?

- STEP 1: Use **implicit differentiation** to find expressions for y'', y''' etc., in terms of x, y and lower-order derivatives of y
  - The number of derivatives you need to find depends on how many terms of the Maclaurin series you want to find
  - For example, if you want the Maclaurin series up to the term, then you will need to find derivatives up to $y^{(4)}$ (the fourth derivative of y)
- STEP 2: Using the given initial value for $y(0)$, find the values of $y'(0), y''(0), y'''(0),$ etc., one by one
  - Each value you find will then allow you to find the value for the next higher derivative
- STEP 3: Put the values found in STEP 2 into the **general Maclaurin series formula**

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots$$

  - This formula is in your exam formula booklet
  - $y = f(x)$ is the solution to the differential equation, so $y(0)$ corresponds to $f(0)$ in the formula, $y'(0)$ corresponds to $f'(0)$, and so on
- STEP 4: Simplify the coefficients for each of the powers of x in the resultant Maclaurin series

Your notes

Worked example

Consider the differential equation $y' = y^2 - x$ with the initial condition $y(0) = 2$.

a) Use implicit differentiation to find expressions for y'', y''' and $y^{(4)}$.

STEP 1 :

$$y'' = \frac{d}{dx}(y') = \frac{d}{dx}(y^2 - x) = 2yy' - 1$$

$$\boxed{y'' = 2yy' - 1}$$

$$y''' = \frac{d}{dx}(y'') = \frac{d}{dx}(2yy' - 1) = 2yy'' + 2(y')^2$$

$$\boxed{y''' = 2yy'' + 2(y')^2}$$

$$y^{(4)} = \frac{d}{dx}(y''') = \frac{d}{dx}(2yy'' + 2(y')^2)$$

$$= 2y'y'' + 2yy''' + 4y'y''$$

$$\boxed{y^{(4)} = 6y'y'' + 2yy'''}$$

b) Use the given initial condition to find the values of $y'(0)$, $y''(0)$, $y'''(0)$ and $y^{(4)} = 0$.

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STEP 2:

$y(0) = 2$, so $y'(0) = 2^2 - 0 = 4$ $\quad y' = y^2 - x$

Then $y''(0) = 2(2)(4) - 1 = 15$ $\quad y'' = 2yy' - 1$

$y'''(0) = 2(2)(15) + 2(4)^2 = 92$ $\quad y''' = 2yy'' + 2(y')^2$

$y^{(4)}(0) = 6(4)(15) + 2(2)(92) = 728$ $\quad y^{(4)} = 6y'y'' + 2yy'''$

$$\boxed{\begin{array}{ll} y'(0) = 4 & y''(0) = 15 \\ y'''(0) = 92 & y^{(4)}(0) = 728 \end{array}}$$

Let $y = f(x)$ be the solution to the differential equation with the given initial condition.

c) Find the first five terms of the Maclaurin series for $f(x)$.

Your notes

Maclaurin series	$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots$

from exam formula booklet

STEP 3: $f(x) = 2 + x(4) + \frac{x^2}{2!}(15) + \frac{x^3}{3!}(92) + \frac{x^4}{4!}(728) + \ldots$

STEP 4:

$$\boxed{f(x) = 2 + 4x + \frac{15}{2}x^2 + \frac{46}{3}x^3 + \frac{91}{3}x^4 + \ldots}$$