


# DP IB Maths: AA HL



Your notes

## 5.2 Further Differentiation

### Contents

- \* 5.2.1 Differentiating Special Functions
- \* 5.2.2 Techniques of Differentiation
- \* 5.2.3 Higher Order Derivatives
- \* 5.2.4 Further Applications of Differentiation
- \* 5.2.5 Concavity & Points of Inflection
- \* 5.2.6 Derivatives & Graphs



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## 5.2.1 Differentiating Special Functions

### Differentiating Trig Functions

How do I differentiate sin, cos and tan?

- The derivative of  $y = \sin x$  is  $\frac{dy}{dx} = \cos x$
- The derivative of  $y = \cos x$  is  $\frac{dy}{dx} = -\sin x$
- The derivative of  $y = \tan x$  is  $\frac{dy}{dx} = \sec^2 x$ 
  - This result can be derived using **quotient rule**
- For the **linear** function  $ax + b$ , where  $a$  and  $b$  are constants,
  - the derivative of  $y = \sin(ax + b)$  is  $\frac{dy}{dx} = a \cos(ax + b)$
  - the derivative of  $y = \cos(ax + b)$  is  $\frac{dy}{dx} = -a \sin(ax + b)$
  - the derivative of  $y = \tan(ax + b)$  is  $\frac{dy}{dx} = a \sec^2(ax + b)$
- For the **general** function  $f(x)$ ,
  - the derivative of  $y = \sin(f(x))$  is  $\frac{dy}{dx} = f'(x) \cos(f(x))$
  - the derivative of  $y = \cos(f(x))$  is  $\frac{dy}{dx} = -f'(x) \sin(f(x))$
  - the derivative of  $y = \tan(f(x))$  is  $\frac{dy}{dx} = f'(x) \sec^2(f(x))$
- These last three results can be derived using the **chain rule**
- For calculus with trigonometric functions angles must be measured in **radians**
  - Ensure you know how to change the angle mode on your GDC

#### Examiner Tip

- As soon as you see a question involving differentiation and trigonometry put your GDC into radians mode



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### Worked example

a) Find  $f'(x)$  for the functions

i.  $f(x) = \sin x$

ii.  $f(x) = \cos(5x + 1)$

i.  $f'(x) = \cos x$

ii.  $f'(x) = -5\sin(5x + 1)$

(Linear function  $ax + b$ )b) A curve has equation  $y = \tan\left(6x^2 - \frac{\pi}{4}\right)$ .Find the gradient of the tangent to the curve at the point where  $x = \frac{\sqrt{\pi}}{2}$ .

Give your answer as an exact value.



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This is of the form  $y = \tan(f(x))$   
so  $\frac{dy}{dx} = f'(x) \sec^2(f(x))$

$$f(x) = 6x^2 - \frac{\pi}{4}$$

$$\therefore f'(x) = 12x$$

$$\therefore \frac{dy}{dx} = 12x \sec^2\left(6x^2 - \frac{\pi}{4}\right)$$

$$\text{At } x = \frac{\sqrt{\pi}}{2}, \quad \frac{dy}{dx} = 12\left(\frac{\sqrt{\pi}}{2}\right) \sec^2\left[6\left(\frac{\sqrt{\pi}}{2}\right)^2 - \frac{\pi}{4}\right]$$

$$= \frac{6\sqrt{\pi}}{\cos^2\left(\frac{5\pi}{4}\right)}$$

$$\left(\sec^2 x = \frac{1}{\cos^2 x}\right)$$

$$\therefore \frac{dy}{dx} = 12\sqrt{\pi} \text{ at } x = \frac{\sqrt{\pi}}{2}$$



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## Differentiating $e^x$ & $\ln x$

How do I differentiate exponentials and logarithms?

- The derivative of  $y = e^x$  is  $\frac{dy}{dx} = e^x$  where  $x \in \mathbb{R}$
- The derivative of  $y = \ln x$  is  $\frac{dy}{dx} = \frac{1}{x}$  where  $x > 0$
- For the **linear** function  $ax + b$ , where  $a$  and  $b$  are constants,
  - the derivative of  $y = e^{(ax+b)}$  is  $\frac{dy}{dx} = ae^{(ax+b)}$
  - the derivative of  $y = \ln(ax+b)$  is  $\frac{dy}{dx} = \frac{a}{(ax+b)}$ 
    - in the special case  $b = 0$ ,  $\frac{dy}{dx} = \frac{1}{x}$  ( $a$ 's cancel)
- For the **general** function  $f(x)$ ,
  - the derivative of  $y = e^{f(x)}$  is  $\frac{dy}{dx} = f'(x)e^{f(x)}$
  - the derivative of  $y = \ln(f(x))$  is  $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
- The last two sets of results can be derived using the **chain rule**

### Examiner Tip

- Remember to avoid the common mistakes:
  - the derivative of  $\ln kx$  with respect to  $x$  is  $\frac{1}{x}$ , NOT  $\frac{k}{x}$
  - the derivative of  $e^{kx}$  with respect to  $x$  is  $ke^{kx}$ , NOT  $kxe^{kx-1}$



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### Worked example

A curve has the equation  $y = e^{-3x+1} + 2\ln 5x$ .

Find the gradient of the curve at the point where  $x = 2$  giving your answer in the form  $y = a + be^c$ , where  $a$ ,  $b$  and  $c$  are integers to be found.

$$y = e^{-3x+1} + 2(\ln 5x)$$

$$\therefore \frac{dy}{dx} = -3e^{-3x+1} + 2\left(\frac{1}{x}\right)$$

$\uparrow$   
 $y = e^{ax+b}, \frac{dy}{dx} = ae^{ax+b}$

$\uparrow$   
 $y = \ln(ax+b), \text{ special case } b=0, \frac{dy}{dx} = \frac{1}{x}$

$$\text{At } x=2, \frac{dy}{dx} = -3e^{-3(2)+1} + \frac{2}{2} = -3e^{-5} + 1$$

$$\therefore \text{Gradient at } x=2 \text{ is } 1-3e^{-5}$$

i.e.  $a=1, b=-3, c=-5$

$\leftarrow$  Your GDC may be able to find gradients but probably not in the exact form required. It is still helpful to check approximate answers though.



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## 5.2.2 Techniques of Differentiation

### Chain Rule

#### What is the chain rule?

- The **chain rule** states if  $y$  is a function of  $u$  and  $u$  is a function of  $x$  then

$$y = f(u(x))$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

- This is given in the **formula booklet**
- In **function notation** this could be written

$$y = f(g(x))$$

$$\frac{dy}{dx} = f'(g(x))g'(x)$$

#### How do I know when to use the chain rule?

- The chain rule is used when we are trying to differentiate **composite functions**
  - “function of a function”
  - these can be identified as the variable (usually  $x$ ) does not ‘appear alone’
    - $\sin x$  - **not** a composite function,  $x$  ‘appears alone’
    - $\sin(3x + 2)$  is a **composite function**;  $x$  is tripled and has 2 added to it before the sine function is applied

#### How do I use the chain rule?

##### STEP 1

Identify the two functions

Rewrite  $y$  as a function of  $u$ ;  $y = f(u)$

Write  $u$  as a function of  $x$ ;  $u = g(x)$

##### STEP 2

Differentiate  $y$  with respect to  $u$  to get  $\frac{dy}{du}$

Differentiate  $u$  with respect to  $x$  to get  $\frac{du}{dx}$

##### STEP 3

Obtain  $\frac{dy}{dx}$  by applying the formula  $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$  and substitute  $u$  back in for  $g(x)$

- In trickier problems **chain rule** may have to be applied **more than once**

### Are there any standard results for using chain rule?

- There are **five** general results that can be useful
  - If  $y = (f(x))^n$  then  $\frac{dy}{dx} = nf'(x)f(x)^{n-1}$
  - If  $y = e^{f(x)}$  then  $\frac{dy}{dx} = f'(x)e^{f(x)}$
  - If  $y = \ln(f(x))$  then  $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
  - If  $y = \sin(f(x))$  then  $\frac{dy}{dx} = f'(x)\cos(f(x))$
  - If  $y = \cos(f(x))$  then  $\frac{dy}{dx} = -f'(x)\sin(f(x))$

### Examiner Tip

- You should aim to be able to spot and carry out the chain rule mentally (rather than use substitution)
  - every time you use it, say it to yourself in your head  
“differentiate the first function ignoring the second, then multiply by the derivative of the second function”



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### Worked example

a) Find the derivative of  $y = (x^2 - 5x + 7)^7$ .

STEP 1 Identify the two functions and rewrite

$$y = u^7$$

$$\text{i.e. } f(u) = u^7$$

$$u = x^2 - 5x + 7$$

$$\text{i.e. } g(x) = x^2 - 5x + 7$$

STEP 2 Find  $\frac{dy}{du}$  and  $\frac{du}{dx}$ .

$$\frac{dy}{du} = 7u^6$$

$$\frac{du}{dx} = 2x - 5$$

STEP 3 Apply chain rule,  $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Chain rule is in the formula booklet

$$\frac{dy}{dx} = 7u^6(2x - 5)$$

and substitute  $u$  back for  $g(x)$

$$\frac{dy}{dx} = 7(2x - 5)(x^2 - 5x + 7)^6$$

b) Find the derivative of  $y = \sin(e^{2x})$ .



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$$y = \sin(e^{2x})$$

"... differentiate sin  $\square$ , ignore  $e^{2x}$ "

$$\frac{dy}{dx} = \cos(e^{2x}) \times 2e^{2x}$$

"... multiply by derivative of  $e^{2x}$  ..."

↑  
" $y = e^{ax+b}$ ,  $\frac{dy}{dx} = ae^{ax+b}$ "  
or by applying chain rule again

$$\therefore \frac{dy}{dx} = 2e^{2x} \cos(e^{2x})$$



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## Product Rule

### What is the product rule?

- The **product rule** states if  $y$  is the product of two functions  $u(x)$  and  $v(x)$  then

$$y = uv$$

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

- This is given in the **formula booklet**
- In **function notation** this could be written as

$$y = f(x)g(x)$$

$$\frac{dy}{dx} = f(x)g'(x) + g(x)f'(x)$$

- '**Dash notation**' may be used as a **shorter** way of writing the rule

$$y = uv$$

$$y' = uv' + vu'$$

- Final answers should match the notation used throughout the question

### How do I know when to use the product rule?

- The **product rule** is used when we are trying to **differentiate** the **product** of **two functions**
  - these can easily be confused with composite functions (see **chain rule**)
    - $\sin(\cos x)$  is a composite function, "sin of cos of  $x$ "
    - $\sin x \cos x$  is a product, "sin  $x$  times cos  $x$ "

### How do I use the product rule?

- Make it clear what  $u$ ,  $v$ ,  $u'$  and  $v'$  are
  - arranging them in a square can help
    - opposite diagonals match up

#### STEP 1

Identify the two functions,  $u$  and  $v$

Differentiate both  $u$  and  $v$  with respect to  $x$  to find  $u'$  and  $v'$

#### STEP 2

Obtain  $\frac{dy}{dx}$  by applying the product rule formula  $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

Simplify the answer if straightforward to do so or if the question requires a particular form

- In trickier problems **chain rule** may have to be used when finding  $u'$  and  $v'$

### Examiner Tip

- Use  $u$ ,  $v$ ,  $u'$  and  $v'$  for the elements of product rule
  - lay them out in a 'square' (imagine a  $2 \times 2$  grid)
  - those that are paired together are then on opposite diagonals ( $u$  and  $v'$ ,  $v$  and  $u'$ )
- For trickier functions chain rule may be required inside product rule
  - i.e. chain rule may be needed to differentiate  $u$  and  $v$



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### Worked example

a) Find the derivative of  $y = e^x \sin x$ .

$$y = e^x \sin x$$

STEP 1 Identify functions and differentiate

$$\begin{array}{l} u = e^x \\ u' = e^x \end{array} \quad \begin{array}{l} v = \sin x \\ v' = \cos x \end{array}$$

arranging  $u, v, u', v'$  in a square makes product rule 'diagonal pairs'

STEP 2 Apply product rule:  $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

(As it is given in the formula booklet)

$$y' = e^x \cos x + e^x \sin x$$

$$\therefore \frac{dy}{dx} = e^x (\cos x + \sin x)$$

It is straightforward to take a factor of  $e^x$  out

b) Find the derivative of  $y = 5x^2 \cos 3x^2$ .

$$y = 5x^2 \cos 3x^2$$

STEP 1  $u = 5x^2$   $v = \cos 3x^2$  (chain rule)

$$\begin{array}{l} u' = 10x \\ v' = -\sin 3x^2 \times 6x \\ v' = -6x \sin 3x^2 \end{array}$$

STEP 2  $y' = -30x^3 \sin 3x^2 + 10x \cos 3x^2$

$$\therefore \frac{dy}{dx} = 10x (\cos 3x^2 - 3x^2 \sin 3x^2)$$



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## Quotient Rule

### What is the quotient rule?

- The **quotient rule** states if  $y$  is the quotient  $\frac{u(x)}{v(x)}$  then

$$y = \frac{u}{v}$$

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

- This is given in the **formula booklet**
- In **function notation** this could be written

$$y = \frac{f(x)}{g(x)}$$

$$\frac{dy}{dx} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

- As with product rule, '**dash notation**' may be used

$$y = \frac{u}{v}$$

$$y' = \frac{vu' - uv'}{v^2}$$

- Final answers should match the notation used throughout the question

### How do I know when to use the quotient rule?

- The **quotient rule** is used when trying to differentiate a fraction where **both** the **numerator** and **denominator** are **functions** of  $X$ 
  - if the **numerator** is a **constant**, **negative powers** can be used
  - if the **denominator** is a **constant**, treat it as a **factor** of the expression

### How do I use the quotient rule?

- Make it clear what  $u$ ,  $v$ ,  $u'$  and  $v'$  are
  - arranging them in a square can help
    - opposite diagonals match up (like they do for product rule)

#### STEP 1

Identify the two functions,  $u$  and  $v$

Differentiate both  $u$  and  $v$  with respect to  $x$  to find  $u'$  and  $v'$

## STEP 2

Obtain  $\frac{dy}{dx}$  by applying the quotient rule formula  $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

Be careful using the formula – because of the **minus sign** in the **numerator**, the **order** of the functions is important

Simplify the answer if straightforward or if the question requires a particular form

- In trickier problems **chain rule** may have to be used when finding  $u'$  and  $v'$ ,

### Examiner Tip

- Use  $u$ ,  $v$ ,  $u'$  and  $v'$  for the elements of quotient rule
  - lay them out in a 'square' (imagine a 2x2 grid)
  - those that are paired together are then on opposite diagonals ( $v$  and  $u'$ ,  $u$  and  $v'$ )
- Look out for functions of the form  $y = f(x)(g(x))^{-1}$ 
  - These can be differentiated using a combination of **chain rule** and **product rule** (it would be good practice to try!)
  - ... but it can also be seen as a quotient rule question in disguise
  - ... and vice versa!
    - A quotient could be seen as a product by rewriting the denominator as  $(g(x))^{-1}$



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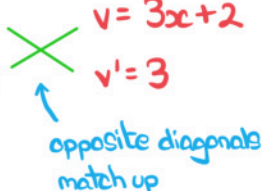
Your notes

### Worked example

Differentiate  $f(x) = \frac{\cos 2x}{3x+2}$  with respect to  $x$ .

STEP 1 Identify  $u$  and  $v$ , differentiate

$$\begin{array}{l}
 u = \cos 2x \qquad v = 3x + 2 \\
 u' = -2\sin 2x \qquad v' = 3
 \end{array}$$


  
 chain rule opposite diagonals match up

STEP 2 Apply quotient rule:  $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

(As it is given in the formula booklet)

$$f'(x) = \frac{(3x+2)(-2\sin 2x) - (\cos 2x)(3)}{(3x+2)^2}$$

$$\therefore f'(x) = \frac{-2(3x+2)\sin 2x - 3\cos 2x}{(3x+2)^2}$$

(Nothing obvious/easy to simplify and question does not specify a particular form)





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## 5.2.3 Higher Order Derivatives

### Second Order Derivatives

#### What is the second order derivative of a function?

- If you **differentiate** the **derivative** of a **function** (i.e. differentiate the function a second time) you get the **second order derivative** of the function
- There are two forms of **notation** for the **second order derivative**
  - $y = f(x)$
  - $\frac{dy}{dx} = f'(x)$  (First order derivative)
  - $\frac{d^2y}{dx^2} = f''(x)$  (Second order derivative)
- Note the position of the superscript 2's
  - differentiating twice (so  $d^2$ ) with respect to  $x$  twice (so  $x^2$ )
- The **second order derivative** can be referred to simply as the **second derivative**
  - Similarly, the **first order derivative** can be just the **first derivative**
- A **first order derivative** is the **rate of change** of a function
  - a **second order derivative** is the **rate of change** of the **rate of change** of a function
    - i.e. the **rate of change** of the function's **gradient**
- **Second order derivatives** can be used to
  - test for local minimum and maximum points
  - help determine the nature of stationary points
  - help determine the concavity of a function
  - graph derivatives

#### How do I find a second order derivative of a function?

- By **differentiating twice!**
- This may involve
  - rewriting **fractions**, **roots**, etc as **negative** and/or **fractional powers**
  - differentiating **trigonometric** functions, **exponentials** and **logarithms**
  - using **chain rule**
  - using **product** or **quotient** rule

#### Examiner Tip

- Negative and/or fractional powers can cause problems when finding second derivatives so work carefully through each term



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### Worked example

Given that  $f(x) = 4 - \sqrt{x} + \frac{3}{\sqrt{x}}$

a) Find  $f'(x)$  and  $f''(x)$ .

a)  $f(x) = 4 - x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}$

REWRITE AS POWERS OF  $x$

$$f'(x) = -\left(\frac{1}{2}\right)x^{\frac{1}{2}-1} + 3\left(-\frac{1}{2}\right)x^{\frac{1}{2}-1}$$

DIFFERENTIATE ONCE TO FIND  $f'(x)$

$$f'(x) = -\frac{1}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}}$$

$$f''(x) = -\frac{1}{2}\left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1} - \frac{3}{2}\left(-\frac{3}{2}\right)x^{-\frac{3}{2}-1}$$

DIFFERENTIATE A SECOND TIME TO FIND  $f''(x)$

$$f''(x) = \frac{1}{4}x^{-\frac{3}{2}} + \frac{9}{4}x^{-\frac{5}{2}}$$

b) Evaluate  $f''(3)$ .

Give your answer in the form  $a\sqrt{b}$ , where  $b$  is an integer and  $a$  is a rational number.

b)  $f''(x) = \frac{1}{4x\sqrt{x}} + \frac{9}{4x^2\sqrt{x}}$

$$x^{\frac{3}{2}} = x\sqrt{x} \quad x^{\frac{5}{2}} = x^2\sqrt{x}$$

$$f''(3) = \frac{1}{12\sqrt{3}} + \frac{9}{36\sqrt{3}}$$

$$= \frac{12}{36\sqrt{3}} = \frac{1}{3\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{9}$$

$$f''(3) = \frac{1}{9}\sqrt{3}$$

RATIONALISE DENOMINATOR



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## Higher Order Derivatives

### What is meant by higher order derivatives of a function?

- Many functions can be **differentiated** numerous times
  - The third, fourth, fifth, etc derivatives of a function are generally called **higher order derivatives**
- It may not be possible, or practical to (algebraically) differentiate complicated functions more than once or twice
- Polynomials** will, eventually, have higher order derivatives of zero
  - Since powers of  $x$  reduce by 1 each time

### What is the notation for higher order derivatives?

- The notation for higher order derivatives follows the logic from the first and second derivatives

$$f^{(n)}(x) \text{ or } \frac{d^n y}{dx^n}$$

except the 'dash' (prime) notation is replaced with numbers as this would become cumbersome after the first few

- e.g. the fifth derivative would be

$$f^{(5)}(x) \text{ or } \frac{d^5 y}{dx^5}$$

### How do I find a higher order derivative of a function?

- By differentiating as many times as required!
- This may involve
  - rewriting fractions, roots, etc as negative and/or fractional powers
  - differentiating trigonometric functions, exponentials and logarithms
  - using chain rule
  - using product or quotient rule

#### Examiner Tip

- If you are required to evaluate a higher order derivative at a specific point your GDC can help
  - Typically a GDC will only work out the first and second derivative directly from the original function
  - But, if you wanted the fourth derivative, say, you only need differentiate twice algebraically, then call this the 'original' function on your GDC



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### Worked example

It is given that  $f(x) = \sin 2x$ .

a) Show that  $f^4(x) = 16f(x)$ .

$f^4(x)$  is the FOURTH derivative

$$f(x) = \sin 2x$$

$$f'(x) = 2 \cos 2x \quad (\text{'sin' } \rightarrow \text{'cos', chain rule})$$

$$f''(x) = -4 \sin 2x \quad (\text{'cos' } \rightarrow \text{'-sin', chain rule})$$

$$f^3(x) = -8 \cos 2x \quad \text{You should notice a pattern by now...}$$

$$f^4(x) = 16 \sin 2x$$

$$\therefore f^4(x) = 16 \sin 2x = 16f(x) \text{ as required}$$

b) Without further working, write down an expression for  $f^8(x)$ .

We can see from part (a)

- the coefficient of each derivative is a power of 2
- $\sin 2x$  ( $f(x)$ ) is involved in every even derivative
- $\sin 2x$  is positive in every other even derivative

$$\therefore f^8(x) = 256 \sin 2x$$



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## 5.2.4 Further Applications of Differentiation

### Stationary Points & Turning Points

What is the difference between a stationary point and a turning point?

- A **stationary point** is a point at which the **gradient function** is equal to zero
  - The **tangent** to the **curve** of the **function** is **horizontal**
- A **turning point** is a type of stationary point, but in addition the **function changes** from **increasing** to **decreasing**, or **vice versa**
  - The curve '**turns**' from '**going upwards**' to '**going downwards**' or **vice versa**
  - **Turning points** will either be (**local**) **minimum** or **maximum** points
- A **point of inflection** *could* also be a **stationary point** but is **not** a turning point

How do I find stationary points and turning points?

- For the function  $y = f(x)$ , **stationary points** can be found using the following process

#### STEP 1

Find the **gradient function**,  $\frac{dy}{dx} = f'(x)$

#### STEP 2

Solve the equation  $f'(x) = 0$  to find the  $x$ -coordinate(s) of any stationary points

#### STEP 3

If the  $y$ -coordinates of the stationary points are also required then substitute the  $x$ -coordinate(s) into  $f(x)$

- A GDC will solve  $f'(x) = 0$  and most will find the coordinates of turning points (minimum and maximum points) in graphing mode



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## Testing for Local Minimum & Maximum Points

### What are local minimum and maximum points?

- Local **minimum** and **maximum** points are two types of **stationary** point
  - The **gradient function** (derivative) at such points equals zero
    - i.e.  $f'(x) = 0$
- A **local minimum** point,  $(x, f(x))$  will be the lowest value of  $f(x)$  in the **local** vicinity of the value of  $x$ 
  - The function may reach a **lower** value further afield
- Similarly, a **local maximum** point,  $(x, f(x))$  will be the highest value of  $f(x)$  in the **local** vicinity of the value of  $x$ 
  - The function may reach a **greater** value further afield
- The graphs of many functions **tend to infinity** for **large** values of  $x$  (and/or **minus infinity** for **large negative** values of  $x$ )
- The **nature** of a stationary point refers to whether it is a **local minimum** point, a **local maximum** point or a **point of inflection**
- A **global** minimum point would represent the **lowest** value of  $f(x)$  for **all values** of  $x$ 
  - similar for a **global** maximum point

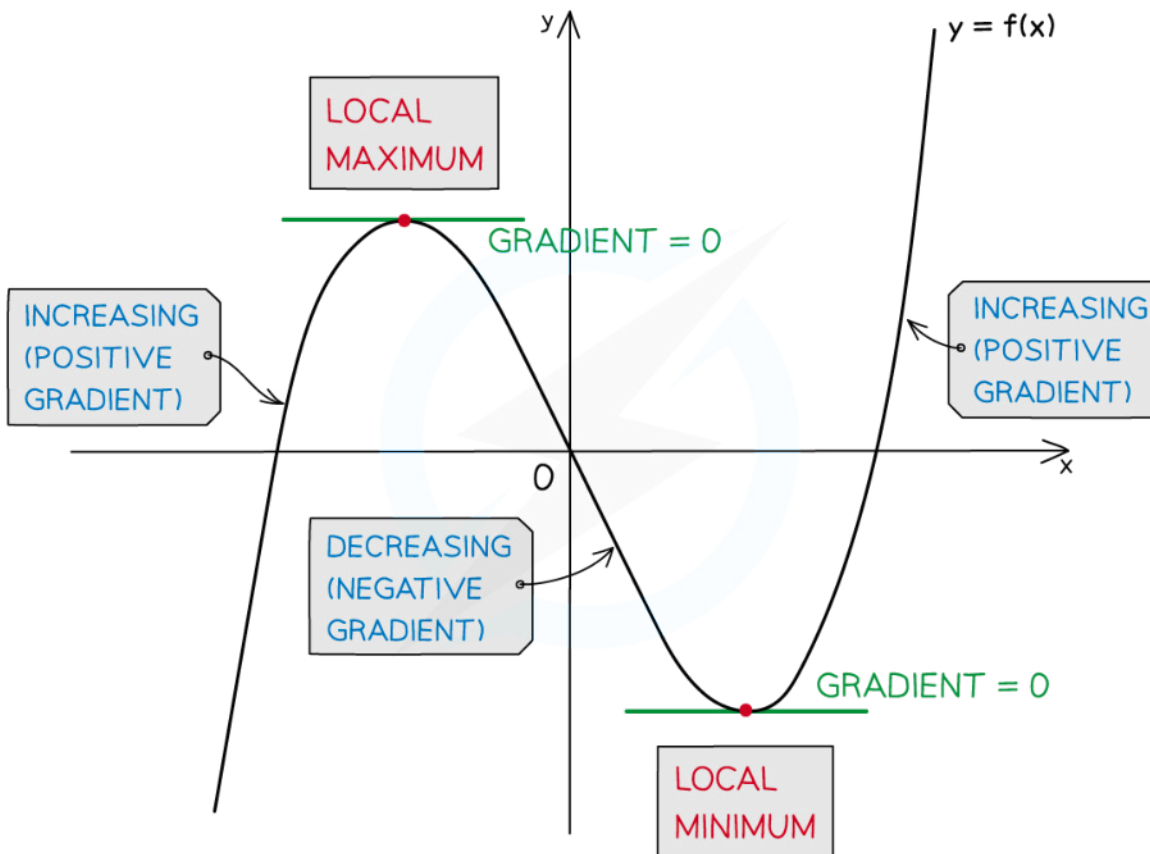
### How do I find local minimum & maximum points?

- The **nature** of a **stationary point** can be determined using the **first derivative** but it is *usually* quicker and easier to use the **second derivative**
  - only in cases when the second derivative is **zero** is the first derivative method needed
- For the function  $f(x)$  ...
  - STEP 1**  
Find  $f'(x)$  and solve  $f'(x) = 0$  to find the  $x$ -coordinates of any stationary points
  - STEP 2** (Second derivative)  
Find  $f''(x)$  and evaluate it at each of the stationary points found in **STEP 1**
  - STEP 3** (Second derivative)
    - If  $f''(x) = 0$  then the nature of the stationary point **cannot** be determined; use the **first derivative** method (**STEP 4**)
    - If  $f''(x) > 0$  then the curve of the graph of  $y = f(x)$  is **concave up** and the stationary point is a **local minimum** point
    - If  $f''(x) < 0$  then the curve of the graph of  $y = f(x)$  is **concave down** and the stationary point is a **local maximum** point
  - STEP 4** (First derivative)  
Find the sign of the first derivative just either side of the stationary point; i.e. evaluate  $f'(x - h)$  and  $f'(x + h)$  for small  $h$



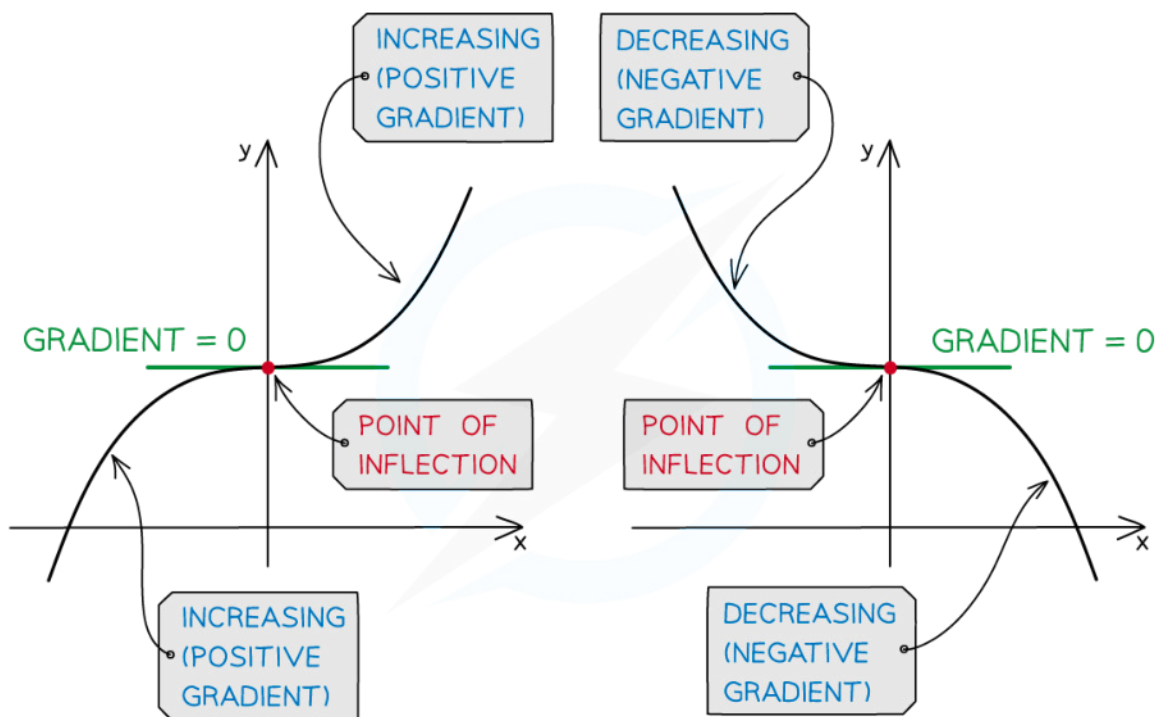
Your notes

- A **local minimum point** changes the function from **decreasing** to **increasing**
  - the **gradient** changes from **negative** to **positive**
  - $f'(x-h) < 0$ ,  $f'(x) = 0$ ,  $f'(x+h) > 0$
- A **local maximum point** changes the function from **increasing** to **decreasing**
  - the **gradient** changes from **positive** to **negative**
  - $f'(x-h) > 0$ ,  $f'(x) = 0$ ,  $f'(x+h) < 0$



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- A **stationary point of inflection** results from the function **either increasing or decreasing** on **both sides** of the stationary point
  - the **gradient** does **not** change sign
  - $f'(x-h) > 0$ ,  $f'(x+h) > 0$  or  $f'(x-h) < 0$ ,  $f'(x+h) < 0$
  - a **point of inflection** does **not** necessarily have  $f'(x) = 0$ 
    - this method will only find those that do - and are often called **horizontal points of inflection**



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 **Examiner Tip**

- Exam questions may use the phrase “classify turning points” instead of “find the nature of turning points”
- Using your GDC to sketch the curve is a valid test for the nature of a stationary point in an exam unless the question says “show that...” or asks for an algebraic method
- Even if required to show a full algebraic solution you can still use your GDC to tell you what you’re aiming for and to check your work



 **Worked example**

Find the coordinates and the nature of any stationary points on the graph of  $y = f(x)$  where  $f(x) = 2x^3 - 3x^2 - 36x + 25$ .



Your notes



Your notes

At stationary points,  $f'(x) = 0$

$$f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6)$$

$$6(x^2 - x - 6) = 0$$

$$(x-3)(x+2) = 0$$

$$x = 3, \quad y = f(3) = 2(3)^3 - 3(3)^2 - 36(3) + 25 = -56$$

$$x = -2, \quad y = f(-2) = 2(-2)^3 - 3(-2)^2 - 36(-2) + 25 = 69$$

Using the second derivative to determine their nature

$$f''(x) = 12x - 6 = 6(2x - 1)$$

$$f''(3) = 6(2 \times 3 - 1) = 30 > 0$$

$\therefore x = 3$  is a local minimum point

$$f''(-2) = 6(2 \times -2 - 1) = -30 < 0$$

$\therefore x = -2$  is a local maximum point

(Note: In this case, both stationary points are turning points)

Turning points are:

$(3, -56)$  local minimum point

$(-2, 69)$  local maximum point

Use a GDC to graph  $y = f(x)$  and the max/min solving feature to check the answers.



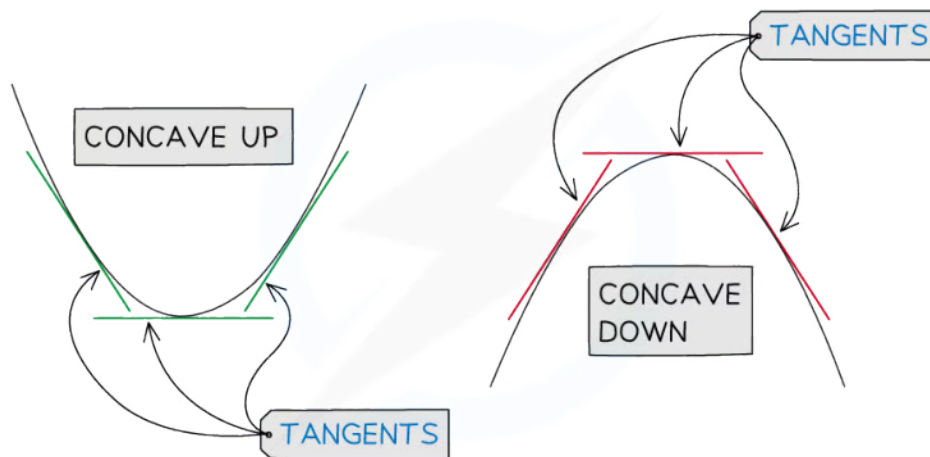
Your notes

## 5.2.5 Concavity & Points of Inflection

### Concavity of a Function

#### What is concavity?

- **Concavity** is the way in which a **curve** (or surface) **bends**
- Mathematically,
  - a curve is **CONCAVE DOWN** if  $f''(x) < 0$  for all values of  $x$  in an interval
  - a curve is **CONCAVE UP** if  $f''(x) > 0$  for all values of  $x$  in an interval



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#### Examiner Tip

- In an exam an easy way to remember the difference is:
  - Concave **down** is the shape of (the mouth of) a sad smiley 😞
  - **Concave up** is the shape of (the mouth of) a happy smiley 😊



Your notes

### Worked example

The function  $f(x)$  is given by  $f(x) = x^3 - 3x + 2$ .

- a) Determine whether the curve of the graph of  $y = f(x)$  is concave down or concave up at the points where  $x = -2$  and  $x = 2$ .

$$f(x) = x^3 - 3x + 2$$

$$f'(x) = 3x^2 - 3$$

$$f''(x) = 6x$$

$$f''(-2) = 6 \times -2 = -12 < 0 \quad (\text{concave down})$$

$$f''(2) = 6 \times 2 = 12 > 0 \quad (\text{concave up})$$

At  $x = -2$ ,  $y = f(x)$  is concave down  
 At  $x = 2$ ,  $y = f(x)$  is concave up

Use your GDC to plot the graph of  $y = f(x)$   
 and to help see if your answers are sensible

- b) Find the values of  $x$  for which the curve of the graph  $y = f(x)$  of is concave up.

$$f''(x) = 6x \quad \text{from part (a)}$$

$$\text{Concave up is } f''(x) > 0$$

$$6x > 0 \quad \text{when } x > 0$$

$\therefore y = f(x)$  is concave up for  $x > 0$

Use your GDC to check your answer



Your notes

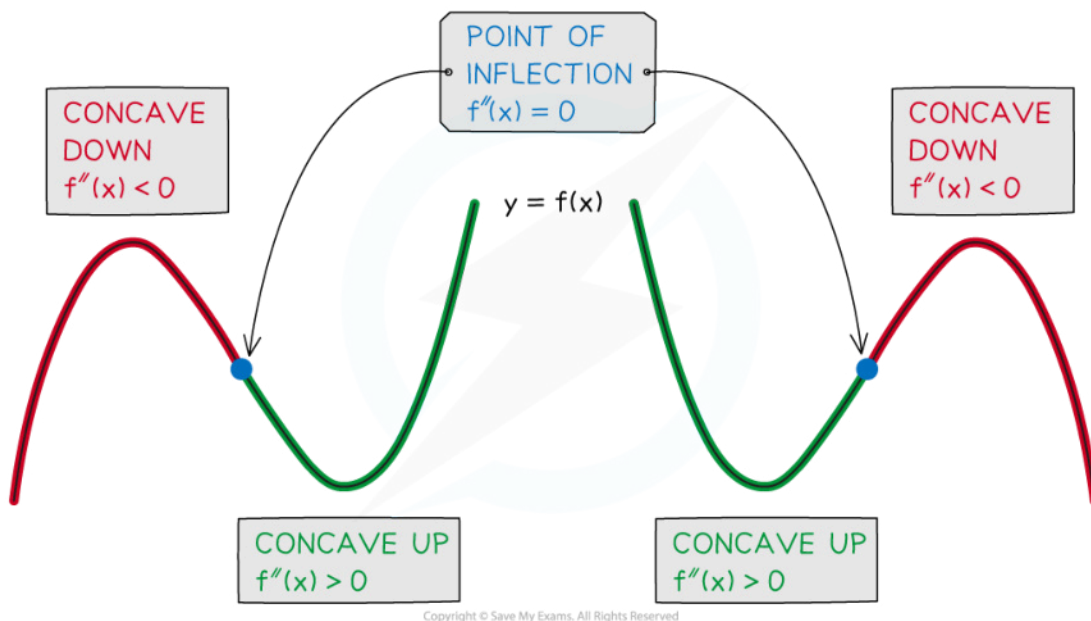
## Points of Inflection

### What is a point of inflection?

- A point at which the curve of the graph of  $y = f(x)$  changes **concavity** is a **point of inflection**
- The alternative spelling, **inflexion**, may sometimes be used

### What are the conditions for a point of inflection?

- A point of inflection requires **BOTH** of the following two conditions to hold
  - the **second derivative** is zero
    - $f''(x) = 0$
  - AND
  - the graph of  $y = f(x)$  changes **concavity**
    - $f''(x)$  changes **sign** through a **point of inflection**



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- It is important to understand that the first condition is **not** sufficient on its own to locate a point of inflection
  - points where  $f''(x) = 0$  could be **local minimum** or **maximum** points
    - the **first derivative** test would be needed
  - However, if it is already known  $f(x)$  has a point of inflection at  $x = a$ , say, then  $f''(a) = 0$

### What about the first derivative, like with turning points?



Your notes

- A **point of inflection**, unlike a turning point, does not necessarily have to have a first derivative value of 0 ( $f'(x) = 0$ )
  - If it does, it is also a **stationary point** and is often called a **horizontal point of inflection**
    - the tangent to the curve at this point would be horizontal
  - The **normal distribution** is an example of a commonly used function that has a graph with two non-stationary points of inflection

### How do I find the coordinates of a point of inflection?

- For the function  $f(x)$

#### STEP 1

Differentiate  $f(x)$  **twice** to find  $f''(x)$  and solve  $f''(x) = 0$  to find the  $x$ -coordinates of possible points of inflection

#### STEP 2

Use the **second derivative** to **test** the **concavity** of  $f(x)$  either side of  $x = a$

- If  $f''(x) < 0$  then  $f(x)$  is concave down
- If  $f''(x) > 0$  then  $f(x)$  is concave up

If concavity changes,  $x = a$  is a **point of inflection**

#### STEP 3

If required, the  $y$ -coordinate of a point of inflection can be found by substituting the  $x$ -coordinate into  $f(x)$

### Examiner Tip

- You can find the  $x$ -coordinates of the point of inflections of  $y = f(x)$  by drawing the graph  $y = f'(x)$  and finding the  $x$ -coordinates of any local maximum or local minimum points
- Another way is to draw the graph  $y = f''(x)$  and find the  $x$ -coordinates of the points where the graph crosses (not just touches) the  $x$ -axis



Your notes

### Worked example

Find the coordinates of the point of inflection on the graph of  $y = 2x^3 - 18x^2 + 24x + 5$ .  
Fully justify that your answer is a point of inflection.

STEP 1: Differentiate twice, solve  $f''(x) = 0$

$$f(x) = 2x^3 - 18x^2 + 24x + 5$$

$$f'(x) = 6x^2 - 36x + 24$$

$$f''(x) = 12x - 36$$

$$12x - 36 = 0 \text{ when } x = 3$$

STEP 2: Use the second derivative to test concavity

$$f''(3) = 0$$

$$f''(2.9) < 0 \quad (\text{concave down})$$

$$f''(3.1) > 0 \quad (\text{concave up})$$

$\therefore$  Concavity changes through  $x = 3$

STEP 3: The  $y$ -coordinate is required

$$f(3) = 2(3)^3 - 18(3)^2 + 24(3) + 5 = -31$$

Since  $f''(3) = 0$  AND the graph of  $y = f(x)$  changes concavity through  $x = 3$ , the point  $(3, -31)$  is a point of inflection.

Use your GDC to plot the graph of  $y = f(x)$   
and to help see if your answer is sensible



Your notes

## 5.2.6 Derivatives & Graphs

### Derivatives & Graphs

#### How are derivatives and graphs connected?

- If the graph of a function  $y = f(x)$  is known, or can be sketched, then it is also possible to sketch the graphs of the derivatives  $y = f'(x)$  and  $y = f''(x)$
- The key properties of a graph include
  - the **y-axis intercept**
  - the **x-axis intercepts** - the **roots** of the function; where  $f(x) = 0$
  - **stationary points**; where  $f'(x) = 0$ 
    - turning points - (local) **minimum** and **maximum** points
    - (horizontal) **points of inflection**
  - (non-stationary,  $f'(x) \neq 0$ ) points of inflection
  - **asymptotes** - **vertical** and **horizontal**
  - intervals where the graph is **increasing** and **decreasing**
  - intervals where the graph is **concave down** and **concave up**
- **Not** all graphs have all of these properties and **not** all can be determined without knowing the expression of the function
- However questions will provide enough information to sketch
  - the **shape** of the graph
  - **some** of the **key properties** such as **roots** or **turning points**

#### How do I sketch the graph of $y = f'(x)$ from the graph of $y = f(x)$ ?

- The graph of  $y = f'(x)$  will have its
  - **x-axis intercepts** at the **x**-coordinates of the **stationary points** of  $y = f(x)$
  - **turning points** at the **x**-coordinates of the **points of inflection** of  $y = f(x)$
- For intervals where  $y = f(x)$  is **concave up**,  $y = f'(x)$  will be **increasing**
- For intervals where  $y = f(x)$  is **concave down**,  $y = f'(x)$  will be **decreasing**
- For intervals where  $y = f(x)$  is **increasing**,  $y = f'(x)$  will be **positive**
- For intervals where  $y = f(x)$  is **decreasing**,  $y = f'(x)$  will be **negative**

#### How do I sketch the graph of $y = f''(x)$ from the graph of $y = f(x)$ ?

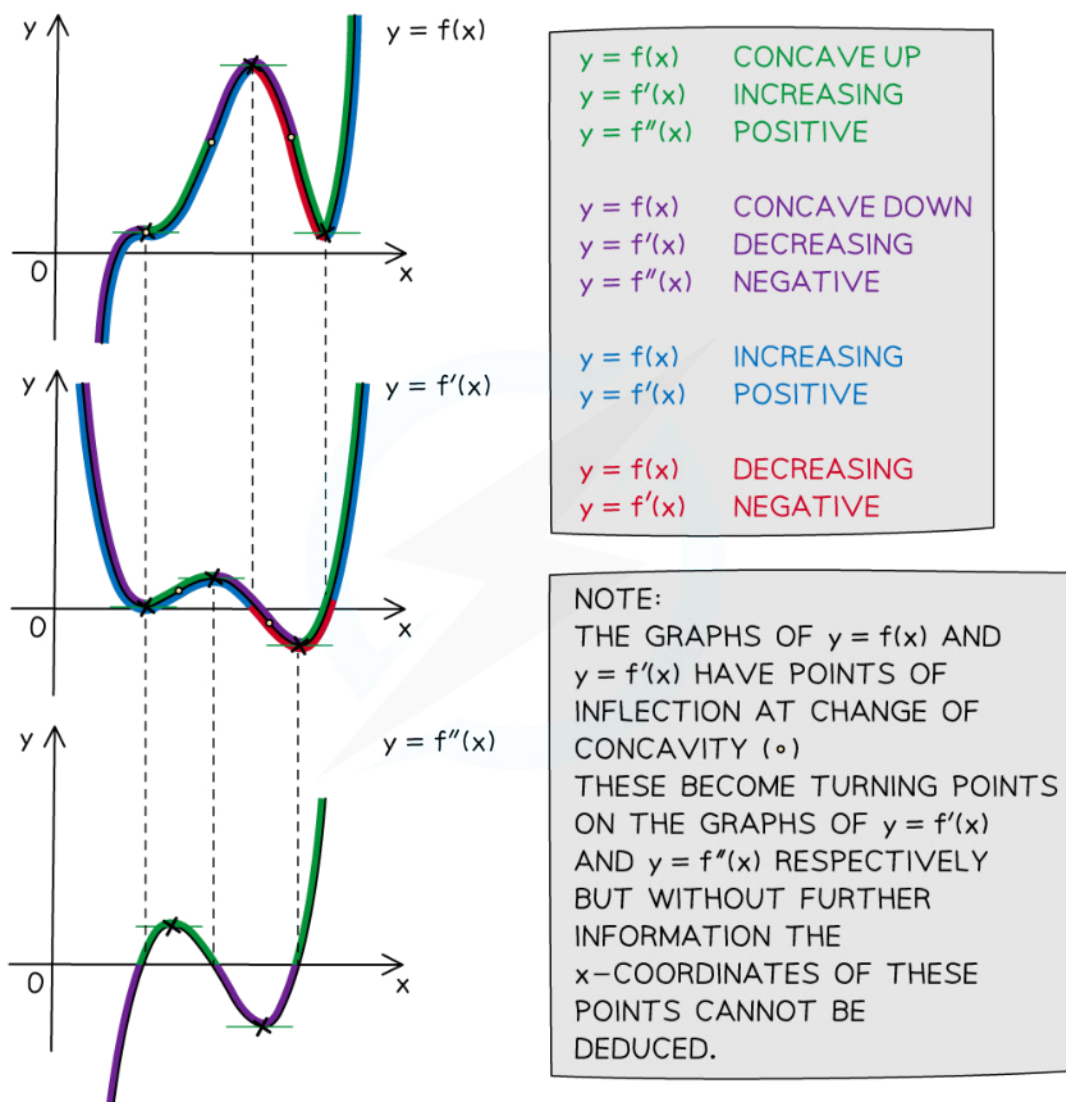
- First sketch the graph of  $y = f'(x)$  from  $y = f(x)$ , as per the above process
- Then, using the same process, sketch the graph of  $y = f''(x)$  from the graph of  $y = f'(x)$
- There are a couple of things you can deduce about the graph of  $y = f''(x)$  directly from the graph of  $y = f(x)$





Your notes

- The graph of  $y = f''(x)$  will have its  $x$ -axis **intercepts** at the  $x$ -coordinates of the **points of inflection** of  $y = f(x)$
- For intervals where  $y = f(x)$  is **concave up**,  $y = f'(x)$  will be **positive**
- For intervals where  $y = f(x)$  is **concave down**,  $y = f'(x)$  will be **negative**



Is it possible to sketch the graph of  $y = f(x)$  from the graph of a derivative?

- It is possible to **sketch** a graph of  $y = f(x)$  by considering the reverse of the above
  - For intervals where  $y = f'(x)$  is **positive**,  $y = f(x)$  will be **increasing** but is **not** necessarily positive

- For intervals where  $y = f'(x)$  is **negative**,  $y = f(x)$  will be **decreasing** but is **not** necessarily negative
- **Roots** of  $y = f'(x)$  give the  $x$ -coordinates of the **stationary points** of  $y = f(x)$
- There are some properties of the graph of  $y = f(x)$  that **cannot** be determined from the graph of  $y = f'(x)$ 
  - the  **$y$ -axis intercept**
  - the **intervals** for which  $y = f(x)$  is **positive** and **negative**
  - the **roots** of  $y = f(x)$
- Unless a **specific** point the **curve passes through** is **known**, the **constant of integration** cannot be determined
  - the exact location of the curve will remain unknown
  - but it will still be possible to **sketch** its **shape**
- If starting from the graph of the **second derivative**,  $y = f''(x)$ , it is easier to sketch the graph of  $y = f'(x)$  first, then sketch  $y = f(x)$



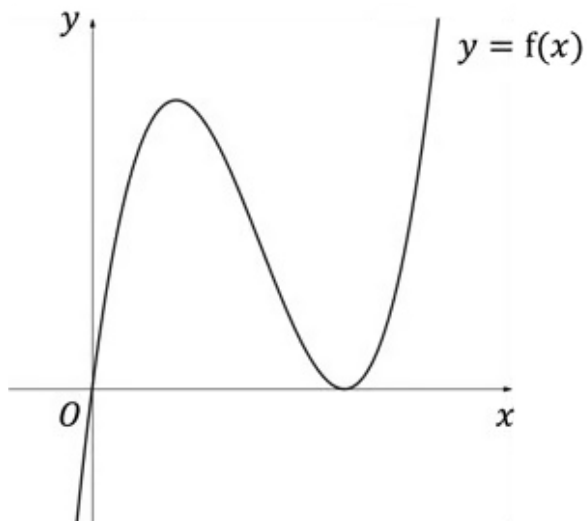
Your notes



Your notes

### Worked example

The graph of  $y = f(x)$  is shown in the diagram below.



On separate diagrams sketch the graphs of  $y = f'(x)$  and  $y = f''(x)$ , labelling any roots and turning points.

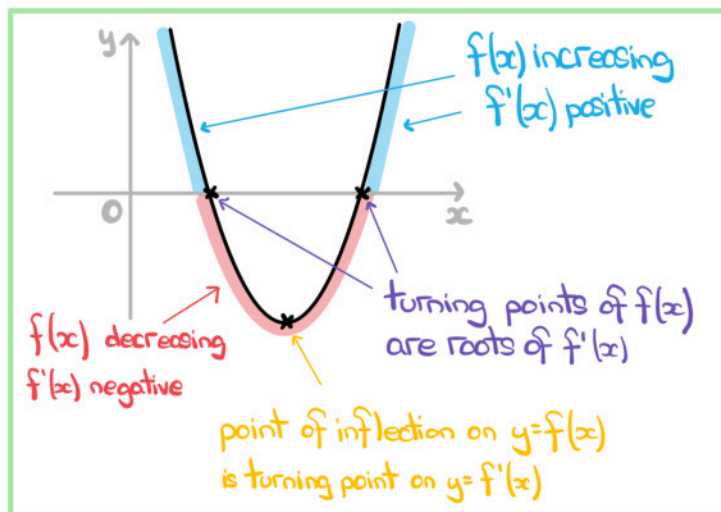


Your notes

Key features from graph of  $y=f(x)$  are:

- local maximum and local minimum
- $f(x)$  is increasing 'at either end'
- $f(x)$  is decreasing between max. and min.
- point of inflexion (non-stationary), graph changes from concave down to concave up

Graph of  $y=f'(x)$ :



Graph of  $y=f''(x)$ :

