SL IB Physics

Your notes

Simple Harmonic Motion

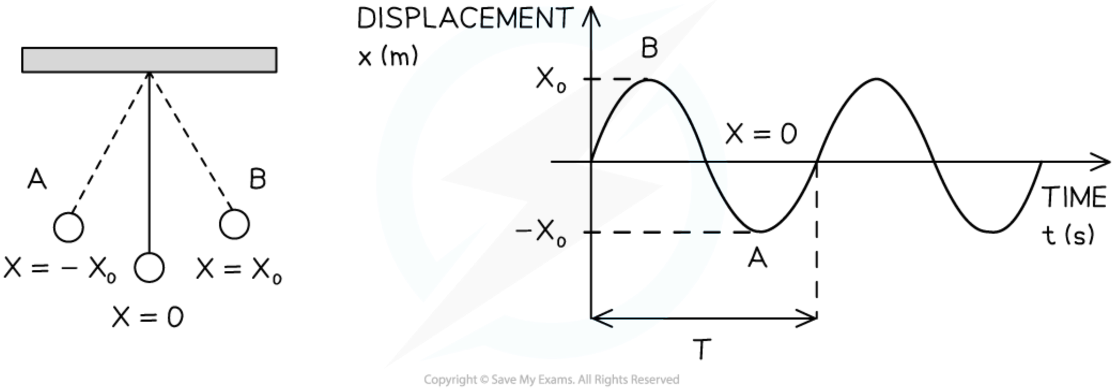
Contents

Describing Oscillations

Properties of Oscillations

- An **oscillation** is defined as follows:

> **The repetitive variation with time *t* of the displacement *x* of an object about the equilibrium position (*x* = 0)**



A pendulum oscillates between A and B. On a displacement-time graph, the oscillating motion of the pendulum is represented by a wave, with an amplitude equal to x_0

- A particle undergoing an oscillation can be described using the following properties:
  - **Equilibrium position (*x* = 0)** is the position when there is no resultant force acting on an object
    - This is the **fixed central point** that the object oscillates around
  - **Displacement (*x*)** is the **horizontal or vertical distance** of a point on the wave from its equilibrium position
    - It is a vector quantity
    - It can be positive or negative depending on which side of the oscillation it is
    - It is measured in metres (m)
  - **Period (*T*)** or time period, is the **time interval** for one complete oscillation measured in seconds (s)
    - If the oscillations have a **constant period**, they are said to be **isochronous**

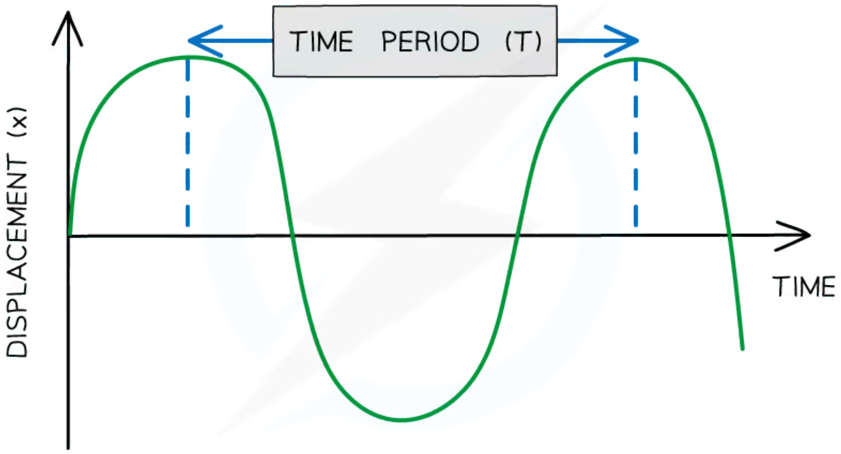
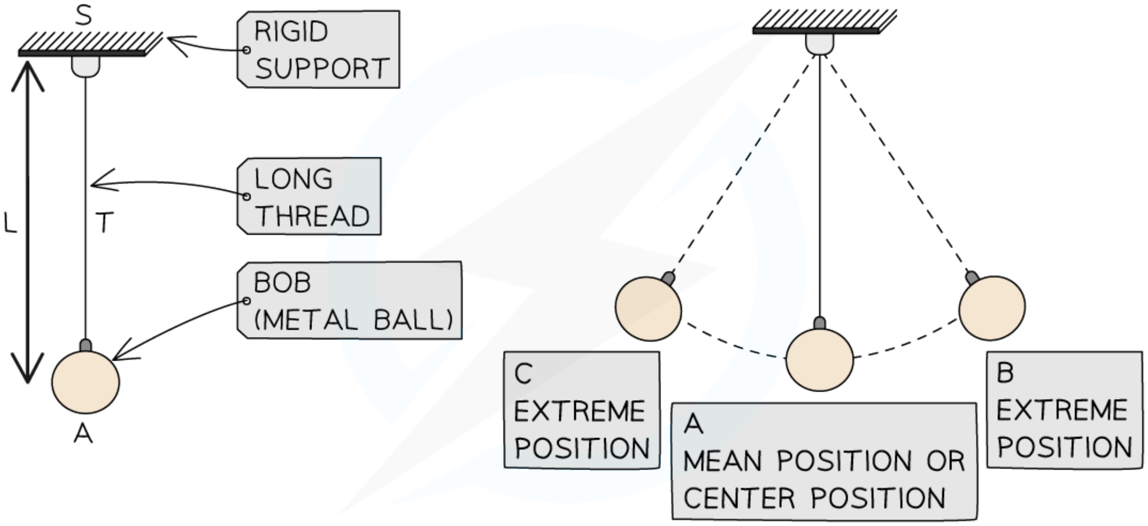


Diagram showing the time period of a wave

- **Amplitude (x_0)** is the **maximum value of the displacement** on either side of the equilibrium position
  - Amplitude is measured in metres (m)

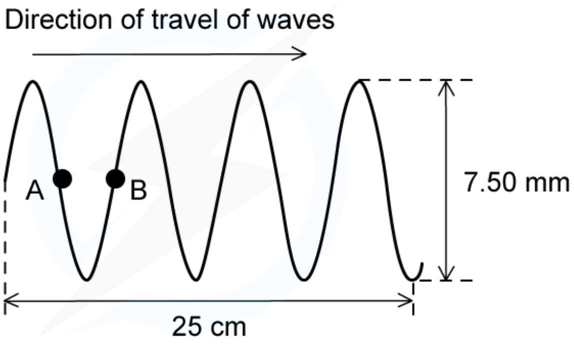


When the pendulum is in its extreme position this is its amplitude

- **Frequency** (*f*) is the **number of oscillations per second** and it is measured in hertz (Hz)
  - Hz has the SI units per second s^{-1} because $f = \frac{1}{T}$ see below
- **Angular frequency** (*ω*) is the rate of change of angular displacement with respect to time
  - It is measured in radians per second (rad s^{-1})

Worked example

The diagram below shows plane waves on the surface of water at a particular instant. A and B are two points on the wave.



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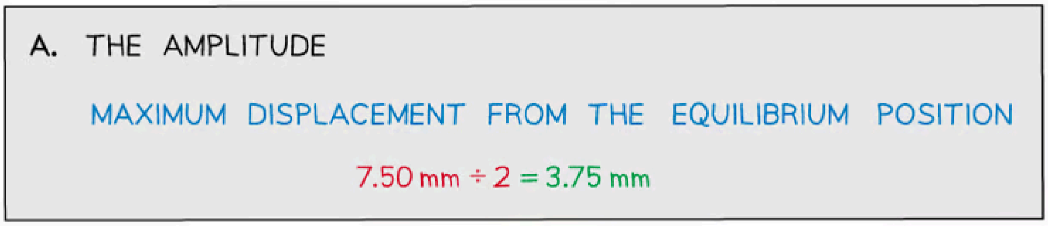
Determine:

(a) The amplitude

(b) The wavelength

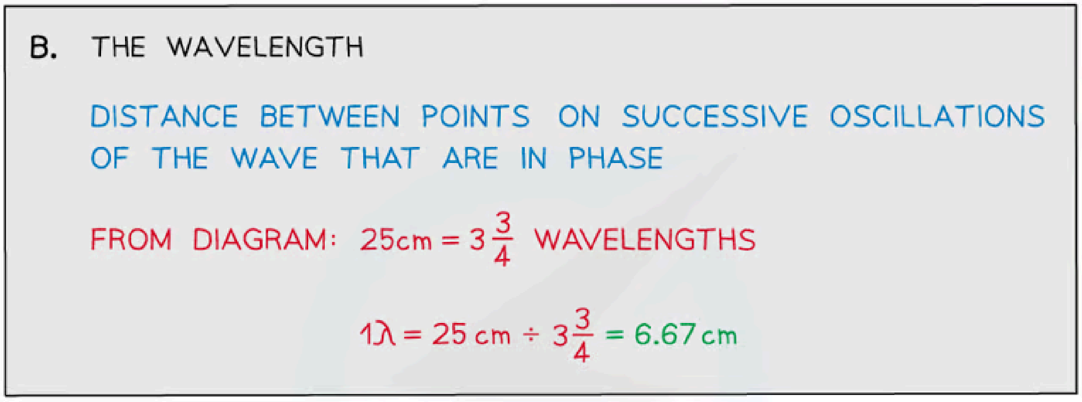
Answer:



A. THE AMPLITUDE

MAXIMUM DISPLACEMENT FROM THE EQUILIBRIUM POSITION

$7.50 \text{ mm} \div 2 = 3.75 \text{ mm}$



B. THE WAVELENGTH

DISTANCE BETWEEN POINTS ON SUCCESSIVE OSCILLATIONS OF THE WAVE THAT ARE IN PHASE

FROM DIAGRAM: $25\text{cm} = 3\frac{3}{4}$ WAVELENGTHS

$1\lambda = 25 \text{ cm} \div 3\frac{3}{4} = 6.67 \text{ cm}$

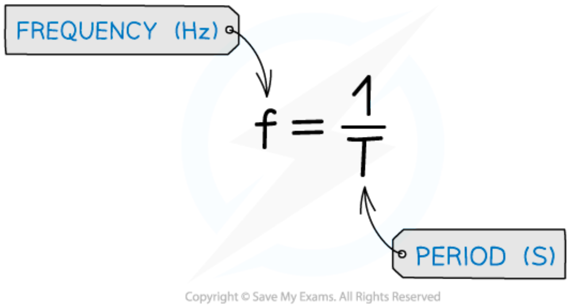
Examiner Tip

When labelling the amplitude and time period on a diagram:

- Make sure that your arrows go from the **very top** of a wave to the very top of the next one
- If your arrow is too short, you will lose marks
- The same goes for labelling amplitude, don't draw an arrow from the bottom to the top of the wave, this will lose you marks too.

Calculating Time Period of an Oscillation

- This equation relates the frequency and the time period of an oscillation:

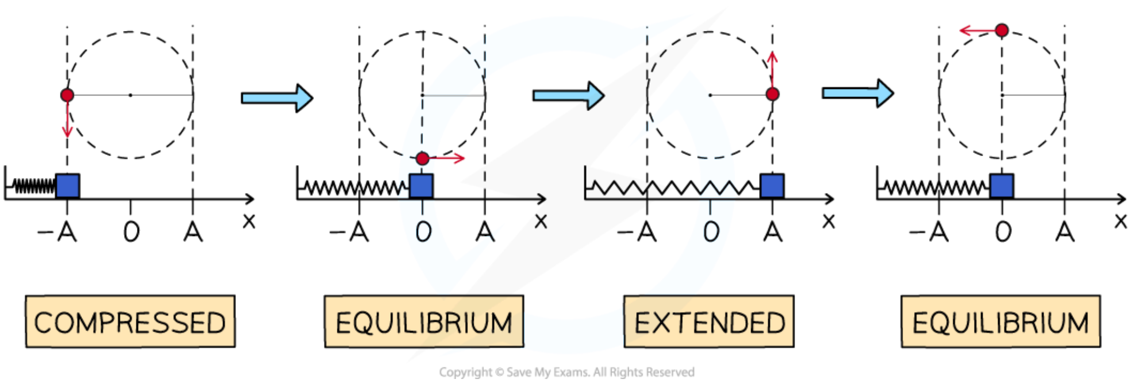


The equation linking time period and frequency

- **Angular frequency** (ω) can be calculated using the equation:

$$\omega = \frac{2\pi}{T} = 2\pi f$$

- Where:
  - ω = angular frequency (rad s^{-1})
  - 2π = circumference of a circle
  - T = time period (s)
  - f = frequency of oscillation (Hz)
- The **angular displacement** of objects in oscillation can be determined by matching the **displacement** to an object in **circular motion**:
  - After moving from one amplitude position $x = -A$ to the equilibrium position $x = 0$ the mass on the spring has moved an angular displacement of $\frac{1}{4}$ of a circle = $\frac{1}{4} \times 2\pi = \frac{\pi}{2}$ radians
  - Continuing the oscillation from the **equilibrium position** to the **other amplitude position** the angular displacement is also $\frac{\pi}{2}$ radians
  - Continuing the oscillation **back to the starting point** means the mass travels a further angular displacement of $\frac{\pi}{2} + \frac{\pi}{2} = \pi$ radians
  - Hence, the **total angular displacement** in **one oscillation** is $\pi + \pi = 2\pi$ radians



The motion of an oscillating object can be analysed in terms of a fraction of an object in circular motion

Worked example

A child on a swing performs 0.2 oscillations per second.

Calculate the time period of the oscillation.

Answer:

Step 1: Write down the known quantities

- Frequency, $f = 0.2$ Hz

Step 2: Write down the relationship between the period T and the frequency f

$$T = \frac{1}{f}$$

Step 3: Substitute the value of the frequency into the above equation and calculate the period

$$T = \frac{1}{0.2} = 5.0 \text{ s}$$

Worked example

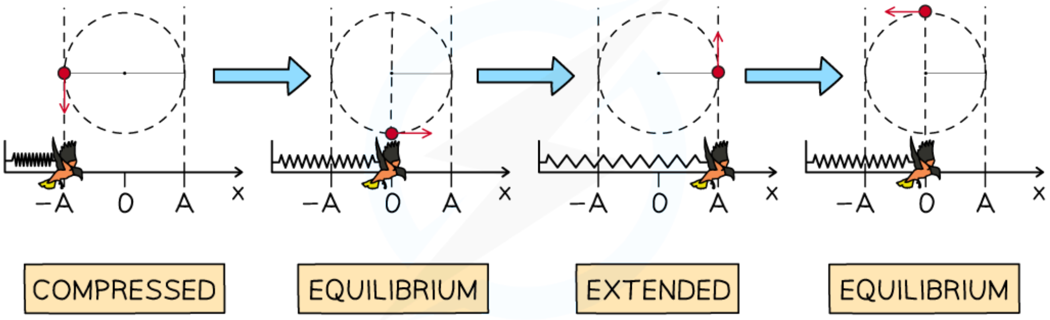
A cuckoo in a cuckoo clock emerges from a fully compressed position to a fully extended position in 1.5 seconds.

Calculate the angular frequency of the cuckoo as it emerges from the clock.

Answer:

Step 1: Consider the motion of the cuckoo

- The cuckoo goes from being fully compressed to fully extended which means that it travels for an **angular displacement** of half a circle and not a full circle
- So, the angular displacement will be π



Step 2: Substitute into the equation for angular velocity and time period

$$\omega = \frac{2\pi}{T} = \frac{\pi}{1.5} = 2.09 \text{ rad s}^{-1}$$

Step 3: State the final answer

- The angular frequency of the cuckoo as it emerges from the clock is 2.1 rad s^{-1} (2 s.f.)

Your notes

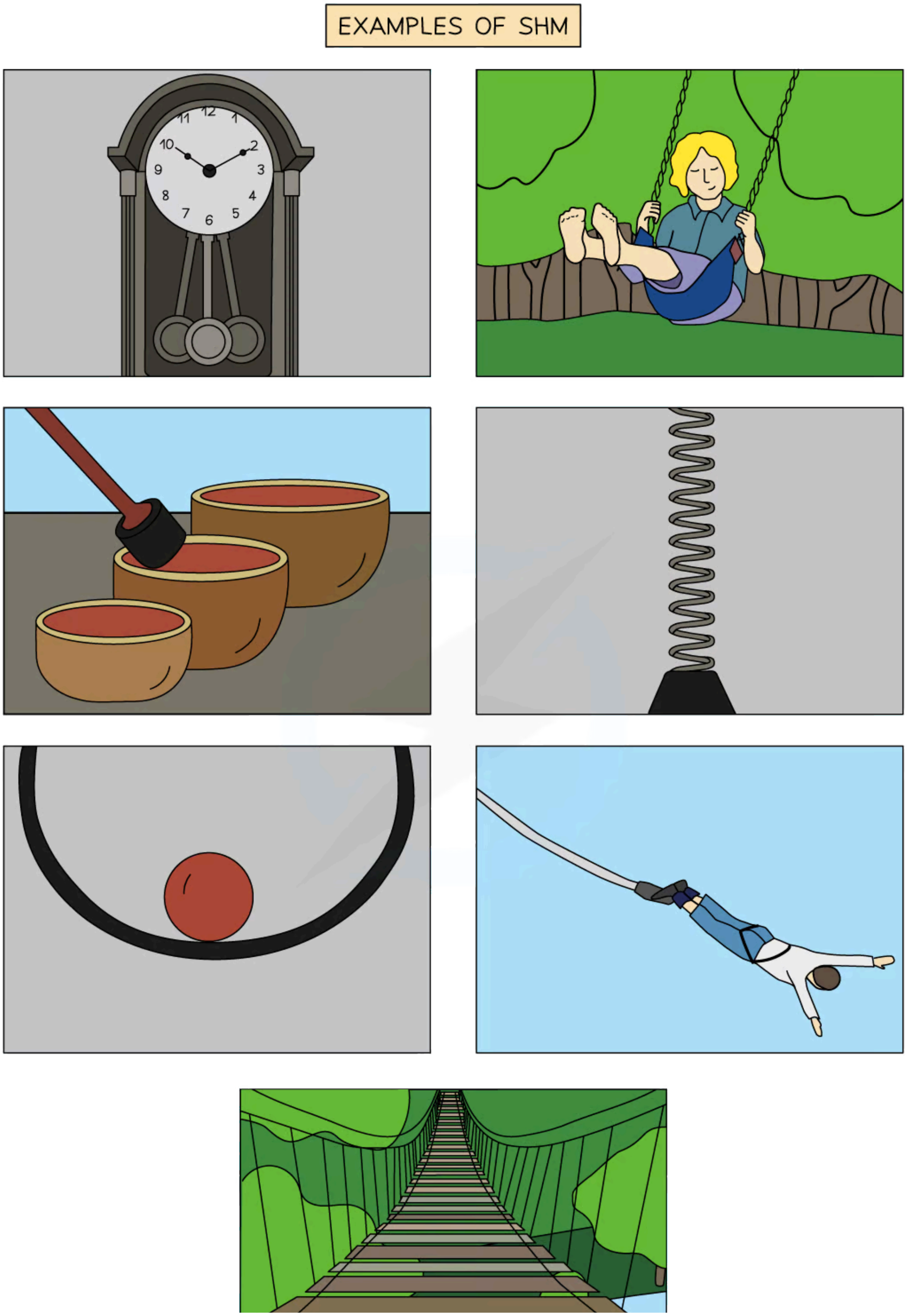
Simple Harmonic Motion (SHM)

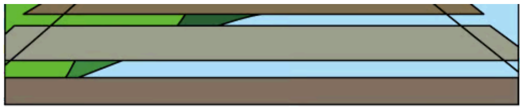
Conditions for Simple Harmonic Motion

- **Simple harmonic motion (SHM)** is a specific type of oscillation where:
  - There is **repetitive movement** back and forth through an equilibrium, or central, position, so the **maximum horizontal or vertical displacement** on one side of this position is equal to the maximum horizontal or vertical displacement on the other
  - The **time** interval of each complete vibration is the **same** (periodic)
  - The **force** responsible for the motion (**restoring force**) is always **directed horizontally or vertically towards** the **equilibrium** position and is directly proportional to the distance from it

Examples of SHM

- Examples of oscillators that undergo SHM are:
  - The pendulum of a clock
  - A child on a swing
  - The vibrations of a bowl
  - A bungee jumper reaching the bottom of his fall
  - A mass on a spring
  - Guitar strings vibrating
  - A ruler vibrating off the end of a table
  - The electrons in alternating current flowing through a wire
  - The movement of a swing bridge when someone crosses
  - A marble dropped into a bowl


EXAMPLES OF SHM



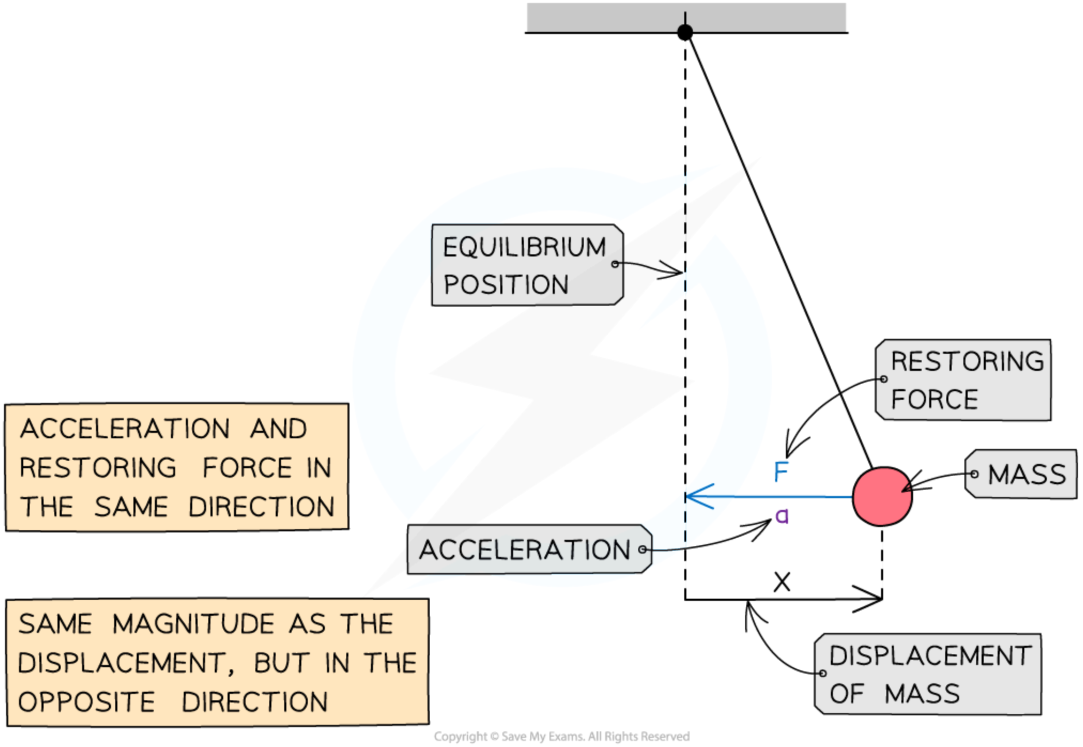
Examples of objects that undergo SHM

Modelling SHM

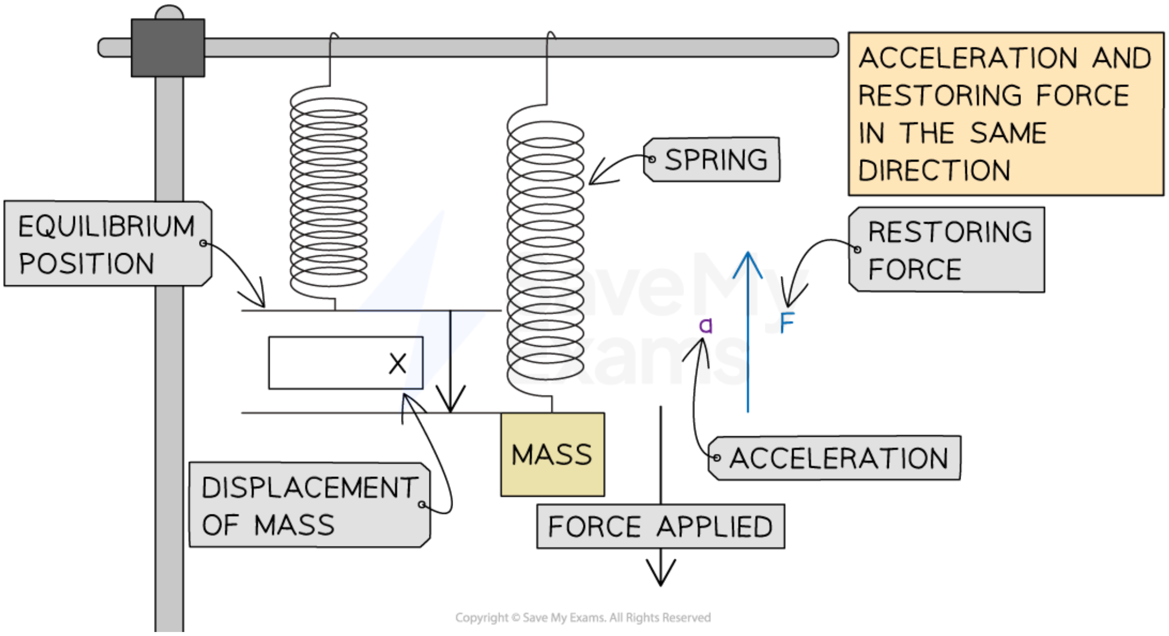
- Not all oscillations are as simple as SHM
    - This is a particularly simple kind
    - It is relatively easy to analyse mathematically
    - Many other types of oscillatory motion can be broken down into a combination of SHMs
- An oscillation is defined to be SHM when:
    - **The acceleration is proportional to the horizontal or vertical displacement**
    - **The acceleration is in the opposite direction to the displacement** (directed towards the equilibrium position)
- The time period of oscillation is independent of the amplitude of the oscillation, for small angles of oscillation
- So, for acceleration *a* and horizontal displacement *x*

$$a \propto -x$$

- You will be required to perform calculations on and explain **two models** of simple harmonic motion:
    - A **simple pendulum** oscillating from side to side attached to a fixed point above
    - A **mass-spring system** oscillating **vertically** up and down or **horizontally** back and forth



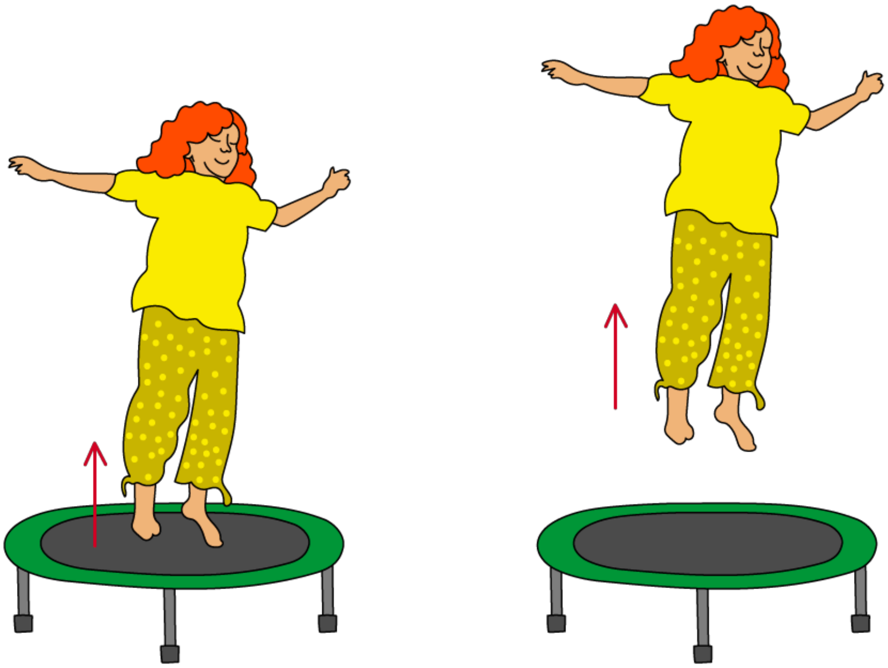
Force, acceleration and displacement of a simple pendulum in SHM



Force, acceleration and displacement of a mass-spring system in SHM

An Example of not SHM

- A person jumping on a trampoline is not an example of simple harmonic motion because:
    - The **restoring force** on the person is **not proportional** to their **displacement** from the equilibrium position and always acts down
    - When the person is **not in contact** with the trampoline, the restoring force is equal to their weight, which is constant
    - This **does not change**, even if they jump higher



The restoring force of the person bouncing is equal to their weight and always acts downwards

Worked example

Explain why a person jumping on a trampoline is **not** an example of simple harmonic motion.

Answer:

Step 1: Recall the conditions for simple harmonic motion

- The conditions required for SHM:
  - The restoring force/acceleration is **proportional** to the displacement
  - The restoring force/acceleration is in the **opposite direction** to the displacement

Step 2: Consider the forces in the scenario given

- When the person is not in contact with the trampoline, the restoring force is equal to their weight, which is constant
- The value of their weight does not change, even if they jump higher (increase displacement)

Step 3: Write a concluding sentence

- The restoring force on the person is not proportional to their distance from the equilibrium position, therefore, this scenario does not fulfil the conditions for SHM

The Defining Equation of Simple Harmonic Motion

- The acceleration of an object oscillating in simple harmonic motion is given by the equation:

$$a = -\omega^2 x$$

- Where:
  - a = acceleration (m s^{-2})
  - ω = angular frequency (rad s^{-1})
  - x = displacement (m)

- The equation demonstrates:
  - Acceleration reaches its **maximum** value when the displacement is at a **maximum,** i.e. $x = x_0$ at its amplitude
  - The **minus** sign shows that when the object is displaced to the **right**, the direction of the acceleration is to the **left** and vice versa (a and x are always in opposite directions to each other)

- Consider the oscillation of a **simple pendulum**:
  - The bob **accelerates** as it moves towards the **midpoint**
  - **Velocity** is at a **maximum** when it passes through the **equilibrium position**
  - The **pendulum slows down** as it continues towards the other **extreme of oscillation**
  - **$v = 0$** at **x_0** as it **changes direction**
  - The pendulum then **reverses** and starts to **accelerate again** towards the midpoin

Graphical Representation of SHM

- The displacement, velocity and acceleration of an object in simple harmonic motion can be represented by graphs against time
- All undamped SHM graphs are represented by **periodic functions**
  - This means they can all be described by sine and cosine curves

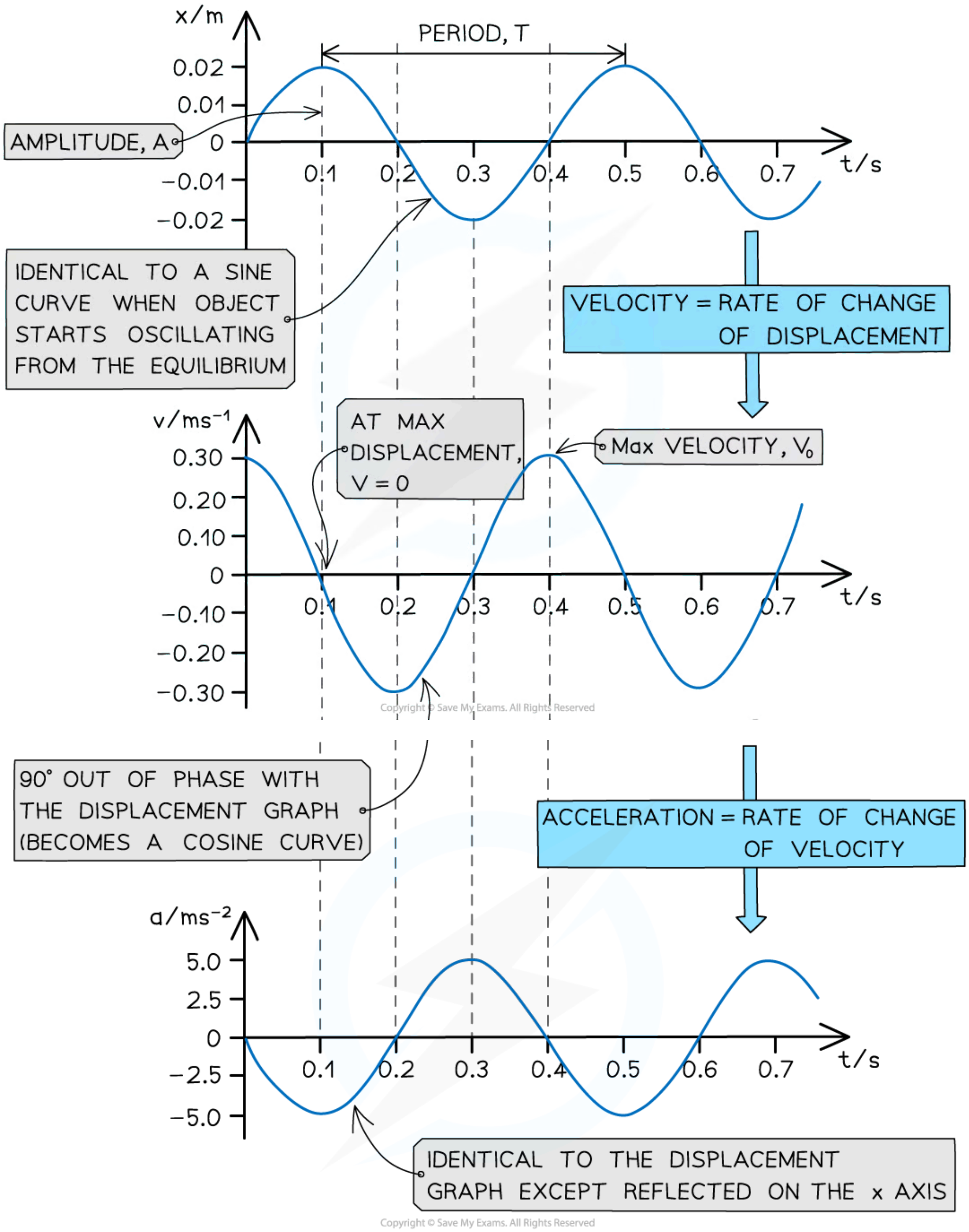
- You need to know what each graph looks like and how it relates to the other graphs
- Remember that:

  - Velocity is the rate of change of displacement $v = \frac{s}{t}$
  - Acceleration is the rate of change of velocity $a = \frac{\Delta v}{t}$

Graphs that Start at the Equilibrium Position

- When oscillations start from the **equilibrium position,** then:
  - The **displacement-time** graph is a **sine curve**
  - The **velocity-time** graph is the **gradient** of the **displacement-time** graph, so a **cosine** graph and 90° out of phase with the displacement-time graph
  - The **acceleration-time** graph is the **gradient** of the **velocity-time graph**, so a **negative sine** graph and 90° out of phase with the velocity-time graph

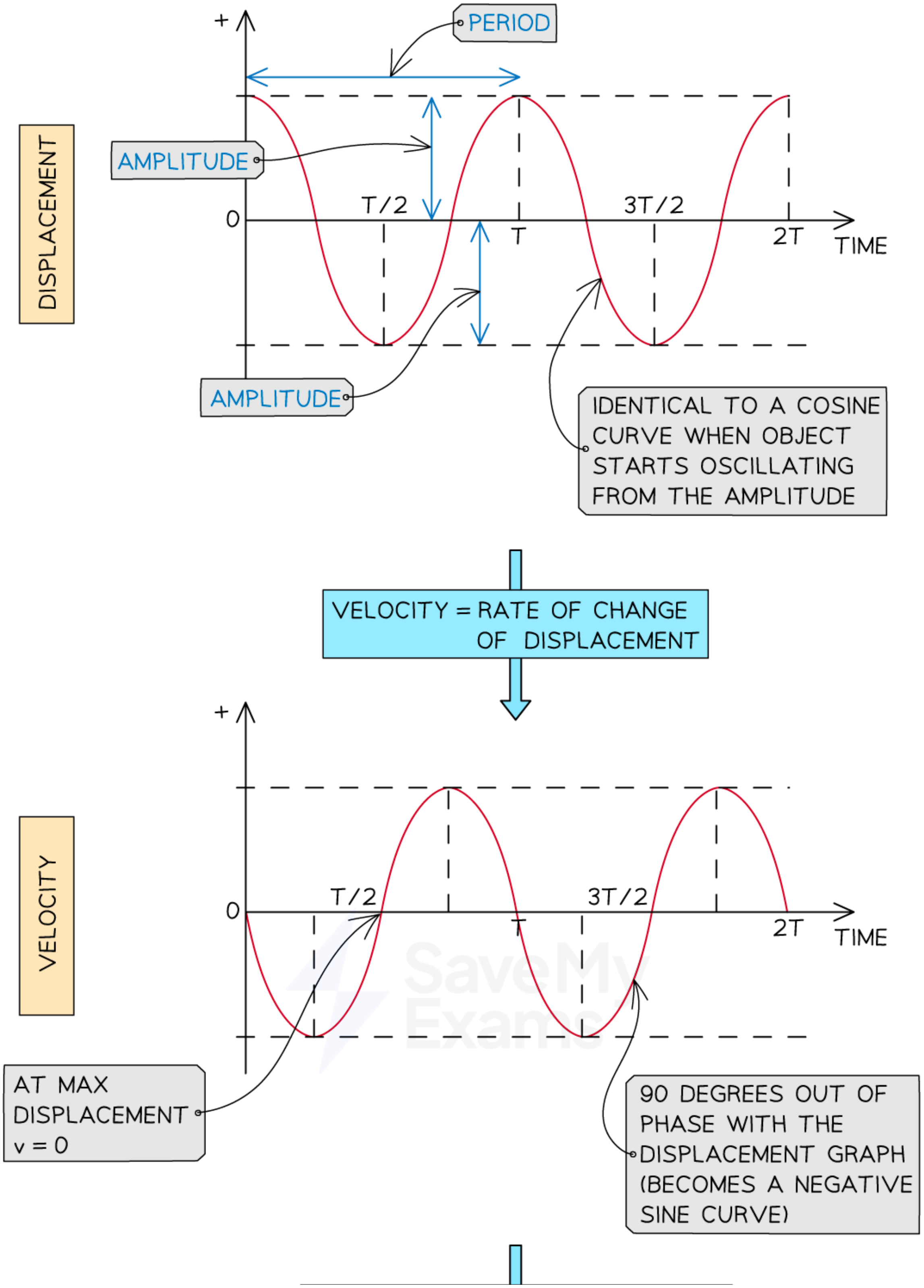
- More information on this can be found in the IB DP Maths Differentiating Special Functions on trigonometric differentiation

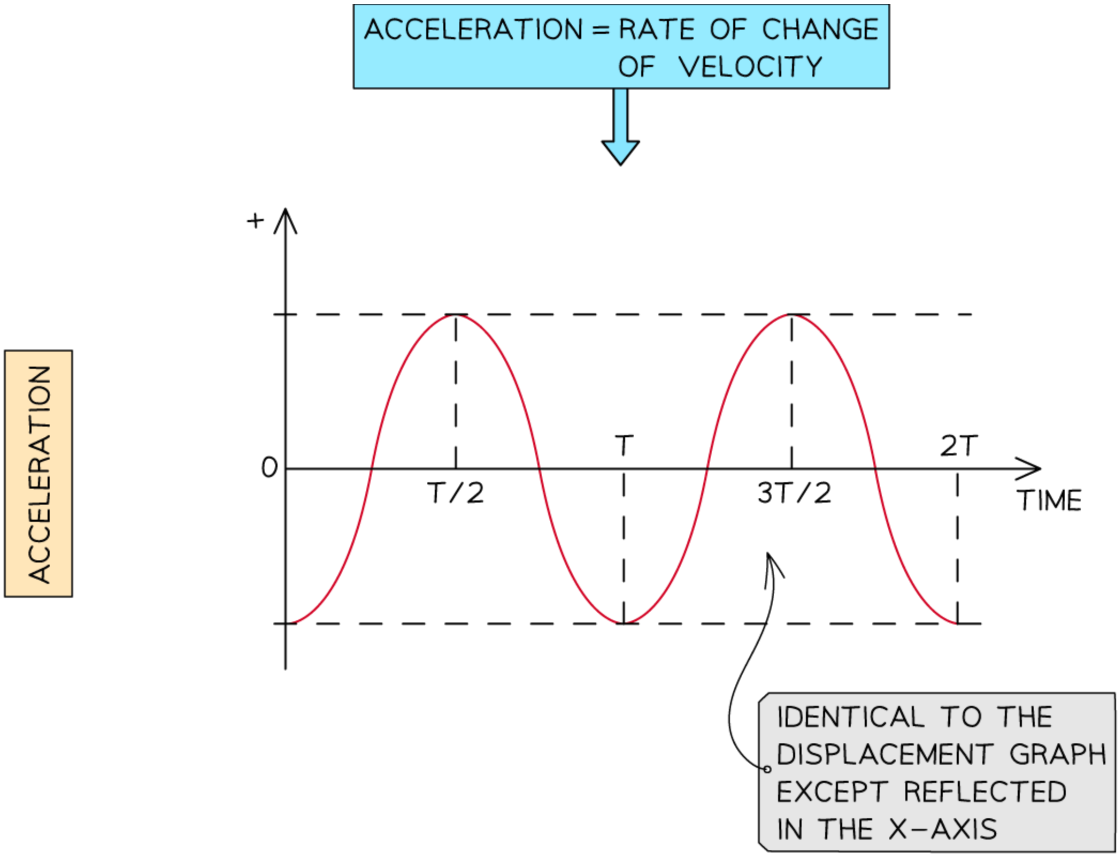


The displacement, velocity and acceleration graphs in SHM are all 90° out of phase with each other

Graphs that Start at the Amplitude Position

- When oscillations start from the **amplitude position,** then:
    - The **displacement-time** graph is a **cosine curve**
    - The **velocity-time** graph is the **gradient** of the **displacement-time** graph, so a **negative sine** graph and 90° out of phase with the displacement-time graph
    - The **acceleration-time** graph is the **gradient** of the **velocity-time graph**, so a **negative cosine** graph and 90° out of phase with the velocity-time graph


DISPLACEMENT
+
PERIOD
AMPLITUDE
0
T/2
T
3T/2
2T
TIME
AMPLITUDE
IDENTICAL TO A COSINE CURVE WHEN OBJECT STARTS OSCILLATING FROM THE AMPLITUDE
VELOCITY = RATE OF CHANGE OF DISPLACEMENT
VELOCITY
+
0
T/2
T
3T/2
2T
TIME
AT MAX DISPLACEMENT v = 0
90 DEGREES OUT OF PHASE WITH THE DISPLACEMENT GRAPH (BECOMES A NEGATIVE SINE CURVE)



The displacement, velocity and acceleration graphs in SHM are all 90° out of phase with each other

Relationship Between Graphs

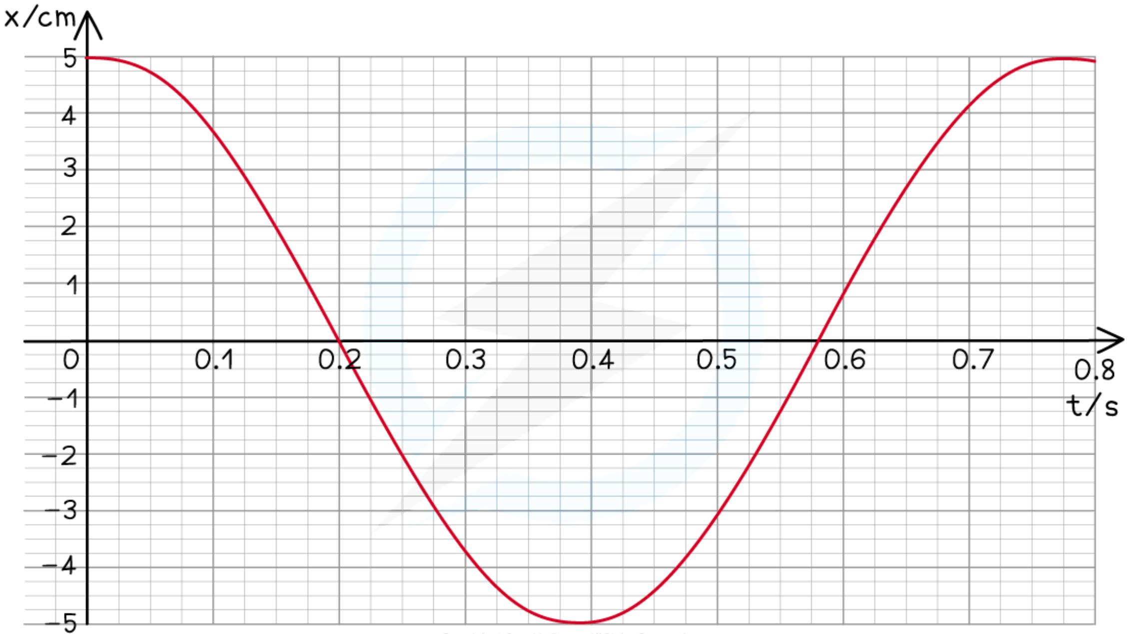
- **Key features of the displacement-time graphs:**
  - The amplitude of oscillations A is the maximum value of x
  - The time period of oscillations T is the time taken for one full wavelength cycle
- **Key features of the velocity-time graphs:**
  - The velocity of an oscillator at any time can be determined from the **gradient of the displacement-time graph:**

$$v = \frac{\Delta x}{\Delta t}$$

- **Key features of the acceleration-time graph:**
  - The acceleration graph is a reflection of the displacement graph on the x-axis
  - This means when a mass has positive displacement (to the right), the acceleration is in the opposite direction (to the left) and vice versa (from $a = -\omega^2 x$)
  - The acceleration of an oscillator at any time can be determined from the **gradient of the velocity-time graph:**

$$a = \frac{\Delta v}{\Delta t}$$

Worked example

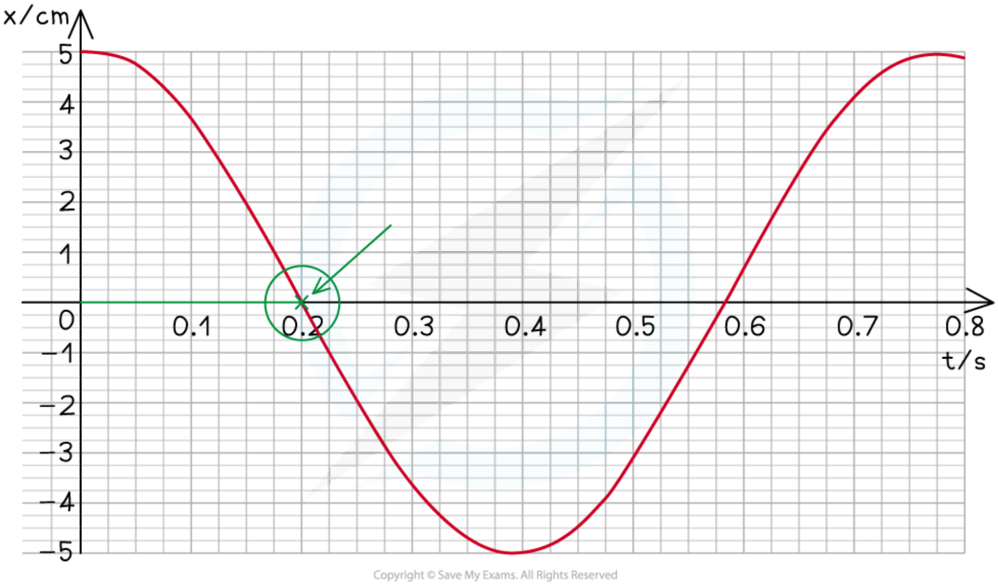


The swing exhibits simple harmonic motion.

Use data from the graph to determine at what time the velocity of the swing is first at its maximum.

Step 1: The velocity is at its maximum when the displacement $x = 0$

Step 2: Reading value of time when $x = 0$



From the graph, this is equal to 0.2 s

Examiner Tip

The defining equation of SHM shows acceleration, as a positive value, and displacement, −*x* as a negative one. This reminds us that acceleration and displacement are **vector** quantities and are always in the opposite direction to each other in SHM.

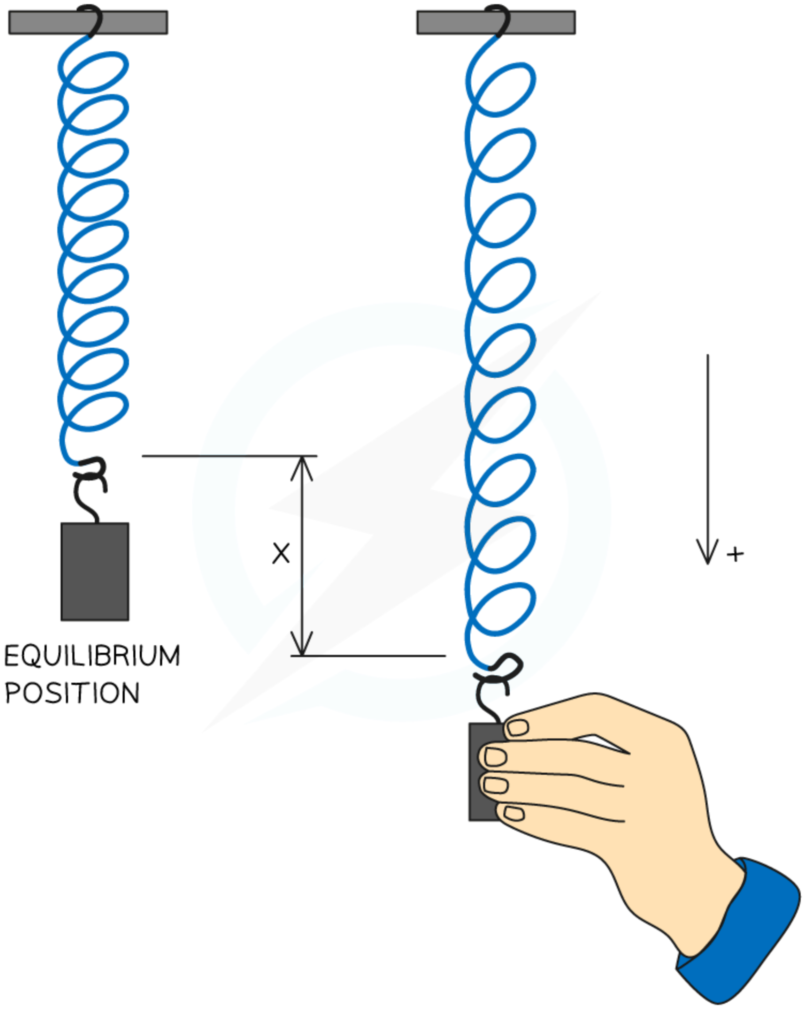
Since displacement is a vector quantity, remember to keep the minus sign in your solutions if they are negative. Getting the marks will depend on keeping your positive and negative numbers distinct from each other! Also remember that your calculator must be in **radians** mode when using the cosine and sine functions. This is because the angular frequency ω is calculated in rad s^{-1}, **not** degrees.

These graphs might not look identical to what is in your textbook, because they depend on the starting position of the oscillation of the object when $t = 0$. If there is no damping, they will be a sine or cosine curve.

Time Period of a Mass–Spring System

Time Period of a Mass–Spring System

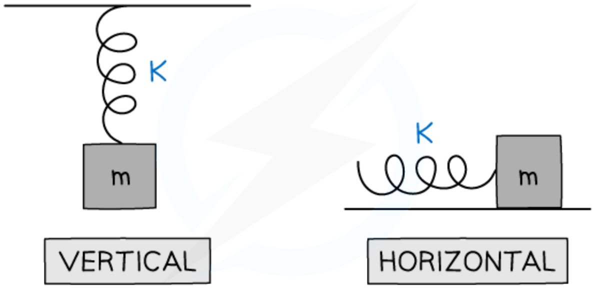
- A **mass-spring** system consists of a mass attached to the end of a spring
- The equation for the **restoring force** (the force responsible for the SHM) is $F_H = -kx$
    - This is the same as the equation for **Hooke's Law**



- The time period of a mass-spring system is given by:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

- Where:
  - T = time period (s)
  - m = mass on the end of the spring (kg)
  - k = spring constant (N m^{-1})
- This equation applies to both horizontal and vertical mass-spring systems:



A mass-spring system can be either vertical or horizontal. The time period equation applies to both

- The equation shows that the time period and frequency, of a mass-spring system, does **not** depend on the force of gravity
  - Therefore, the oscillations would have the same time period on Earth and the Moon
- The higher the spring constant *k*, the stiffer the spring and the shorter the time period of the oscillation

Worked example

A 200 g toy robot is attached to a pole by a spring, which has a spring constant of 90 N m^{-1} and made to oscillate horizontally.

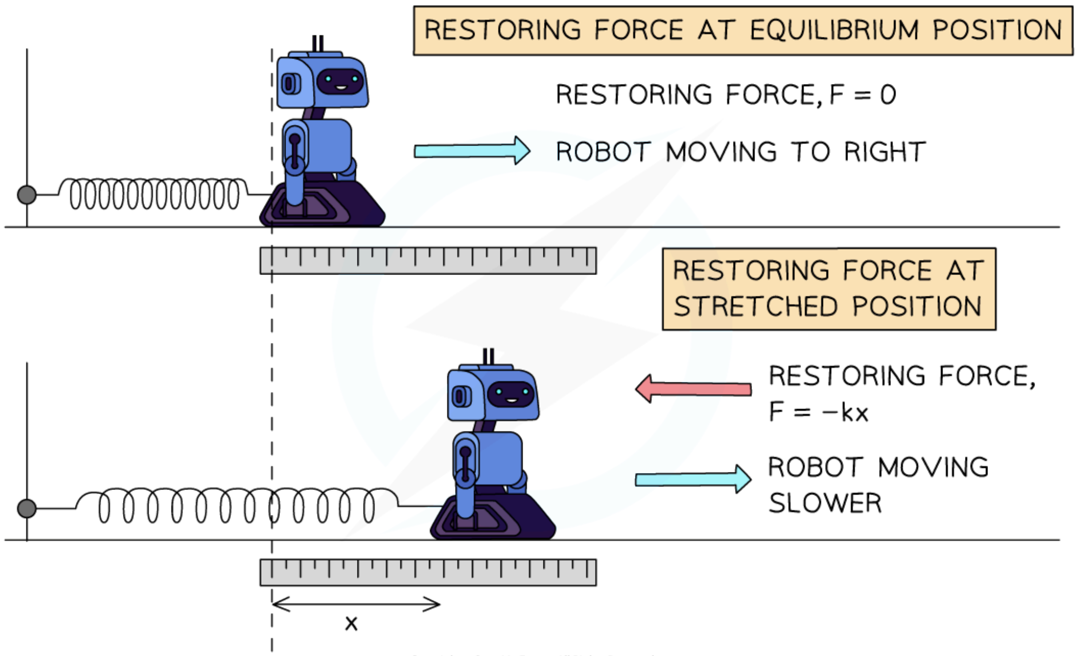
Calculate:

(a) The force that acts on the robot when the spring is extended by 5 cm.
(b) The acceleration of the robot whilst at its amplitude position.

Answer:

(a)

- Consider the motion of the robot at the equilibrium and stretched (amplitude) positions:



- The restoring force is given by

$$F = -kx$$

- Using:
  - Extension, x = 5 cm = 0.05 m
  - Spring constant, k = 90 N m^{-1}

$$F = -90 \times 0.05 = -4.5 \text{ N}$$

- A force of 4.5 N will act on the robot, trying to pull it back towards the equilibrium position.

(b)

- Newton's second law relates force and acceleration by

$$F = ma$$

- Using:
  - Mass, m = 200 g = 0.2 kg

$$a = \frac{F}{m} = \frac{-4.5}{0.2} = -22.5 \text{ m s}^{-2}$$

- The robot will decelerate at a rate of 22.5 m s^{-2} when at this amplitude position

Worked example

Calculate the frequency of a mass of 2.0 kg attached to a spring with a spring constant of 0.9 N m^{-1} oscillating with simple harmonic motion.

Answer:

Step 1: Write down the known quantities

- Mass, m = 2.0 kg
- Spring constant, k = 0.9 N m^{-1}

Step 2: Write down the equation for the time period of a mass-spring system

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Step 3: Combine with the equation relating time period T and frequency, f

$$T = \frac{1}{f}$$

$$\frac{1}{f} = 2\pi\sqrt{\frac{m}{k}} \Rightarrow f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Step 4: Substitute in the values to calculate frequency

$$f = \frac{1}{2\pi}\sqrt{\frac{0.9}{2}}$$

Frequency: f = 0.11 Hz

Examiner Tip

Another area of physics where you may have seen the spring constant *k* is from Hooke's Law. Exam questions commonly merge these two topics together, so make sure you're familiar with the Hooke's Law equation too.

In the second worked example, the frequency calculated is the natural frequency of the mass-spring system, a concept that comes up in the topic of **resonance**.

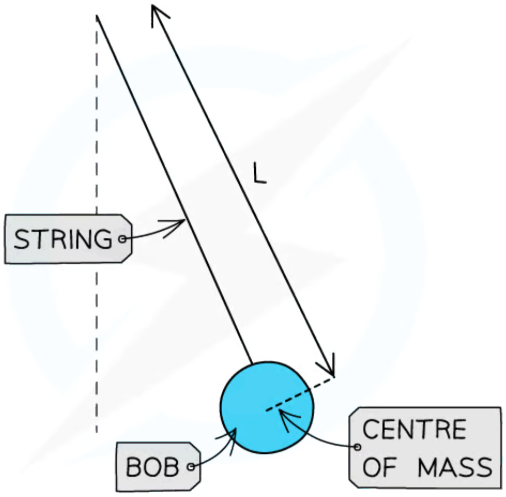
Time Period of a Simple Pendulum

Time Period of a Simple Pendulum

- A simple pendulum consists of a **string** and a **bob** at the end
  - The **bob** is a weight, generally spherical and considered a point mass
  - The bob moves from **side to side**
  - The string is **light and inextensible** remaining in tension throughout the oscillations
  - The string is attached to a **fixed point** above the equilibrium position
- The **time period** of a simple pendulum for small angles of oscillation is given by:

$$T = 2\pi\sqrt{\frac{L}{g}}$$

- Where:
  - T = time period (s)
  - L = length of string (from the pivot to the centre of mass of the bob) (m)
  - g = gravitational field strength (N kg^{-1})



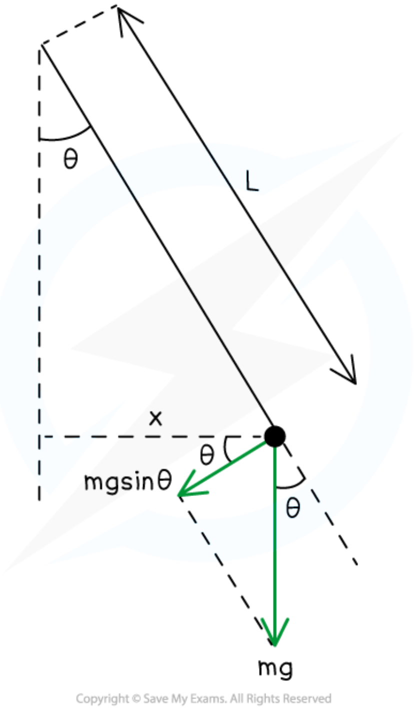
A simple pendulum

- The time period of a **pendulum** does depend on the **gravitational field strength**, meaning its period would be different on the Earth and the Moon

Small Angle Approximation

- This formula for time period is limited to **small angles** ($\theta < 10°$) and therefore **small amplitudes** of oscillation from the equilibrium point

- The **restoring force** of the pendulum is the **weight component** acting along the **arc** of the circle towards the equilibrium position
- It is resolved to act at an angle θ to the horizontal x
- When considering SHM because of small angle approximation it is assumed the **restoring force** acts along the **horizontal**
- So **sin $\theta \cong \theta$**



Forces on a pendulum when it is displaced. Assuming θ < 10°, the small angle approximation can be used to describe the time period of a simple pendulum such as this.

Your notes

Worked example

A swinging pendulum with a length of 80.0 cm has a maximum angle of displacement of 8°.

Determine the angular frequency of the oscillation.

Answer:

Step 1: List the known quantities

- Length of the pendulum, $L = 80 \text{ cm} = 0.8 \text{ m}$
- Acceleration due to gravity, $g = 9.81 \text{ m s}^{-2}$

Step 2: Write down the relationship between angular frequency, *ω*, and period, *T*

$$T = \frac{2\pi}{\omega}$$

Step 3: Write down the equation for the time period of a simple pendulum

$$T = 2\pi\sqrt{\frac{L}{g}}$$

- This equation is valid for this scenario since the maximum angle of displacement is less than 10°

Step 4: Equate the two equations and rearrange for *ω*

$$\frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}} \quad \Rightarrow \quad \omega = \sqrt{\frac{g}{L}}$$

Step 5: Substitute the values to calculate *ω*

$$\omega = \sqrt{\frac{9.81}{0.8}} = 3.50 \text{ rad s}^{-1}$$

Angular frequency: $\omega = 3.5 \text{ rad s}^{-1}$

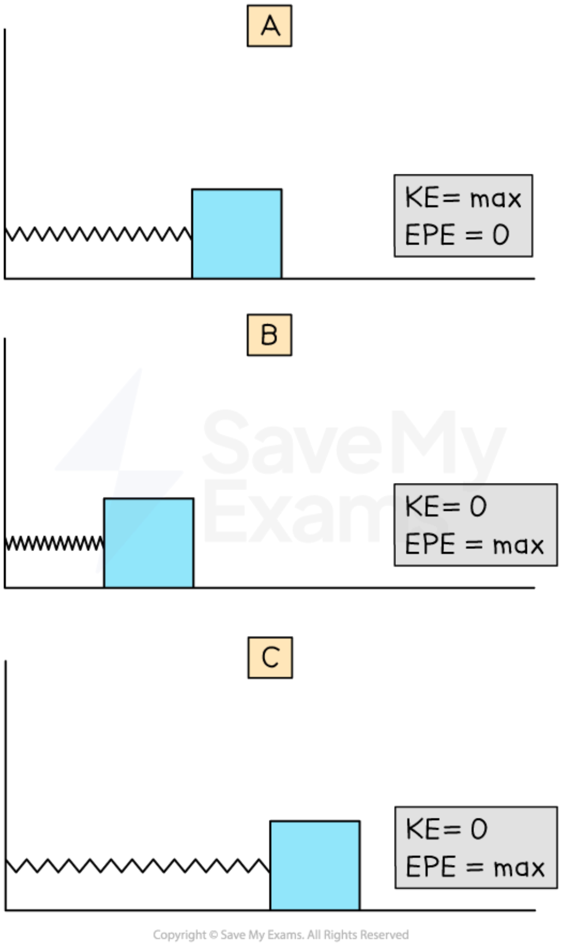
Energy Changes in Simple Harmonic Motion (SHM)

Energy Changes in Simple Harmonic Motion

- Simple harmonic motion also involves an **interplay** between different types of **energy**: potential and kinetic
  - The swinging of a **pendulum** is an interplay between **gravitational potential energy** and **kinetic energy**
  - The **horizontal** oscillation of a **mass on a spring** is an interplay between **elastic potential energy** and **kinetic energy**

Energy of a Horizontal Mass-Spring System

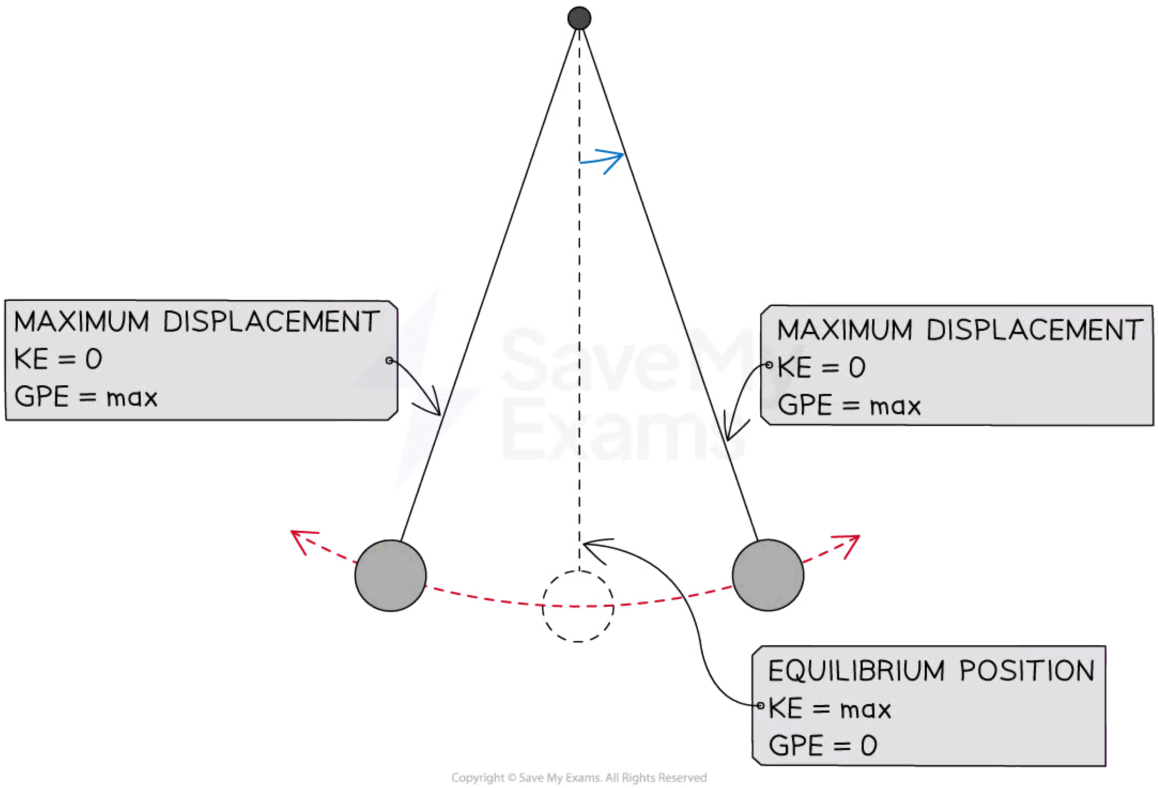
- The system has the maximum amount of **elastic potential energy** when held so the string is stretched beyond its equilibrium position
- When the **mass is released,** it moves back towards the equilibrium position, accelerating as it goes so the **kinetic energy increases**
- At the equilibrium position, **kinetic energy** is at its **maximum** and **elastic potential energy** is at its **minimum**
- Once **past the equilibrium** position, the **kinetic energy decreases** and elastic **potential energy increases**



In a horizontal mass-spring system the kinetic energy is maximum in the equilibrium position and the elastic potential energy is maximum in the amplitude position

Energy of a Simple-Pendulum

- At the **amplitude** at the top of the swing, the pendulum has a **maximum** amount of **gravitational potential energy**
- When the pendulum is **released**, it moves back towards the equilibrium position, **accelerating** as it goes so the **kinetic energy increases**
- As the **height** of the pendulum **decreases,** the **gravitational potential energy** also **decreases**
- Once the mass has passed the equilibrium position, **kinetic energy decreases** and **gravitational potential energy increases**



In a simple pendulum system the kinetic energy is maximum in the equilibrium position and the gravitational potential energy is maximum in the amplitude position

Total Energy of an SHM System

- The **total energy** in the system remains constant, but the amount of energy in **one form goes up** while the amount in the **other form goes down**
  - This constant total energy shows how energy in a **closed system** is never created or destroyed; it is transferred from one store to another
  - This is the **law of conservation of energy**

The total energy of a simple harmonic system always remains constant and is equal to the sum of the kinetic and potential energy

- The **total energy** is calculated using the equation:

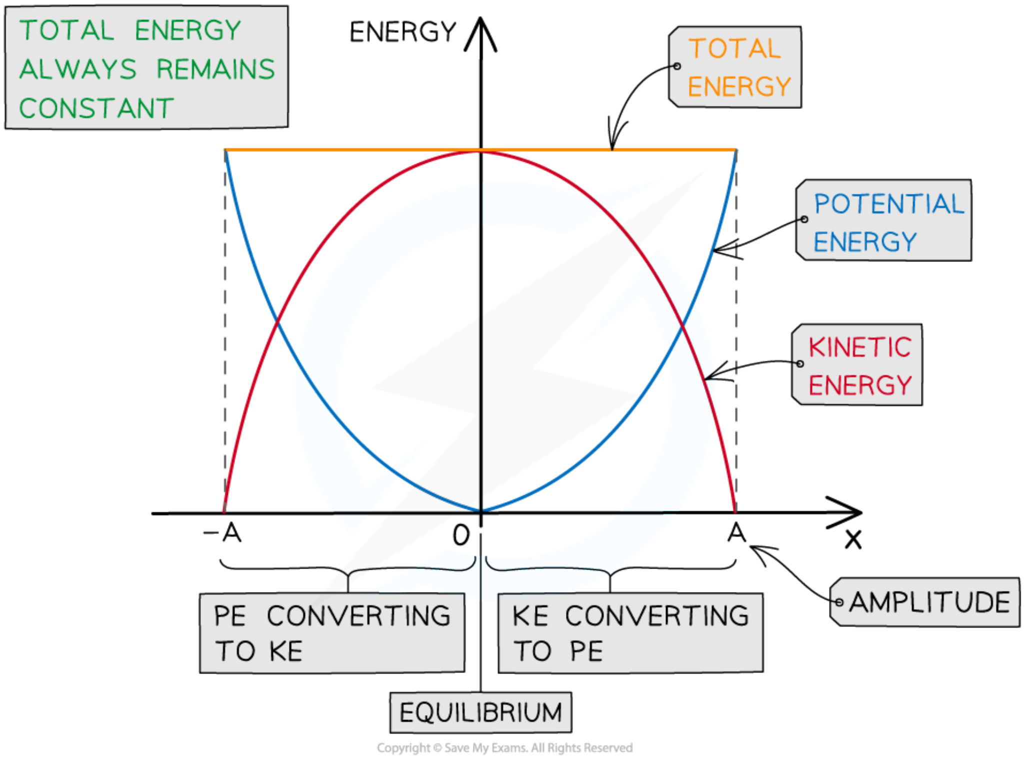
$$E = E_P + E_K$$

- Where:
  - E = total energy in joules (J)
  - E_P = potential energy in joules (J)
  - E_K = kinetic energy in joules (J)

- Remember the equations for potential and kinetic energy:
    - Gravitational potential energy: $E_p = mgh$
    - Elastic potential energy, $E_p = \frac{1}{2}kx^2$
    - Kinetic energy, $E_k = \frac{1}{2}mv^2$

Energy-Displacement Graph

- The **kinetic** and **potential energy transfers** go through **two** complete cycles during one **period** of oscillation
    - One complete oscillation reaches the maximum displacement **twice** (on both the positive and negative sides of the equilibrium position)
- You need to be familiar with the graph showing the total, potential and kinetic energy transfers in **half an SHM oscillation** (half a cycle)



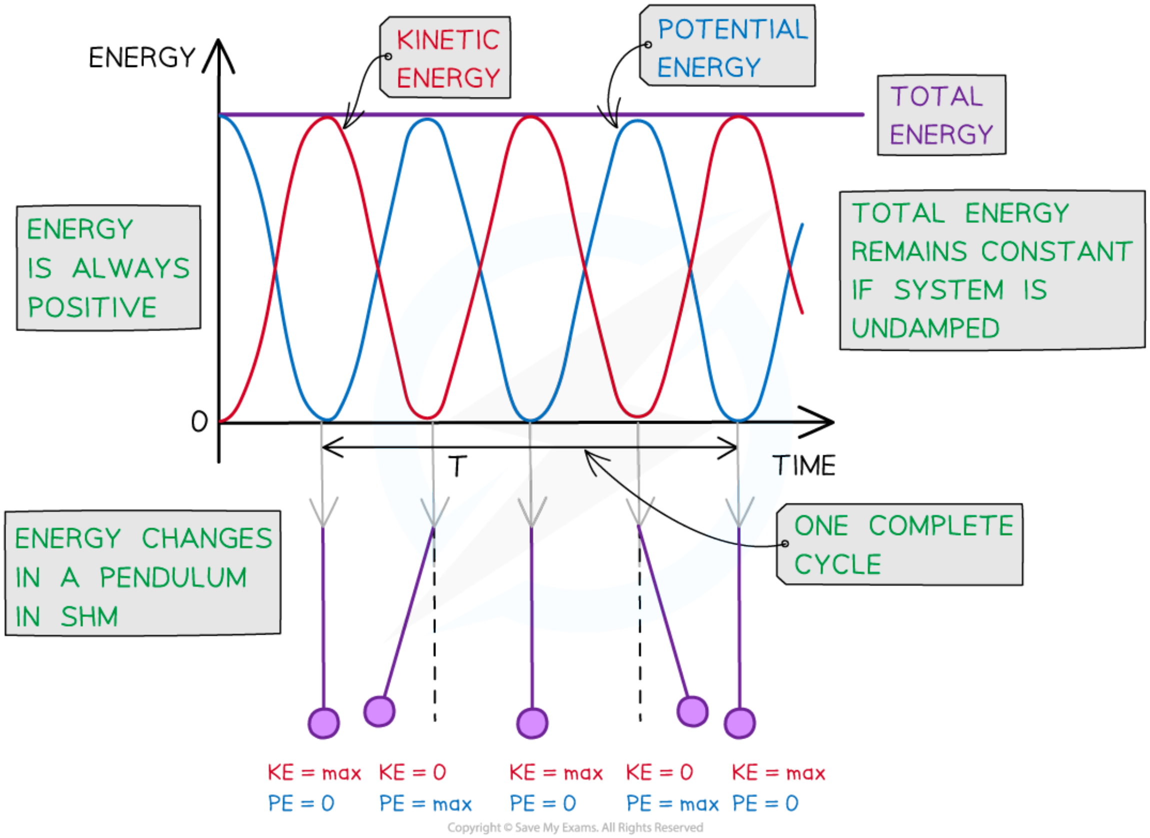
Graph showing the potential and kinetic energy against displacement in half a period of an SHM oscillation

- **The key features of the energy-displacement graph for half a period of oscillation are:**
    - Displacement is a vector, so, the graph has both **positive** and **negative** x values
    - The potential energy is always maximum at the amplitude positions $x = x_0$, and 0 at the equilibrium position $x = 0$

  - This is represented by a **'U' shaped curve**
- The kinetic energy is the opposite: it is 0 at the amplitude positions $x = x_0$, and maximum at the equilibrium position $x = 0$
  - This is represented by an **'n' shaped curve**
- The total energy is represented by a **horizontal straight line** above the curves

Energy-Time Graph for a Simple Pendulum

- You also need to be familiar with the graph showing the total, **gravitational potential** and **kinetic energy** transfers against time for **multiple cycles** of a **simple pendulum** oscillating in **simple harmonic motion**



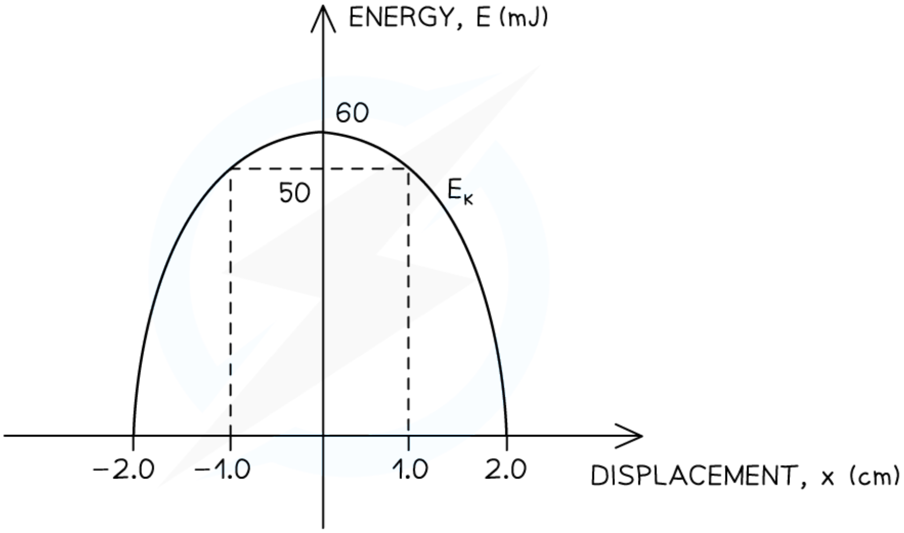
The kinetic and gravitational potential energy of a simple pendulum oscillating in SHM vary periodically

- **The key features of the simple pendulum energy-time graph are**:
  - Both the **kinetic and gravitational potential energy transfers** are represented by periodic functions (sine or cosine) which vary in opposite directions to one another
  - When the gravitational potential energy is 0, the kinetic energy is at its maximum and vice versa
  - The **total energy** is represented by a **horizontal straight line** directly above the energy curves at the **maximum** kinetic and gravitational potential energy value

- Energy is **always positive** so there are no negative values on the y-axis (Any SHM energy graph drawn with negative energy values is incorrect)

Worked example

The following graph shows the variation with displacement of the kinetic energy of an object of mass 0.50 kg oscillating with simple harmonic motion. Energy losses can be neglected.



Determine:

(a) The total energy of the object

(b) The amplitude of the oscillations

(c) The maximum velocity of the object

(d) The potential energy of the object when the displacement is *x* = 1.0 cm

Answer:

(a)

- From the graph, the maximum value of kinetic energy is 60 mJ
- At the equilibrium position $(x = 0)$, the total energy *E* is exactly equal to the maximum value of kinetic energy
- Since energy losses can be neglected, the total energy is constant

Total energy: *E* = 60 mJ

(b)

- The amplitude is equal to the maximum displacement on either side of the equilibrium position (where the kinetic energy is zero)

Your notes

Amplitude: x_0 = 2.0 cm

(c)

- The maximum velocity can be found using the maximum kinetic energy in the equation:

$$E_k = \frac{1}{2}mv^2 \quad \Rightarrow \quad v = \sqrt{\frac{2E_k}{m}}$$

- Using:
  - Mass of the object, m = 0.50 kg
  - Maximum kinetic energy, E_K = 60 mJ = 0.06 J

$$v = \sqrt{\frac{2 \times 0.06}{0.50}}$$

Maximum velocity: v = 0.49 m s^{-1}

(d)

- From the graph, when the displacement is x = 1.0 cm, kinetic energy is E_K = 50 mJ
- The relationship between total energy E, kinetic energy E_K and potential energy E_P is:

$$E = E_P + E_K$$

- Therefore, the potential energy is

$$E_P = E - E_K$$

$$E_P = 60 - 50 = 10 \text{ mJ}$$

Examiner Tip

You may be expected to draw as well as interpret energy graphs against time or displacement in exam questions. Make sure the sketches of the curves are as even as possible and **use a ruler** to draw straight lines, for example, to represent the total energy.