DP IB Maths: AI HL

Your notes

3.4 Further Trigonometry

Contents

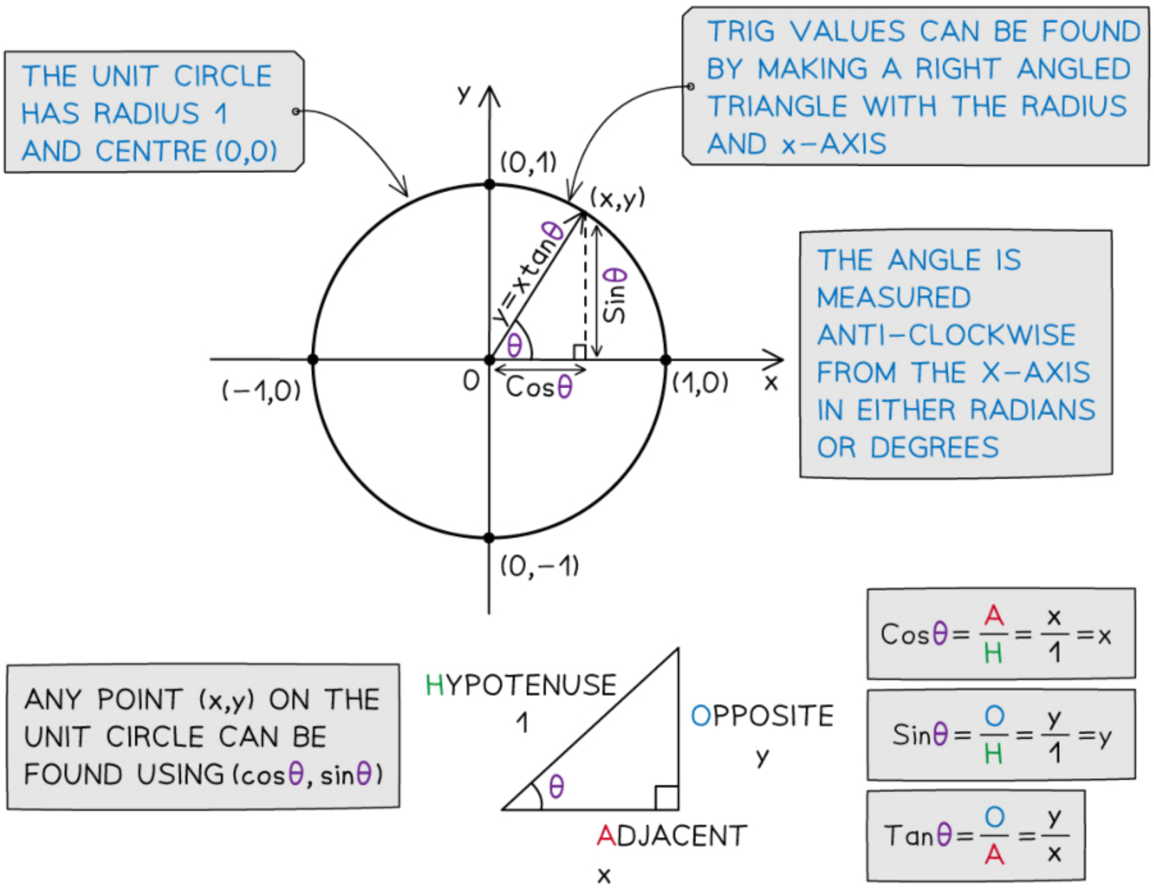
3.4.1 The Unit Circle

Your notes

Defining Sin, Cos and Tan

What is the unit circle?

- The unit circle is a circle with radius 1 and centre (0, 0)
- Angles are always measured from the positive x-axis and turn:
  - **anticlockwise** for **positive** angles
  - **clockwise** for **negative** angles
- It can be used to calculate trig values as a coordinate point (x, y) on the circle
  - Trig values can be found by making a right triangle with the radius as the hypotenuse
  - θ is the angle measured anticlockwise from the positive *x*-axis
  - The *x*-axis will always be adjacent to the angle, θ
- SOHCAHTOA can be used to find the values of sinθ, cosθ and tanθ easily
- As the radius is 1 unit
  - the ***x* coordinate** gives the value of **cosθ**
  - the ***y* coordinate** gives the value of **sinθ**
- As the origin is one of the end points - dividing the y coordinate by the x coordinate gives the gradient
  - the **gradient** of the line gives the value of **tanθ**
- It allows us to calculate sin, cos and tan for angles greater than 90° ($\frac{\pi}{2}$ rad)



How is the unit circle used to construct the graphs of sine and cosine?

- On the unit circle the ***y*-coordinates** give the value of **sine**
  - Plot the *y*-coordinate from the unit circle as the *y*-coordinate on a trig graph for *x*-coordinates of $\theta = 0, \pi/2, \pi, 3\pi/2$ and 2π
  - Join these points up using a smooth curve
    - To get a clearer idea of the shape of the curve the points for *x*-coordinates of $\theta = \pi/4, 3\pi/4, 5\pi/4$ and $7\pi/4$ could also be plotted

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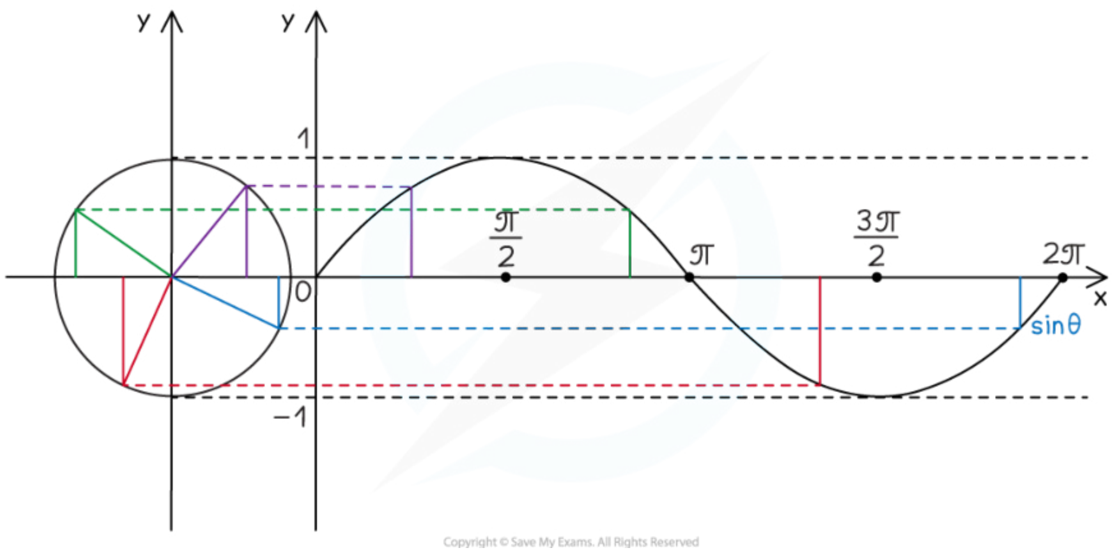
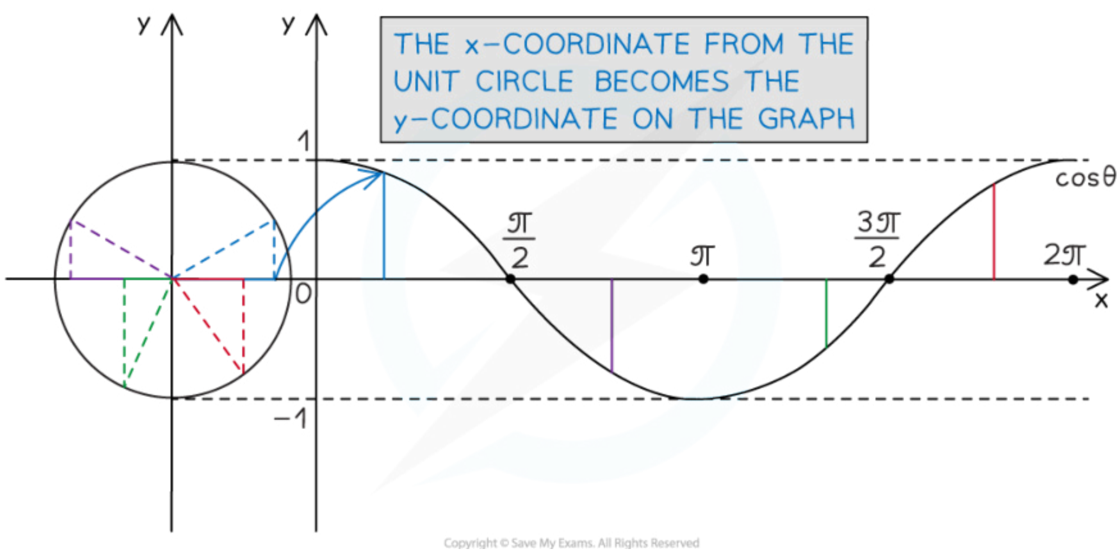


- On the unit circle the ***x*-coordinates** give the value of **cosine**
  - Plot the *x*-coordinate from the unit circle as the *y*-coordinate on a trig graph for *x*-coordinates of θ = 0, π/4, π/2, 3π/4 and 2π
  - Join these points up using a smooth curve
    - To get a clearer idea of the shape of the curve the points for *x*-coordinates of θ = π/4, 3π/4, 5π/4 and 7π/4 could also be plotted



- Looking at the unit circle alongside of the sine or cosine graph will help to visualise this clearer

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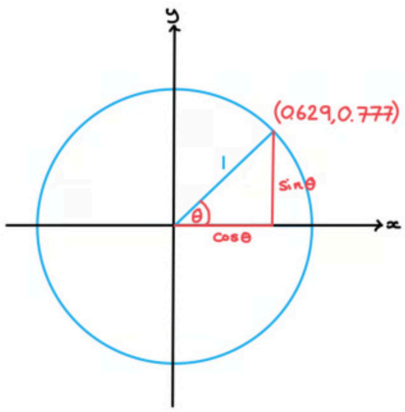
Worked example

The coordinates of a point on a unit circle, to 3 significant figures, are (0.629, 0.777). Find θ° to the nearest degree.



We know $(x, y) = (\cos\theta, \sin\theta)$

so,

$\cos\theta = 0.629$

$\sin\theta = 0.777$

Using either ratio:

$\theta = \cos^{-1}(0.629)$

$= 51.023...$

$\theta = 51°$ (nearest degree)

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Using The Unit Circle

What are the properties of the unit circle?

- The unit circle can be split into four **quadrants** at every 90° ($\frac{\pi}{2}$ rad)
  - The first quadrant is for angles between 0 and 90°
    - **A**ll three of Sinθ, Cosθ and Tanθ are positive in this quadrant
  - The second quadrant is for angles between 90° and 180° ($\frac{\pi}{2}$ rad and π rad)
    - **S**inθ is positive in this quadrant
  - The third quadrant is for angles between 180° and 270° (π rad and $\frac{3\pi}{2}$)
    - **T**anθ is positive in this quadrant
  - The fourth quadrant is for angles between 270° and 360° ($\frac{3\pi}{2}$ rad and 2π)
    - **C**osθ is positive in this quadrant
  - Starting from the **fourth** quadrant (on the bottom right) and working anti-clockwise the positive trig functions spell out **CAST**
    - This is why it is often thought of as the **CAST** diagram
    - You may have your own way of remembering this
    - A popular one starting from the first quadrant is **All Students Take Calculus**
  - To help picture this better try sketching all three trig graphs on one set of axes and look at which graphs are positive in each 90° section

How is the unit circle used to find secondary solutions?

- Trigonometric functions have more than one input to each output
  - For example sin 30° = sin 150° = 0.5
  - This means that trigonometric equations have more than one solution
  - For example both 30° and 150° satisfy the equation sin x = 0.5
- The unit circle can be used to find all solutions to trigonometric equations in a given interval
  - Your calculator will only give you the first solution to a problem such as $x = \sin^{-1}(0.5)$
    - This solution is called the **primary value**
  - However, due to the **periodic** nature of the trig functions there could be an infinite number of solutions
    - Further solutions are called the **secondary values**
  - This is why you will be given a **domain** in which your solutions should be found
    - This could either be in degrees or in radians
    - If you see π or some multiple of π then you must work in radians
- The following steps may help you use the unit circle to find **secondary values**

STEP 1: Draw the angle into the first quadrant using the x or y coordinate to help you

- If you are working with sin $x = k$, draw the line from the origin to the circumference of the circle at the point where the **y coordinate** is k

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- If you are working with cos $x = k$, draw the line from the origin to the circumference of the circle at the point where the **x coordinate** is k
- If you are working with tan $x = k$, draw the line from the origin to the circumference of the circle such that the gradient of the line is k
  - Note that whilst this method works for tan, it is complicated and generally unnecessary, tan x repeats every 180° (π radians) so the quickest method is just to add or subtract multiples of 180° to the primary value
- This will give you the angle which should be measured from the **positive x-axis…**
    - … anticlockwise for a positive angle
    - … clockwise for a negative angle

STEP 2: Draw the radius in the other quadrant which has the same...

- ... x-coordinate if solving cos $x = k$
  - This will be the quadrant which is vertical to the original quadrant
- ... y-coordinate if solving sin $x = k$
  - This will be the quadrant which is horizontal to the original quadrant
- ... gradient if solving tan $x = k$
  - This will be the quadrant diagonally across from the original quadrant

STEP 3: Work out the size of the second angle, measuring from the positive x-axis

- … anticlockwise for a positive angle
- … clockwise for a negative angle
  - You should look at the given range of values to decide whether you need the negative or positive angle

STEP 4: Add or subtract either 360° or 2π radians to both values until you have all solutions in the required range

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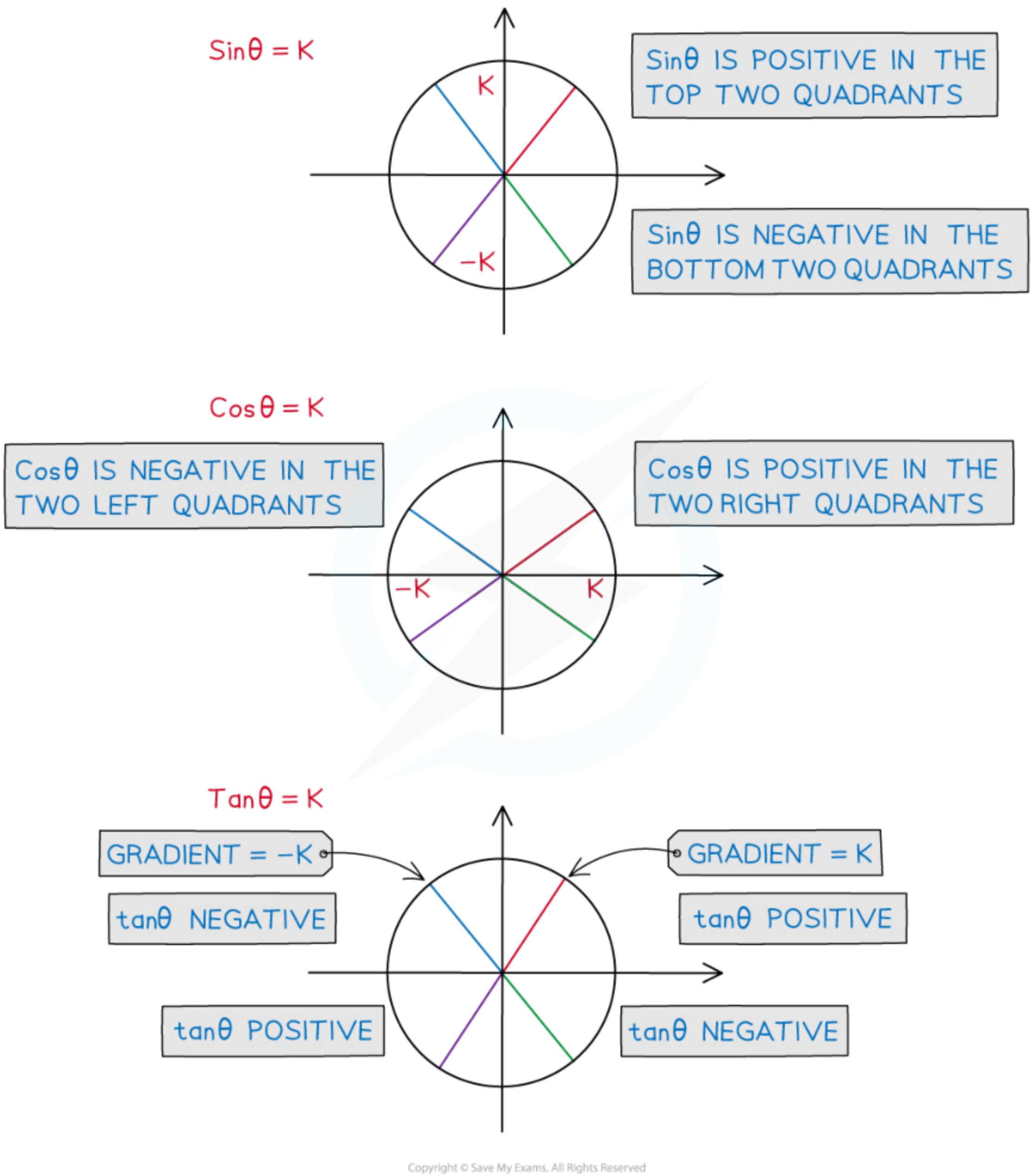
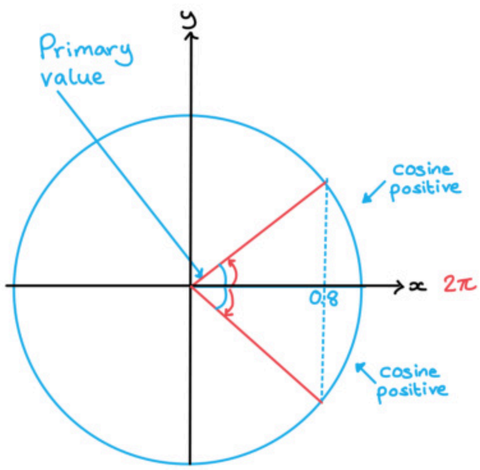


Examiner Tip

- Being able to sketch out the unit circle and remembering CAST can help you to find all solutions to a problem in an exam question

Worked example

Given that one solution of $\cos\theta = 0.8$ is $\theta = 0.6435$ radians correct to 4 decimal places, find all other solutions in the range $-2\pi \leq \theta \leq 2\pi$. Give your answers correct to 3 significant figures.



Cosine is positive in the first and fourth quadrants so draw the angle from the horizontal axis in both quadrants.

Primary value $= 0.6435$

Using diagram, secondary value $= -0.6435$

Therefore all values are: $0.6435 \pm 2\pi n$

and $-0.6435 \pm 2\pi n$

Within given domain: $-2\pi \leq \theta \leq 2\pi$

$$\theta = -5.64^c,\ -0.644^c,\ 0.644^c,\ 5.64^c$$

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3.4.2 Simple Identities

Simple Identities

What is a trigonometric identity?

- Trigonometric identities are statements that are true for all values of x or θ
- They are used to help simplify trigonometric equations before solving them
- Sometimes you may see identities written with the symbol ≡
  - This means 'identical to'

What trigonometric identities do I need to know?

- The two trigonometric identities you must know are
  - $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
    - This is the identity for $\tan\theta$
  - $\sin^2\theta + \cos^2\theta = 1$
    - This is the Pythagorean identity
    - Note that the notation $\sin^2\theta$ is the same as $(\sin\theta)^2$
- Both identities can be found **in the formula booklet**
- Rearranging the second identity often makes it easier to work with
  - $\sin^2\theta = 1 - \cos^2\theta$
  - $\cos^2\theta = 1 - \sin^2\theta$

Where do the trigonometric identities come from?

- You do not need to know the proof for these identities but it is a good idea to know where they come from
- From SOHCAHTOA we know that
  - $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{O}{H}$
  - $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}} = \dfrac{A}{H}$
  - $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{O}{A}$
- The identity for $\tan\theta$ can be seen by diving $\sin\theta$ by $\cos\theta$
  - $\dfrac{\sin\theta}{\cos\theta} = \dfrac{\frac{O}{H}}{\frac{A}{H}} = \dfrac{O}{A} = \tan\theta$
  - This can also be seen from the unit circle by considering a right-triangle with a hypotenuse of 1

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- $\tan\,\theta = \frac{O}{A} = \frac{\sin\,\theta}{\cos\,\theta}$
- The Pythagorean identity can be seen by considering a right-triangle on the unit circle with a hypotenuse of 1
  - Then $(\text{opposite})^2 + (\text{adjacent})^2 = 1$
  - Therefore $\sin^2\,\theta + \cos^2\,\theta = 1$
- Considering the equation of the unit circle also shows the Pythagorean identity
  - The equation of the unit circle is $x^2 + y^2 = 1$
  - The coordinates on the unit circle are $(\cos\,\theta,\ \sin\,\theta)$
  - Therefore the equation of the unit circle could be written $\cos^2\,\theta + \sin^2\,\theta = 1$
- A third very useful identity is $\sin\,\theta = \cos\,(90° - \theta)$ or $\sin\,\theta = \cos\,(\frac{\pi}{2} - \theta)$
  - This is not included in the formula booklet but is useful to remember

How are the trigonometric identities used?

- Most commonly trigonometric identities are used to change an equation into a form that allows it to be solved
- They can also be used to prove further identities such as the **double angle formulae**

Examiner Tip

- If you are asked to show that one thing is identical (≡) to another, look at what parts are missing – for example, if tan x has gone it must have been substituted

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Worked example

Show that the equation $2\sin^2 x - \cos x = 0$ can be written in the form $a\cos^2 x + b\cos x + c = 0$, where a, b and c are integers to be found.

$2\sin^2 x - \cos x = 0$

Equation has both $\sin x$ and $\cos x$ so will need changing before it can be solved.

Use the identity $\sin^2 x = 1 - \cos^2 x$

Substitute: $2(1 - \cos^2 x) - \cos x = 0$

Expand: $2 - 2\cos^2 x - \cos x = 0$

Rearrange: $2\cos^2 x + \cos x - 2 = 0$

$a = 2$, $b = 1$, $c = -2$

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3.4.3 Solving Trigonometric Equations

Graphs of Trigonometric Functions

What are the graphs of trigonometric functions?

- The trigonometric functions sin, cos and tan all have special **periodic graphs**
- You'll need to know their properties and how to sketch them for a given domain in either **degrees** or **radians**
- Sketching the trigonometric graphs can help to
  - Solve trigonometric equations and find all solutions
  - Understand transformations of trigonometric functions

What are the properties of the graphs of sin x and cos x?

- The graphs of sin x and cos x are both **periodic**
  - They **repeat every 360°** (2π radians)
  - The angle will always be on the x-axis
    - Either in degrees or radians
- The graphs of sin x and cos x are always in the **range $-1 \le y \le 1$**
  - **Domain:** $\{x \mid x \in \mathbb{R}\}$
  - **Range:** $\{y \mid -1 \le y \le 1\}$
  - The graphs of sin x and cos x are identical however one is a **translation** of the other
    - sin x passes through the origin
    - cos x passes through (0, 1)
- The **amplitude** of the graphs of sin x and cos x is 1

What are the properties of the graph of tan x?

- The graph of tan x is **periodic**
  - It **repeats every 180°** (π radians)
  - The angle will always be on the x-axis
    - Either in degrees or radians
- The graph of tan x is **undefined** at the points ± 90°, ± 270° etc
  - There are **asymptotes** at these points on the graph
  - In radians this is at the points $\pm\frac{\pi}{2}$, $\pm\frac{3\pi}{2}$ etc
- The range of the graph of tan x is
  - **Domain:** $\left\{x \mid x \neq \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}\right\}$
  - **Range:** $\{y \mid y \in \mathbb{R}\}$

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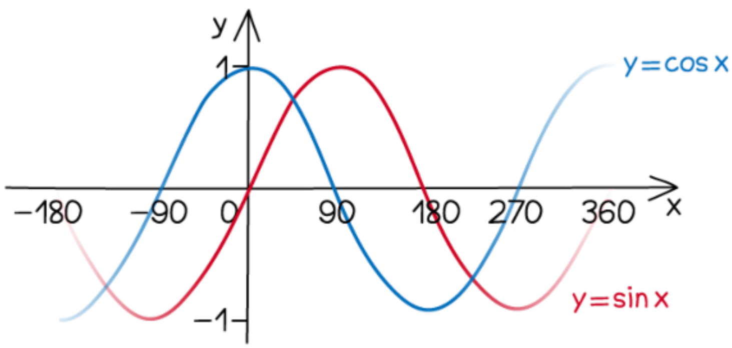
$y = \sin x$ AND $y = \cos x$

Sin x AND Cos x ARE ALWAYS IN THE RANGE −1 TO 1

Sin x PASSES THROUGH THE ORIGIN
Cos x PASSES THROUGH 1

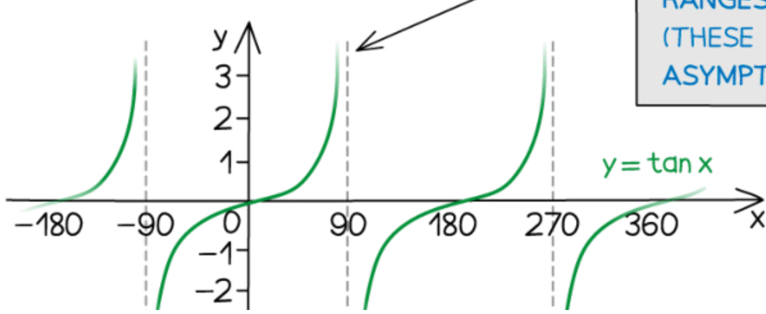


Sin x AND Cos x ARE PERIODIC REPEATING EVERY 360°

Sin x HAS ROTATIONAL SYMMETRY ABOUT THE ORIGIN SO $\sin(-x) = -\sin(x)$
Cos x IS SYMMETRICAL THROUGH THE y-AXIS SO $\cos(-x) = \cos(x)$

$y = \tan x$

Tan x IS UNDEFINED AT ±90°, ±270°, ±450°... MEANING IT RANGES FROM −∞ TO +∞ (THESE POINTS ARE CALLED ASYMPTOTES)



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How do I sketch trigonometric graphs?

- You may need to sketch a trigonometric graph so you will need to remember the key features of each one
- The following steps may help you sketch a trigonometric graph
  - STEP 1: Check whether you should be working in degrees or radians
    - You should check the domain given for this
    - If you see π in the given domain then you should work in radians
  - STEP 2: Label the x-axis in multiples of 90°
    - This will be multiples of $\frac{\pi}{2}$ if you are working in radians
    - Make sure you cover the whole domain on the x-axis
  - STEP 3: Label the y-axis
    - The range for the y-axis will be $-1 \le y \le 1$ for sin or cos
    - For tan you will not need any specific points on the y-axis
  - STEP 4: Draw the graph
    - Knowing exact values will help with this, such as remembering that sin(0) = 0 and cos(0) = 1
    - Mark the important points on the axis first
    - If you are drawing the graph of tan x put the asymptotes in first
    - If you are drawing sin x or cos x mark in where the maximum and minimum points will be
    - Try to keep the symmetry and rotational symmetry as you sketch, as this will help when using the graph to find solutions

Examiner Tip

- Sketch all three trig graphs on your exam paper so you can refer to them as many times as you need to!

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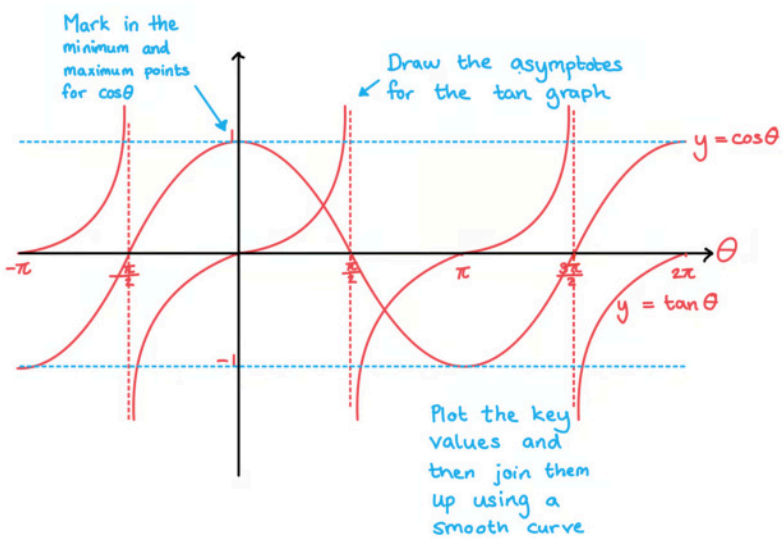
Worked example

Sketch the graphs of y = cosθ and y = tanθ on the same set of axes in the interval -π ≤ θ ≤ 2π. Clearly mark the key features of both graphs.



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Using Trigonometric Graphs

How can I use a trigonometric graph to find extra solutions?

- Your calculator will only give you the first solution to a problem such as $\sin^{-1}(0.5)$
  - This solution is called the **primary value**
- However, due to the **periodic** nature of the trig functions there could be an infinite number of solutions
  - Further solutions are called the **secondary values**
- This is why you will be given a **domain** (interval) in which your solutions should be found
  - This could either be in degrees or in radians
    - If you see π or some multiple of π then you must work in radians
- The following steps will help you use the **trigonometric graphs** to find **secondary values**
  - STEP 1: Sketch the graph for the given function and interval
    - Check whether you should be working in degrees or radians and label the axes with the key values
  - STEP 2: Draw a horizontal line going through the y-axis at the point you are trying to find the values for
    - For example if you are looking for the solutions to $\sin^{-1}(-0.5)$ then draw the horizontal line going through the y-axis at -0.5
    - The number of times this line cuts the graph is the number of solutions within the given interval
  - STEP 3: Find the primary value and mark it on the graph
    - This will either be an exact value and you should know it
    - Or you will be able to use your calculator to find it
  - STEP 4: Use the symmetry of the graph to find all the solutions in the interval by adding or subtracting from the key values on the graph

What patterns can be seen from the graphs of trigonometric functions?

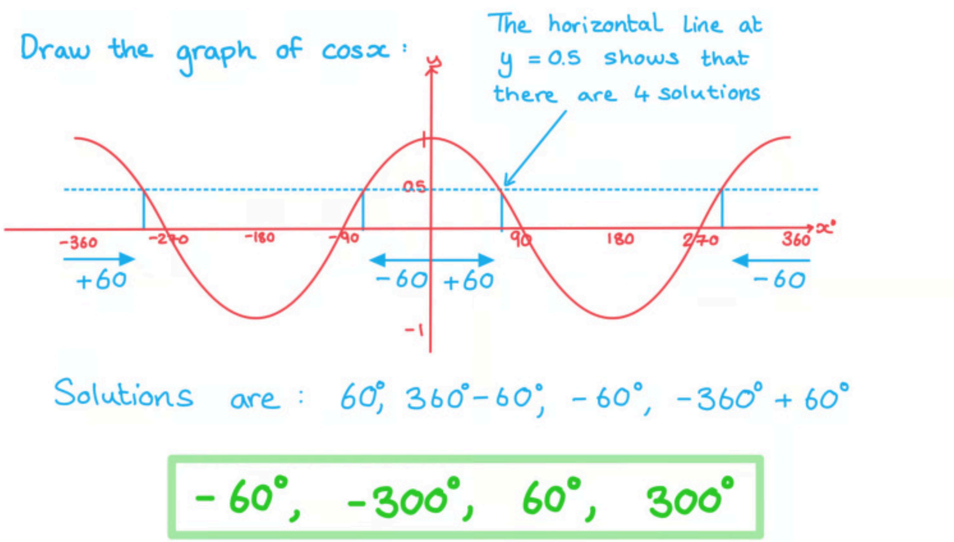
- The graph of sin x has rotational symmetry about the origin
  - So $\sin(-x) = -\sin(x)$
  - $\sin(x) = \sin(180° - x)$ or $\sin(\pi - x)$
- The graph of cos x has reflectional symmetry about the y-axis
  - So $\cos(-x) = \cos(x)$
  - $\cos(x) = \cos(360° - x)$ or $\cos(2\pi - x)$
- The graph of tan x repeats every 180° (π radians)
  - So $\tan(x) = \tan(x \pm 180°)$ or $\tan(x \pm \pi)$
- The graphs of sin x and cos x repeat every 360° (2π radians)
  - So $\sin(x) = \sin(x \pm 360°)$ or $\sin(x \pm 2\pi)$
  - $\cos(x) = \cos(x \pm 360°)$ or $\cos(x \pm 2\pi)$

Examiner Tip

- Take care to always check what the **interval** for the angle is that the question is focused on

Worked example

One solution to cos x = 0.5 is 60°. Find all the other solutions in the range -360° ≤ x ≤ 360°.



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