

DP IB Maths: AI HL



2.5 Transformations of Graphs

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2.5.1 Translations of Graphs



Your notes

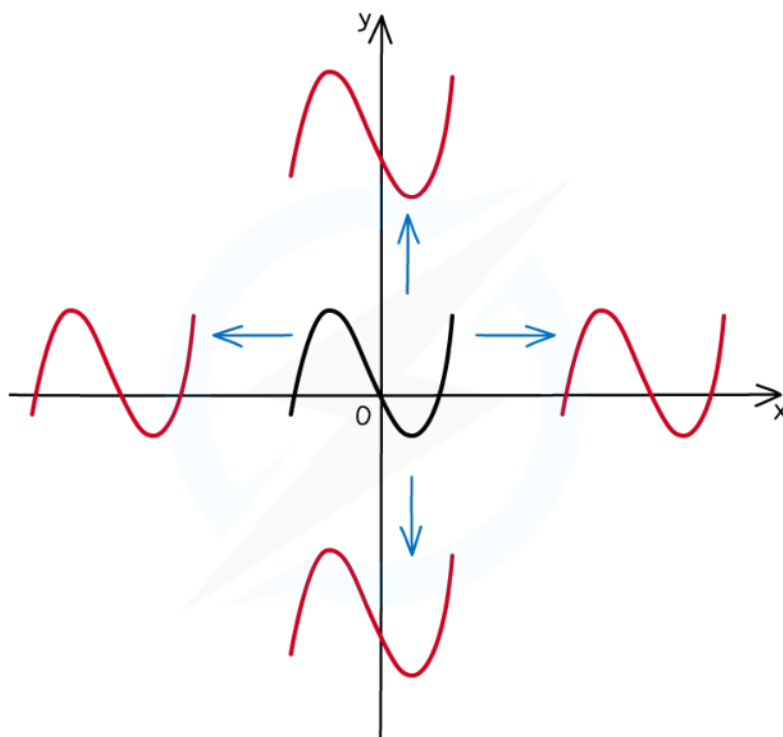
Translations of Graphs

What are translations of graphs?

- When you alter a function in certain ways, the effects on the graph of the function can be described by **geometrical transformations**
- For a **translation**:
 - the graph is **moved** (up or down, left or right) in the xy plane
 - Its position **changes**
 - the shape, size, and orientation of the graph remain **unchanged**
- A particular translation (how far left/right, how far up/down) is specified by a **translation vector** $\begin{pmatrix} x \\ y \end{pmatrix}$:
 - x is the **horizontal** displacement
 - **Positive** moves **right**
 - **Negative** moves **left**
 - y is the **vertical** displacement
 - **Positive** moves **up**
 - **Negative** moves **down**



Your notes



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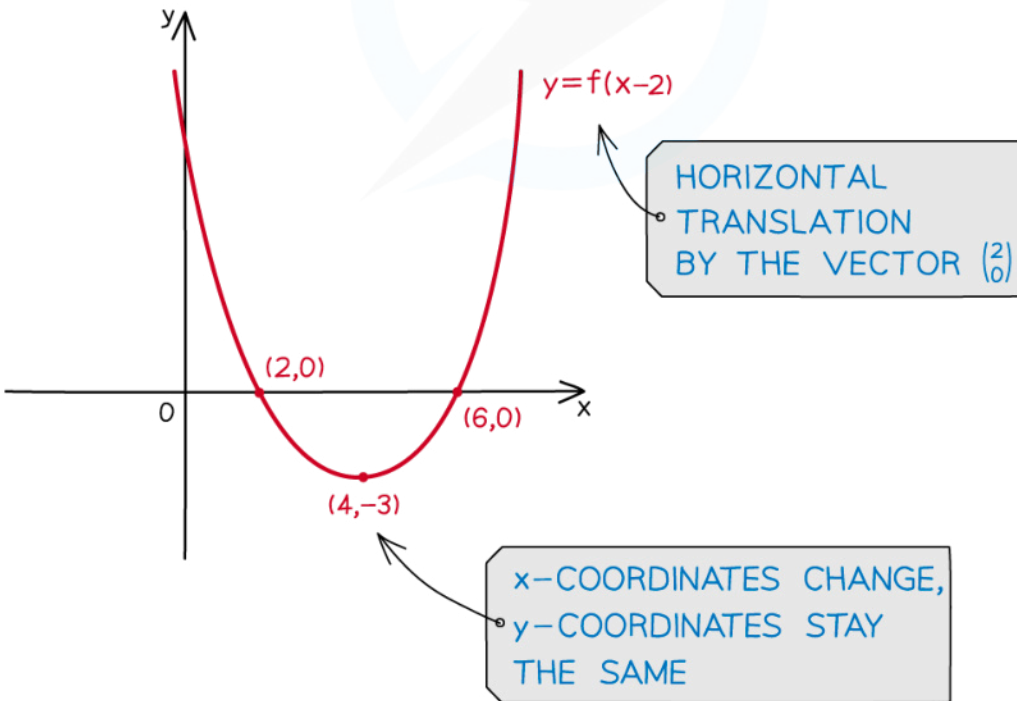
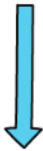
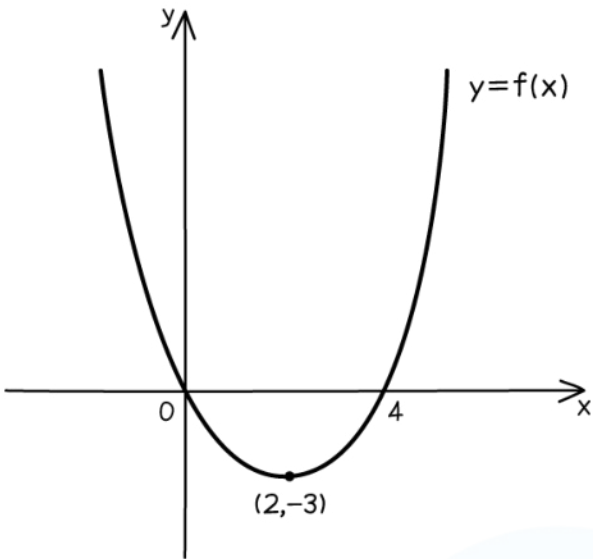


What effects do horizontal translations have on the graphs and functions?

- A **horizontal translation** of the graph $y = f(x)$ by the vector $\begin{pmatrix} a \\ 0 \end{pmatrix}$ is represented by
 - $y = f(x - a)$
- The **x-coordinates change**
 - The value a is **subtracted** from them
- The **y-coordinates stay the same**
- The coordinates (x, y) become $(x + a, y)$
- **Horizontal asymptotes stay the same**
- **Vertical asymptotes change**
 - $x = k$ becomes $x = k + a$



Your notes



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What effects do vertical translations have on the graphs and functions?

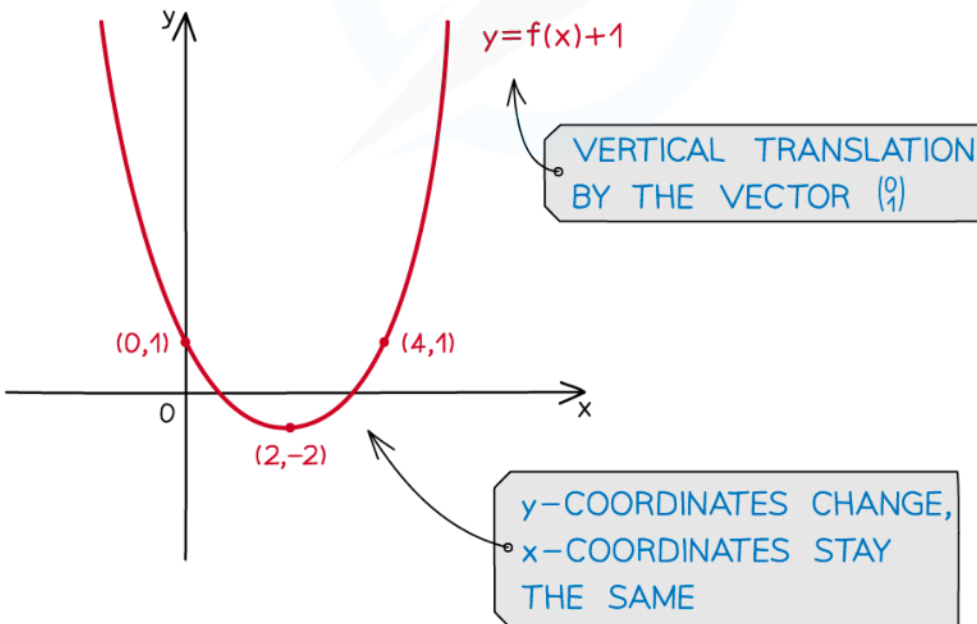
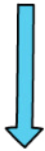
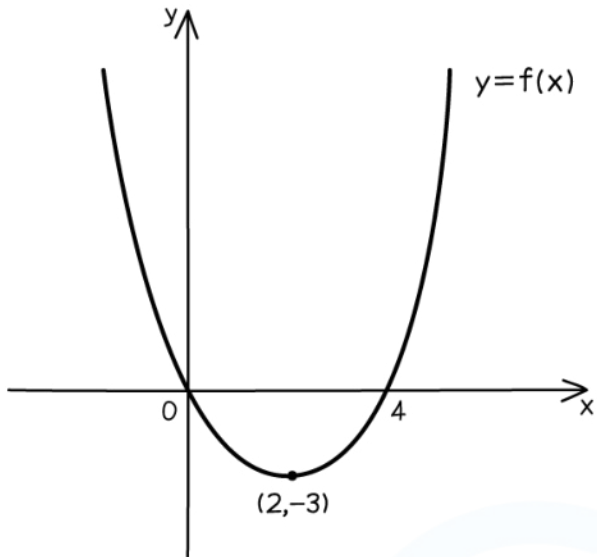
- A **vertical translation** of the graph $y = f(x)$ by the vector $\begin{pmatrix} 0 \\ b \end{pmatrix}$ is represented by
 - $y - b = f(x)$
 - This is often rearranged to $y = f(x) + b$
 - The **x-coordinates stay the same**
 - The **y-coordinates change**
 - The value b is **added** to them
 - The coordinates (x, y) become $(x, y + b)$
 - **Horizontal asymptotes change**
 - $y = k$ becomes $y = k + b$
 - **Vertical asymptotes stay the same**



Your notes



Your notes



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Examiner Tip

- To get full marks in an exam make sure you use correct mathematical terminology
 - For example: Translate by the vector $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$



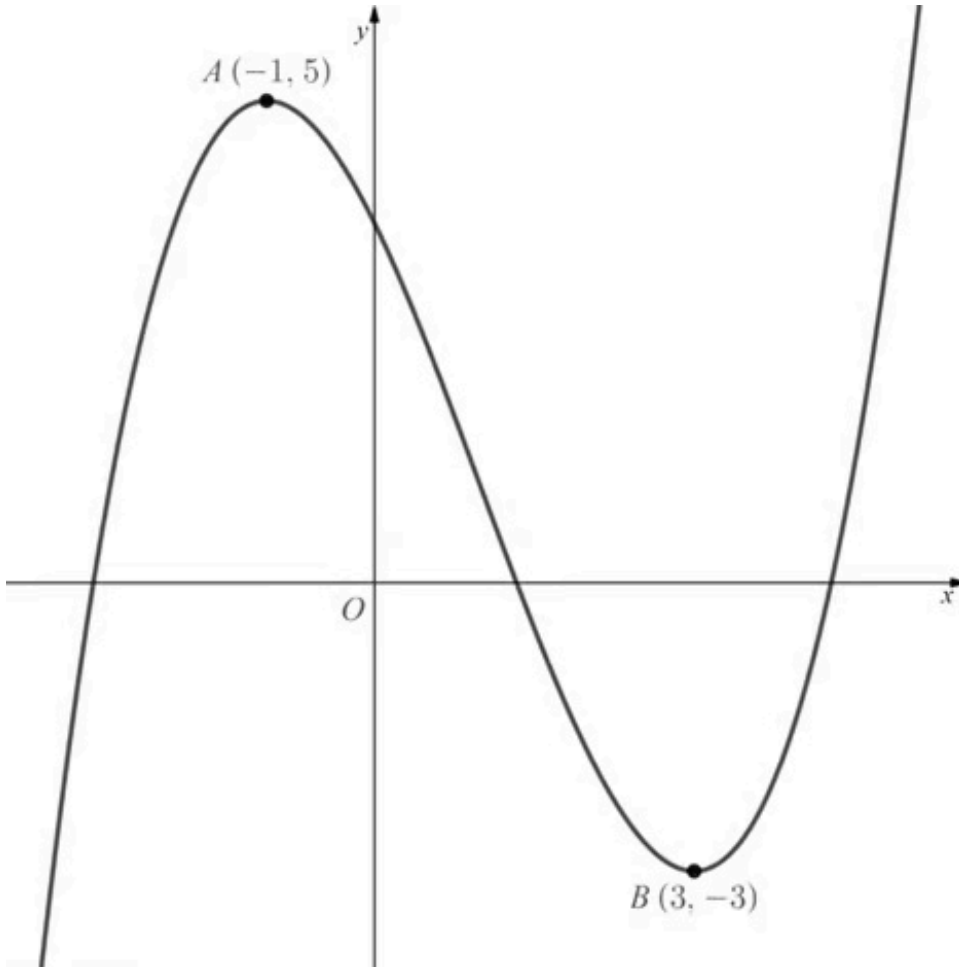
Your notes



Your notes

 **Worked example**

The diagram below shows the graph of $y = f(x)$.



- a) Sketch the graph of $y = f(x + 3)$.



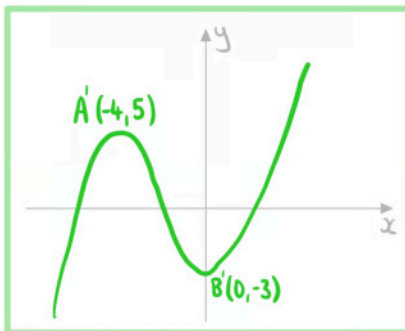
Your notes

$y = f(x+k)$ translation by $\begin{pmatrix} -k \\ 0 \end{pmatrix}$

Translate $y = f(x)$ by $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$

A becomes $(-4, 5)$

B becomes $(0, -3)$



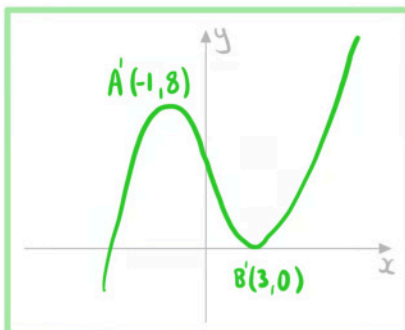
b) Sketch the graph of $y = f(x) + 3$.

$y = f(x) + k$ translation by $\begin{pmatrix} 0 \\ k \end{pmatrix}$

Translate $y = f(x)$ by $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$

A becomes $(-1, 8)$

B becomes $(3, 0)$





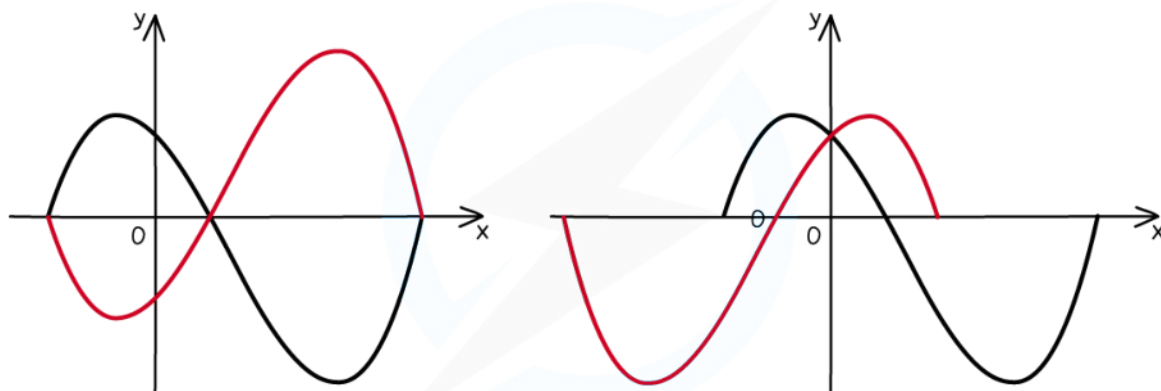
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2.5.2 Reflections of Graphs

Reflections of Graphs

What are reflections of graphs?

- When you alter a function in certain ways, the effects on the graph of the function can be described by **geometrical transformations**
- For a **reflection**:
 - the graph is **flipped** about one of the coordinate axes
 - Its orientation **changes**
 - the size of the graph remains **unchanged**
- A particular reflection is specified by an **axis of symmetry**:
 - $y = 0$
 - This is the x-axis
 - $x = 0$
 - This is the y-axis



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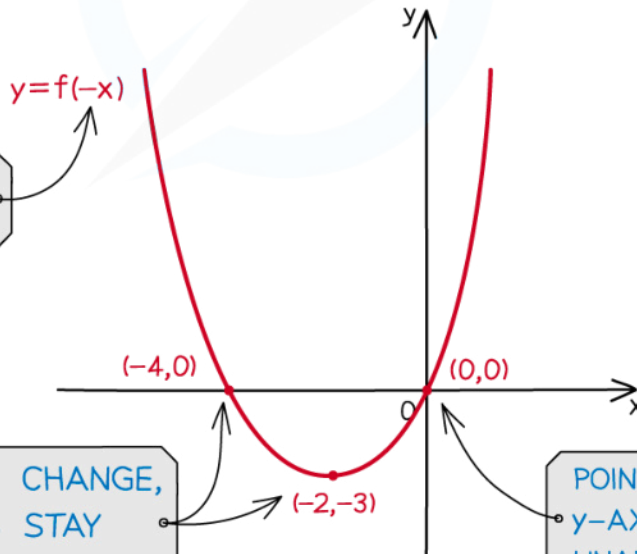
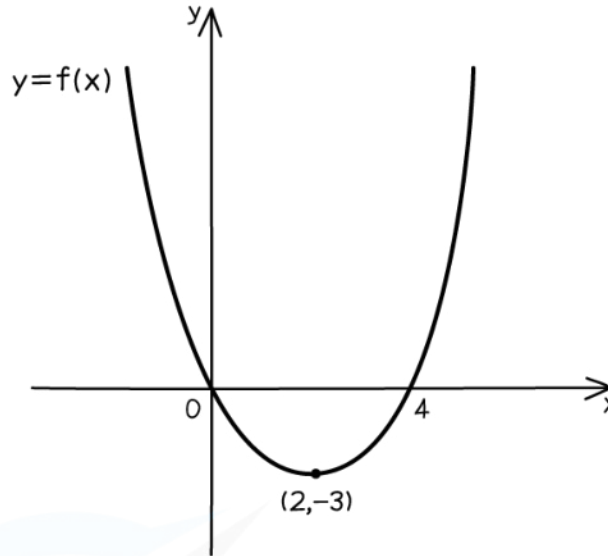
What effects do horizontal reflections have on the graphs and functions?

- A **horizontal reflection** of the graph $y = f(x)$ about the y-axis is represented by
 - $y = f(-x)$
- The **x-coordinates change**
 - Their **sign** changes
- The **y-coordinates stay the same**
- The coordinates (x, y) become $(-x, y)$
- **Horizontal** asymptotes **stay the same**
- **Vertical** asymptotes **change**

- $x = k$ becomes $x = -k$



Your notes



REFLECTION IN THE y -AXIS

x -COORDINATES CHANGE, y -COORDINATES STAY THE SAME

POINTS ON THE y -AXIS ARE UNAFFECTED

What effects do vertical reflections have on the graphs and functions?

- A **vertical reflection** of the graph $y = f(x)$ about the x-axis is represented by
 - $-y = f(x)$
 - This is often rearranged to $y = -f(x)$
- The **x-coordinates stay the same**
- The **y-coordinates change**
 - Their **sign** changes
- The coordinates (x, y) become $(x, -y)$
- **Horizontal asymptotes change**
 - $y = k$ becomes $y = -k$
- **Vertical asymptotes stay the same**



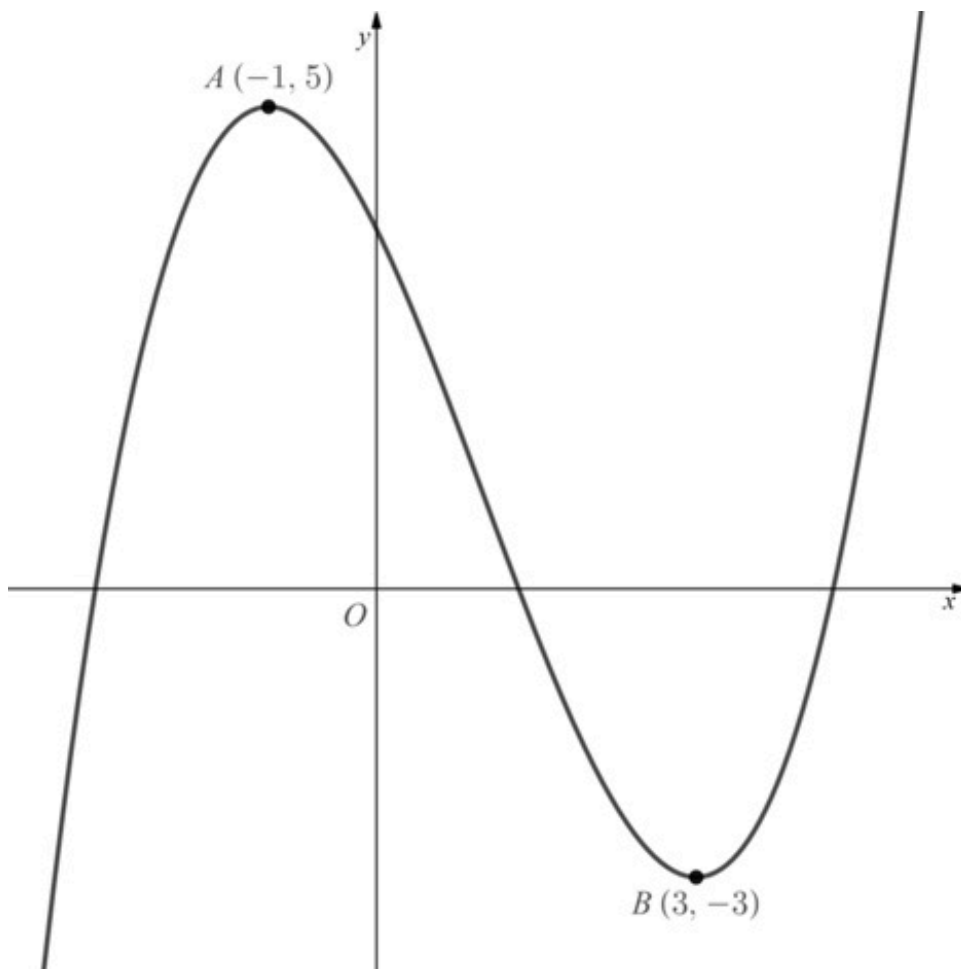
Your notes



Your notes

 **Worked example**

The diagram below shows the graph of $y = f(x)$.



- a) Sketch the graph of $y = -f(x)$.

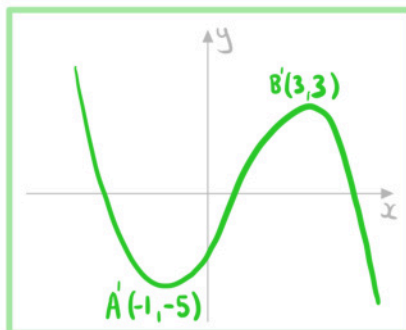


Your notes

$y = -f(x)$ reflection in x -axis

A becomes $(-1, -5)$

B becomes $(3, 3)$

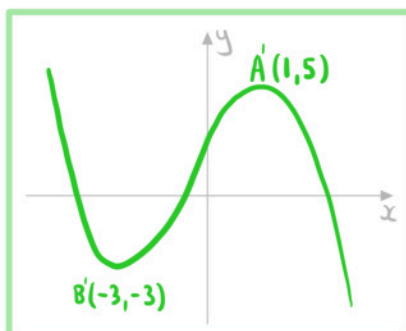


b) Sketch the graph of $y = f(-x)$.

$y = f(-x)$ reflection in y -axis

A becomes $(1, 5)$

B becomes $(-3, -3)$





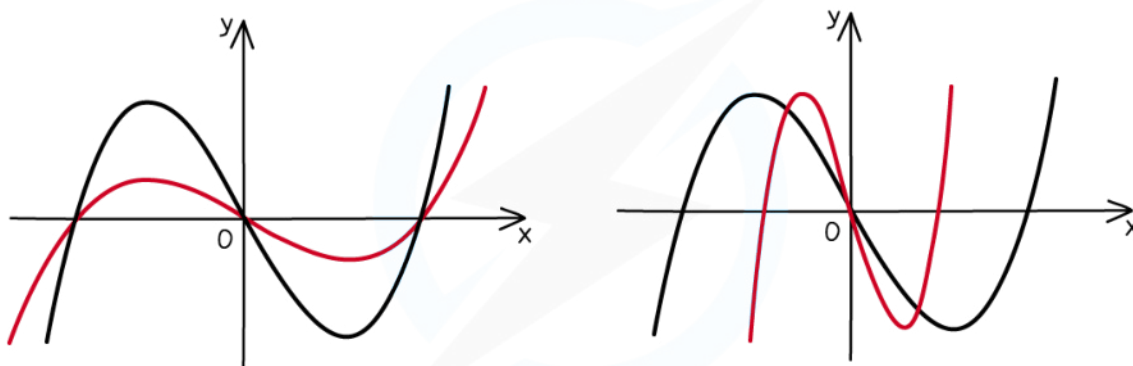
Your notes

2.5.3 Stretches of Graphs

Stretches of Graphs

What are stretches of graphs?

- When you alter a function in certain ways, the effects on the graph of the function can be described by **geometrical transformations**
- For a **stretch**:
 - the graph is **stretched** about one of the coordinate axes by a scale factor
 - Its size **changes**
 - the orientation of the graph remains **unchanged**
- A particular stretch is specified by a **coordinate axis** and a **scale factor**:
 - The **distance** between a **point** on the graph and the **specified coordinate axis** is **multiplied** by the **constant scale factor**
 - The graph is stretched in the **direction** which is **parallel** to the **other coordinate axis**
 - For scale factors **bigger than 1**
 - the points on the graph get **further away** from the **specified coordinate axis**
 - For scale factors **between 0 and 1**
 - the points on the graph get **closer** to the **specified coordinate axis**
 - This is also sometimes called a **compression** but in your exam you must use the term **stretch** with the appropriate scale factor



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What effects do horizontal stretches have on the graphs and functions?

- A **horizontal stretch** of the graph $y = f(x)$ by a scale factor q centred about the y -axis is represented by

$$y = f\left(\frac{x}{q}\right)$$

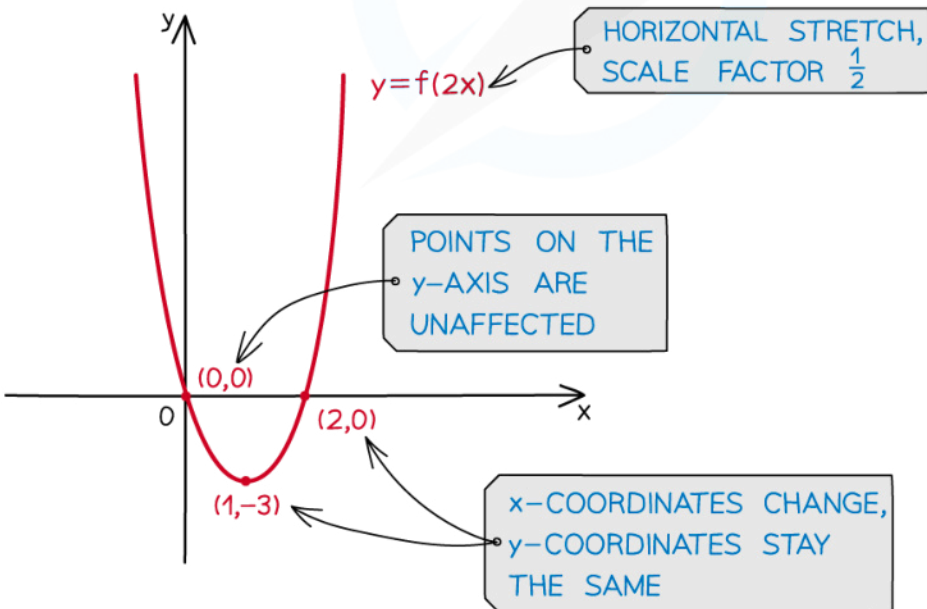
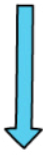
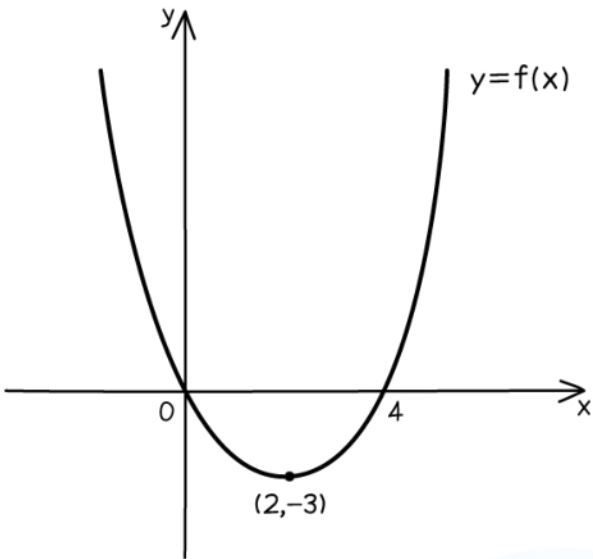
- The **x-coordinates change**
 - They are **divided** by q
- The **y-coordinates stay the same**
- The coordinates (x, y) become (qx, y)
- **Horizontal** asymptotes **stay the same**
- **Vertical** asymptotes **change**
 - $x = k$ becomes $x = qk$



Your notes



Your notes



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What effects do vertical stretches have on the graphs and functions?

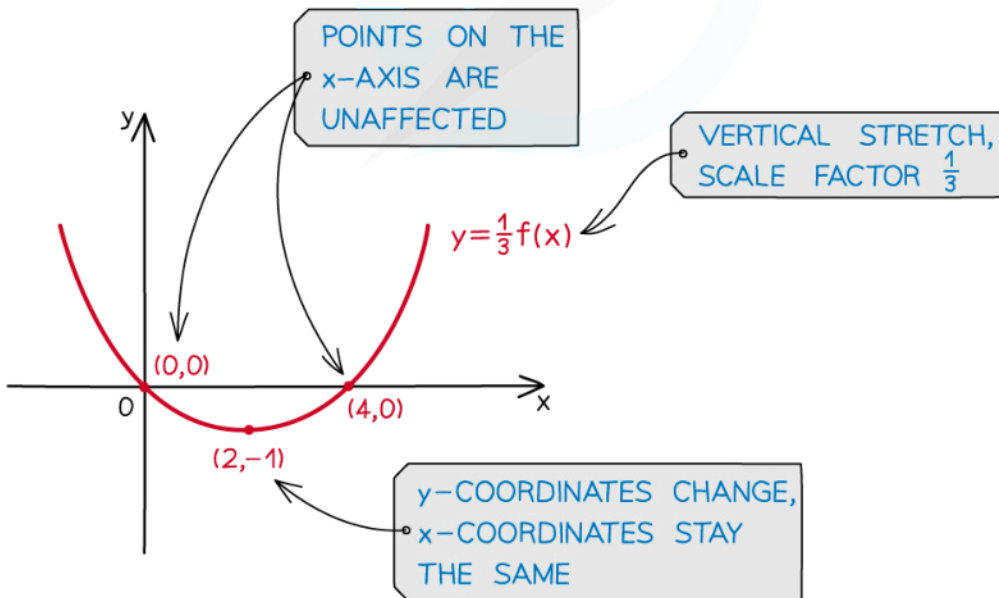
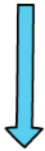
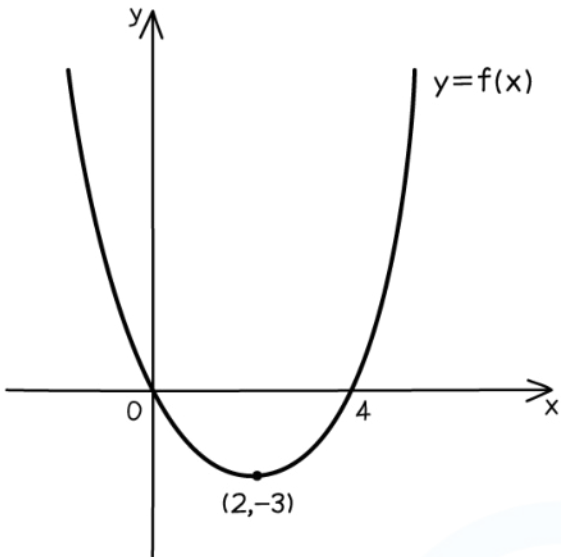
- A **vertical stretch** of the graph $y = f(x)$ by a scale factor p centred about the x -axis is represented by
 - $\frac{y}{p} = f(x)$
 - This is often rearranged to $y = pf(x)$
- The **x -coordinates stay the same**
- The **y -coordinates change**
 - They are **multiplied** by p
- The coordinates (x, y) become (x, py)
- **Horizontal asymptotes change**
 - $y = k$ becomes $y = pk$
- **Vertical asymptotes stay the same**



Your notes



Your notes



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Examiner Tip

- To get full marks in an exam make sure you use correct mathematical terminology
 - For example: Stretch vertically by scale factor $\frac{1}{2}$
 - Do not use the word "compress" in your exam



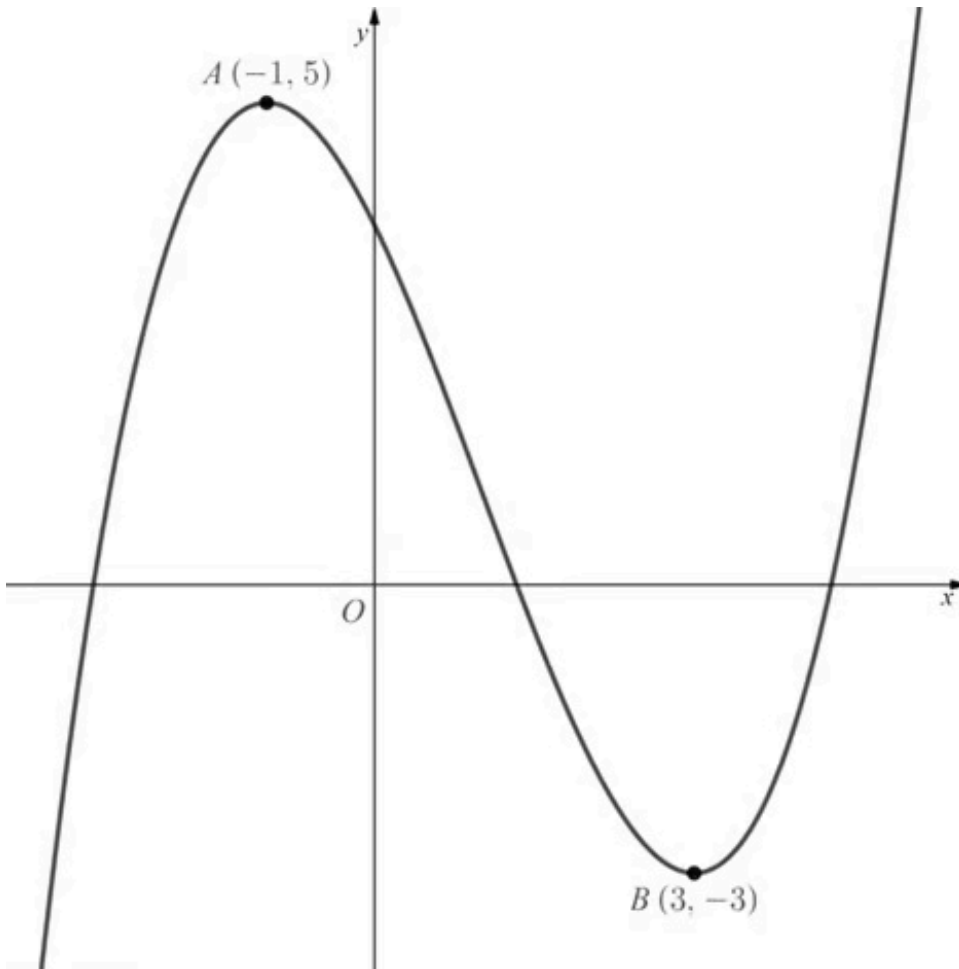
Your notes



Your notes

 **Worked example**

The diagram below shows the graph of $y = f(x)$.



- a) Sketch the graph of $y = 2f(x)$.



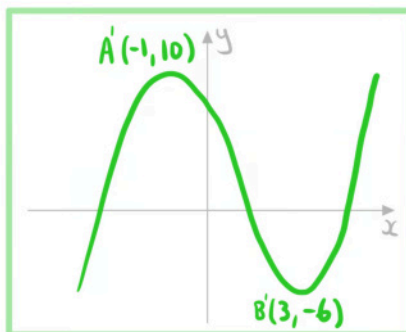
Your notes

$y = kf(x)$ vertical stretch scale factor k

Stretch $y = f(x)$ vertically
scale factor 2

A becomes $(-1, 10)$

B becomes $(3, -6)$



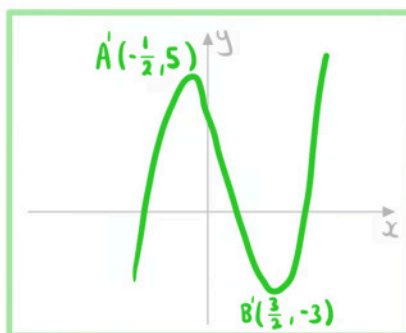
b) Sketch the graph of $y = f(2x)$.

$y = f(kx)$ horizontal stretch scale factor $\frac{1}{k}$

Stretch $y = f(x)$ horizontally
scale factor $\frac{1}{2}$

A becomes $(-\frac{1}{2}, 5)$

B becomes $(\frac{3}{2}, -3)$





Your notes

2.5.4 Composite Transformations of Graphs

Composite Transformations of Graphs

What transformations do I need to know?

- $y = f(x + k)$ is **horizontal translation** by vector $\begin{pmatrix} -k \\ 0 \end{pmatrix}$
 - If k is **positive** then the graph moves **left**
 - If k is **negative** then the graph moves **right**
- $y = f(x) + k$ is **vertical translation** by vector $\begin{pmatrix} 0 \\ k \end{pmatrix}$
 - If k is **positive** then the graph moves **up**
 - If k is **negative** then the graph moves **down**
- $y = f(kx)$ is a **horizontal stretch** by scale factor $\frac{1}{k}$ centred about the y -axis
 - If $k > 1$ then the graph gets **closer** to the y -axis
 - If $0 < k < 1$ then the graph gets **further** from the y -axis
- $y = kf(x)$ is a **vertical stretch** by scale factor k centred about the x -axis
 - If $k > 1$ then the graph gets **further** from the x -axis
 - If $0 < k < 1$ then the graph gets **closer** to the x -axis
- $y = f(-x)$ is a **horizontal reflection** about the y -axis
 - A **horizontal reflection** can be viewed as a special case of a **horizontal stretch**
- $y = -f(x)$ is a **vertical reflection** about the x -axis
 - A **vertical reflection** can be viewed as a special case of a **vertical stretch**

How do horizontal and vertical transformations affect each other?

- **Horizontal and vertical transformations** are **independent** of each other
 - The horizontal transformations involved will need to be applied in their correct order
 - The vertical transformations involved will need to be applied in their correct order
- Suppose there are **two horizontal** transformation H_1 then H_2 and **two vertical** transformations V_1 then V_2 then they can be applied in the following orders:
 - Horizontal then vertical:
 - $H_1 H_2 V_1 V_2$
 - Vertical then horizontal:
 - $V_1 V_2 H_1 H_2$
 - Mixed up (provided that H_1 comes before H_2 and V_1 comes before V_2):
 - $H_1 V_1 H_2 V_2$
 - $H_1 V_1 V_2 H_2$
 - $V_1 H_1 V_2 H_2$

- $V_1H_1H_2V_2$

 **Examiner Tip**

- In an exam you are more likely to get the correct solution if you deal with one transformation at a time and sketch the graph after each transformation



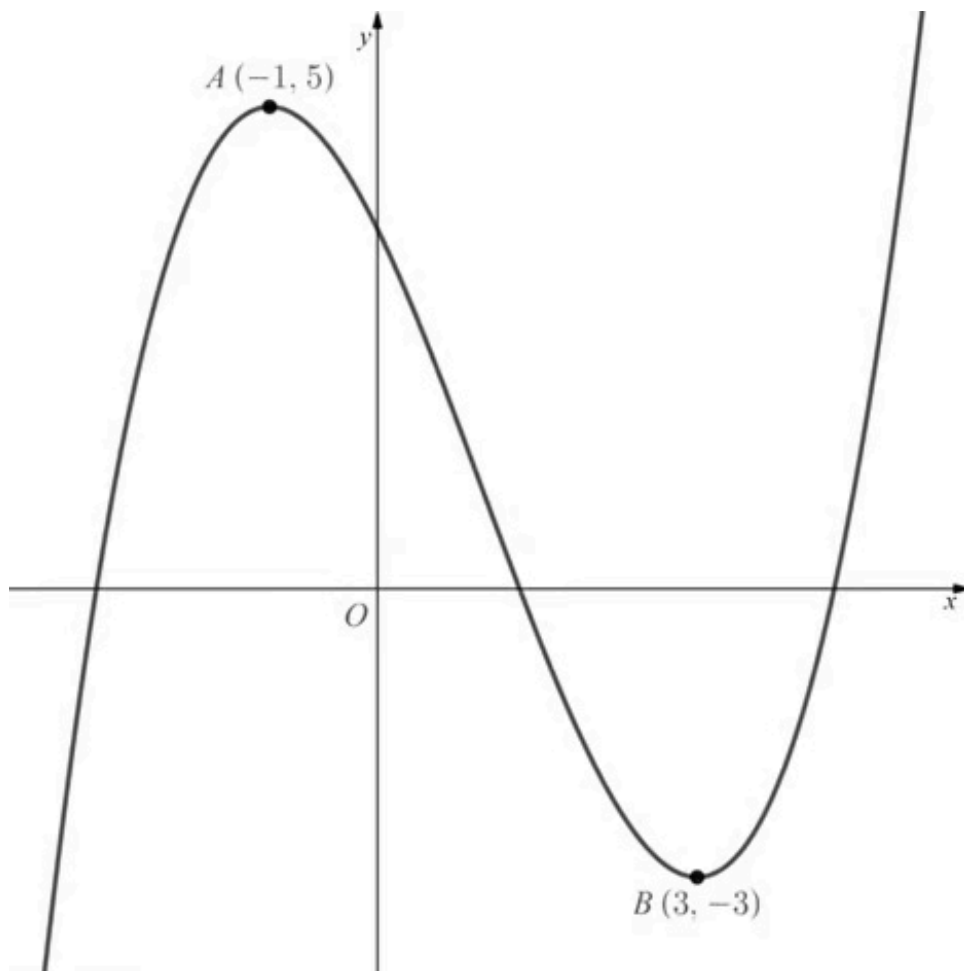
Your notes



Your notes

 **Worked example**

The diagram below shows the graph of $y = f(x)$.



Sketch the graph of $y = \frac{1}{2}f\left(\frac{x}{2}\right)$.



Your notes

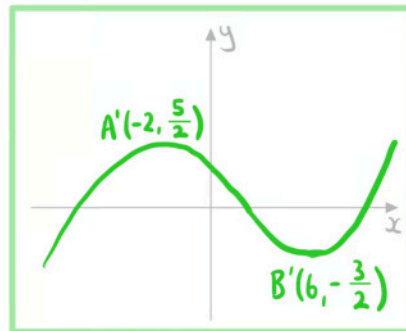
A vertical and horizontal transformation can be done in any order

$y = \frac{1}{2}f(x)$: vertical stretch scale factor $\frac{1}{2}$

$y = f\left(\frac{x}{2}\right)$: horizontal stretch scale factor 2

A becomes $\left(-2, \frac{5}{2}\right)$

B becomes $\left(6, -\frac{3}{2}\right)$





Your notes

Composite Vertical Transformations $af(x)+b$

How do I deal with multiple vertical transformations?

- **Order matters** when you have **more than one vertical transformations**
- If you are asked to find the equation then **build up the equation** by looking at the transformations in order

- A **vertical stretch** by scale factor a followed by a **translation** of $\begin{pmatrix} 0 \\ b \end{pmatrix}$

- Stretch: $y = af(x)$
- Then translation: $y = [af(x)] + b$
- Final equation: $y = af(x) + b$

- A **translation** of $\begin{pmatrix} 0 \\ b \end{pmatrix}$ followed by a **vertical stretch** by scale factor a

- Translation: $y = f(x) + b$
- Then stretch: $y = a[f(x) + b]$
- Final equation: $y = af(x) + ab$

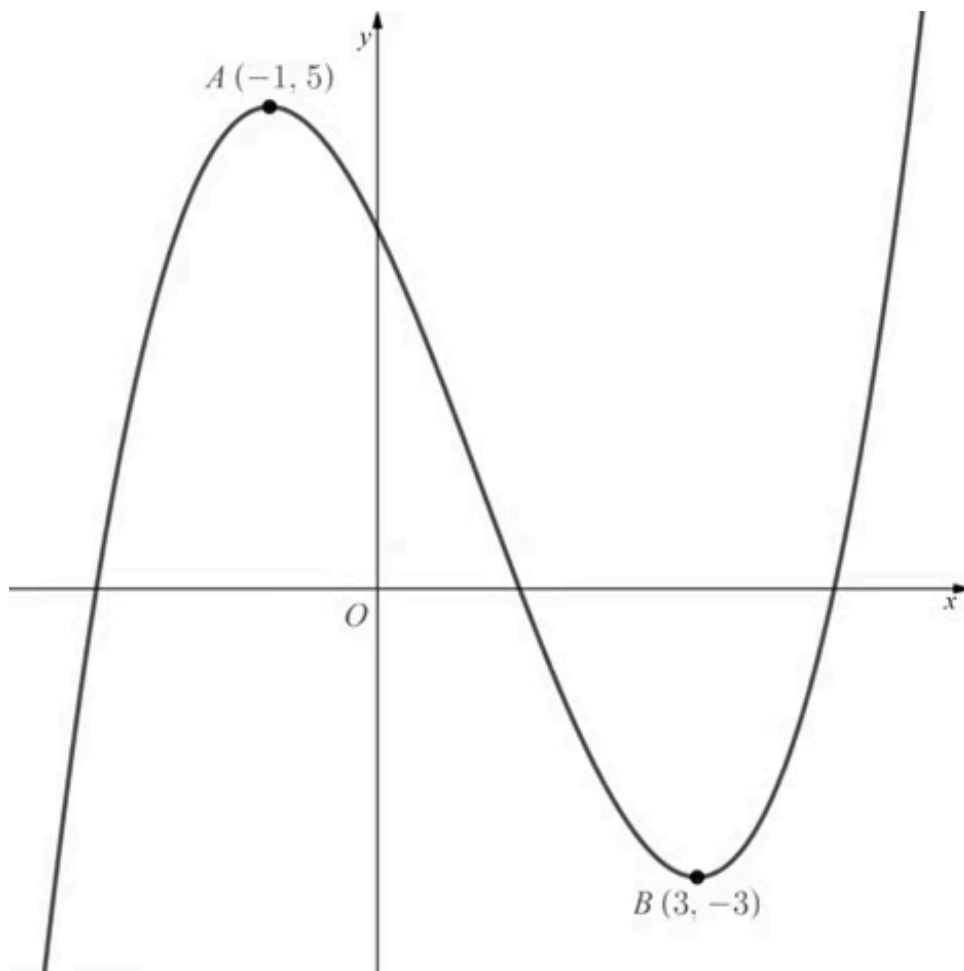
- If you are asked to determine the **order**
 - The order of vertical transformations **follows the order of operations**
 - First write the equation in the form $y = af(x) + b$
 - **First stretch vertically** by scale factor a
 - If a is negative then the **reflection and stretch** can be **done in any order**
 - **Then translate** by $\begin{pmatrix} 0 \\ b \end{pmatrix}$



Your notes

 **Worked example**

The diagram below shows the graph of $y = f(x)$.



Sketch the graph of $y = 3f(x) - 2$.



Your notes

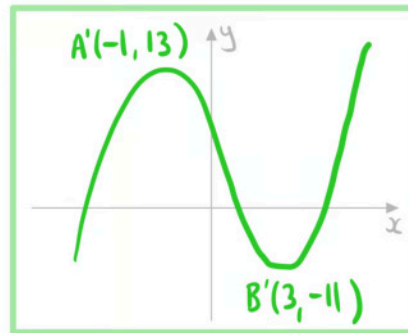
The order vertical transformations follows the order of operations

$y = 3f(x)$: Vertical stretch scale factor 3

$y = f(x) - 2$: Translate $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$

A becomes $(-1, 13)$

B becomes $(3, -11)$





Your notes

Composite Horizontal Transformations $f(ax+b)$

How do I deal with multiple horizontal transformations?

- **Order matters** when you have **more than one horizontal transformations**
- If you are asked to find the equation then **build up the equation** by looking at the transformations in order

- A **horizontal stretch** by scale factor $\frac{1}{a}$ followed by a **translation** of $\begin{pmatrix} -b \\ 0 \end{pmatrix}$

- Stretch: $y = f(ax)$
- Then translation: $y = f(a(x + b))$
- Final equation: $y = f(ax + ab)$

- A **translation** of $\begin{pmatrix} -b \\ 0 \end{pmatrix}$ followed by a **horizontal stretch** by scale factor $\frac{1}{a}$

- Translation: $y = f(x + b)$
- Then stretch: $y = f((ax) + b)$
- Final equation: $y = f(ax + b)$

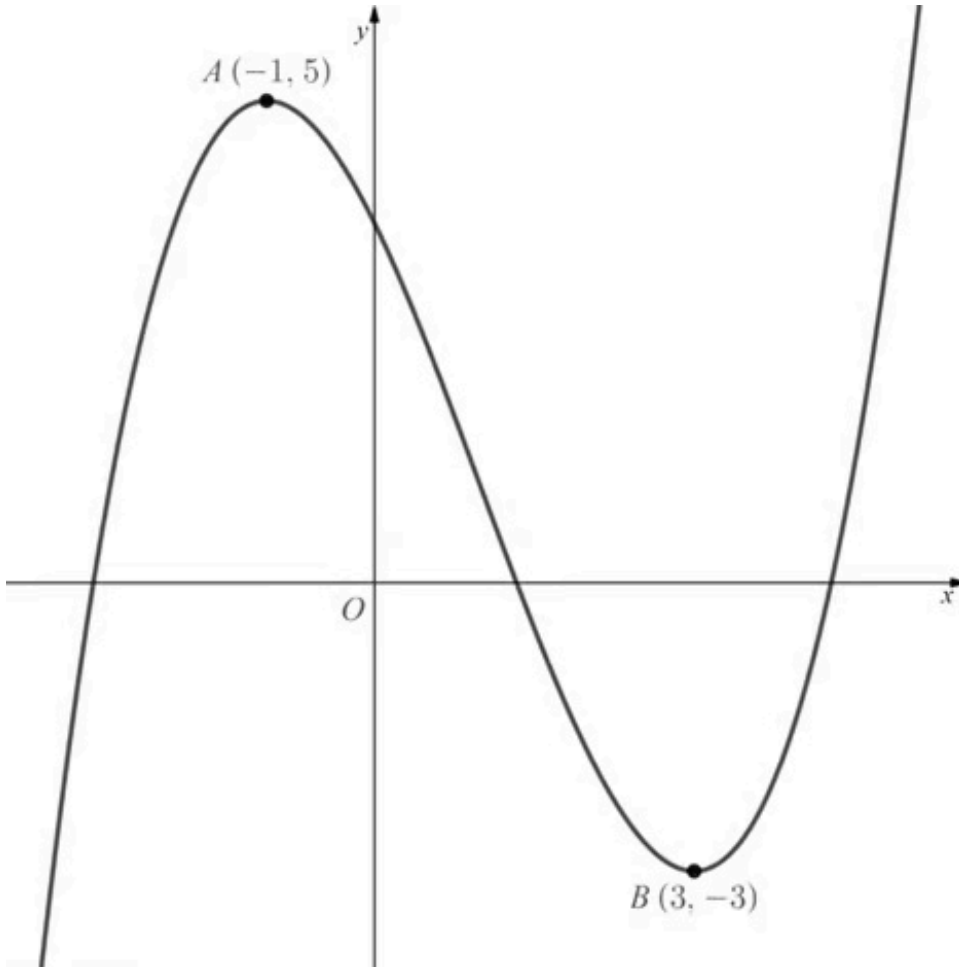
- If you are asked to determine the **order**
 - First write the equation in the form $y = f(ax + b)$
 - The order of horizontal transformations **is the reverse of the order of operations**
 - **First translate** by $\begin{pmatrix} -b \\ 0 \end{pmatrix}$
 - **Then stretch** by scale factor $\frac{1}{a}$
 - If a is negative then the **reflection and stretch** can be **done in any order**



Your notes

 **Worked example**

The diagram below shows the graph of $y = f(x)$.



Sketch the graph of $y = f(2x - 1)$.



Your notes

The order of horizontal transformations is the reverse of the order of operations

$y = f(x-1)$: Translate $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$y = f(2x)$: Horizontal stretch scale factor $\frac{1}{2}$

A becomes $(0, 5)$

B becomes $(2, -3)$

