

# DP IB Maths: AA HL



Your notes

## 3.11 Vector Planes

### Contents

- \* 3.11.1 Vector Equations of Planes
- \* 3.11.2 Intersections of Lines & Planes
- \* 3.11.3 Angles Between Lines & Planes
- \* 3.11.4 Shortest Distances with Planes



Your notes

## 3.11.1 Vector Equations of Planes

### Equation of a Plane in Vector Form

#### How do I find the vector equation of a plane?

- A plane is a flat surface which is two-dimensional
  - Imagine a flat piece of paper that continues on forever in both directions
- A plane is often denoted using the capital Greek letter  $\Pi$
- The vector form of the equation of a plane can be found using **two direction vectors** on the plane
  - The direction vectors must be
    - **parallel** to the plane
    - **not parallel** to each other
  - If **both** direction vectors **lie on the plane** then they will **intersect at a point**
- The formula for finding the **vector equation** of a plane is
  - $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b} + \mu \mathbf{c}$ 
    - Where  $\mathbf{r}$  is the **position vector** of any point on the plane
    - $\mathbf{a}$  is the **position vector** of a known point on the plane
    - $\mathbf{b}$  and  $\mathbf{c}$  are two **non-parallel direction** (displacement) **vectors** parallel to the plane
    - $\lambda$  and  $\mu$  are scalars
  - The formula is **given in the formula booklet** but you must make sure you know what each part means
- As  $\mathbf{a}$  could be the position vector of **any** point on the plane and  $\mathbf{b}$  and  $\mathbf{c}$  could be **any non-parallel** direction vectors on the plane there are infinite vector equations for a single plane

#### How do I determine whether a point lies on a plane?

- Given the equation of a plane  $\mathbf{r} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} + \lambda \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{pmatrix} + \mu \begin{pmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \end{pmatrix}$  then the point  $\mathbf{r}$  with position

vector  $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$  is on the plane if there exists a value of  $\lambda$  and  $\mu$  such that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} + \lambda \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{pmatrix} + \mu \begin{pmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \end{pmatrix}$$

- This means that there exists a single value of  $\lambda$  and  $\mu$  that satisfy the three **parametric** equations:
  - $x = a_1 + \lambda b_1 + \mu c_1$
  - $y = a_2 + \lambda b_2 + \mu c_2$

- $z = a_3 + \lambda b_3 + \mu c_3$

- Solve two of the equations first to find the values of  $\lambda$  and  $\mu$  that satisfy the first two equations and then check that this value also satisfies the third equation
- If the values of  $\lambda$  and  $\mu$  do not satisfy all three equations, then the point  $r$  does not lie on the plane

### Examiner Tip

- The formula for the vector equation of a plane is given in the formula booklet, make sure you know what each part means
- Be careful to use different letters, e.g.  $\lambda$  and  $\mu$  as the scalar multiples of the two direction vectors



Your notes



Your notes

### Worked example

The points A, B and C have position vectors  $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ ,  $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ , and  $\mathbf{c} = 4\mathbf{i} - \mathbf{j} + 3\mathbf{k}$  respectively, relative to the origin O.

(a) Find the vector equation of the plane.

Start by finding the direction vectors  $\vec{AB}$  and  $\vec{AC}$

$$\vec{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ 5 \end{pmatrix}$$

$$\vec{AC} = \mathbf{c} - \mathbf{a} = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$$

All three points lie on the plane, so choose the position vector of one point, e.g.  $\vec{OA}$ , to use as 'a' in the vector equation of a plane formula.

Check that  $\vec{AB}$  and  $\vec{AC}$  are not parallel.

$$\mathbf{r} = \mathbf{a} + \lambda \vec{AB} + \mu \vec{AC}$$

$$\mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -4 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$$

(This is one of many correct answers)

(b) Determine whether the point D with coordinates (-2, -3, 5) lies on the plane.



Your notes

Let  $D$  have position vector  $\underline{d} = \begin{pmatrix} -2 \\ -3 \\ 5 \end{pmatrix}$ , then the point  $D$  lies on the plane if there exists a value of  $\lambda$  and  $\mu$  for which:

$$\begin{pmatrix} -2 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -4 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$$

Find the parametric equations:

$$\begin{aligned} -2 &= 3 - 2\lambda + \mu &\Rightarrow \mu - 2\lambda &= -5 & \textcircled{1} \\ -3 &= 2 - 4\lambda - 3\mu &\Rightarrow 3\mu + 4\lambda &= 5 & \textcircled{2} \\ 5 &= -1 + 5\lambda + 4\mu &\Rightarrow 4\mu + 5\lambda &= 6 & \textcircled{3} \end{aligned}$$

} solve two equations for  $\lambda$  and  $\mu$ .

Find the value of  $\lambda$  and  $\mu$  from two equations:

$$2\textcircled{1}: 2\mu - 4\lambda = -10$$

$$+ \textcircled{2}: \frac{3\mu + 4\lambda = 5}{5\mu = -5}$$

$$\mu = -1 \text{ sub into } \textcircled{1}: (-1) - 2\lambda = -5$$

$$\lambda = 2$$

Check to see if  $\lambda$  and  $\mu$  satisfy the third equation:

$$4(-1) + 5(2) = -4 + 10 = 6 \checkmark$$

The point  $D$  lies on the plane.



Your notes

## Equation of a Plane in Cartesian Form

### How do I find the vector equation of a plane in cartesian form?

- The **cartesian** equation of a plane is given in the form
  - $ax + by + cz = d$
  - This is **given in the formula booklet**
- A **normal vector** to the plane can be used along with a **known point on the plane** to find the cartesian equation of the plane
  - The normal vector will be a vector that is **perpendicular** to the plane
- The **scalar product** of the normal vector and any **direction vector** on the plane will be **zero**
  - The two vectors will be perpendicular to each other
  - The **direction vector** from a fixed-point  $A$  to any point on the plane,  $R$  can be written as  $\mathbf{r} - \mathbf{a}$
  - Then  $\mathbf{n} \cdot (\mathbf{r} - \mathbf{a}) = 0$  and it follows that  $(\mathbf{n} \cdot \mathbf{r}) - (\mathbf{n} \cdot \mathbf{a}) = 0$
- This gives the **equation of a plane using the normal vector**:
  - $\mathbf{n} \cdot \mathbf{r} = \mathbf{a} \cdot \mathbf{n}$ 
    - Where  $\mathbf{r}$  is the **position vector** of any point on the plane
    - $\mathbf{a}$  is the **position vector** of a known point on the plane
    - $\mathbf{n}$  is a vector that is **normal** to the plane
  - This is **given in the formula booklet**

- If the vector  $\mathbf{r}$  is given in the form  $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$  and  $\mathbf{a}$  and  $\mathbf{n}$  are both known vectors given in the form  $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$

and  $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$  then the Cartesian equation of the plane can be found using:

- $\mathbf{n} \cdot \mathbf{r} = ax + by + cz$
- $\mathbf{a} \cdot \mathbf{n} = a_1 a + a_2 b + a_3 c$
- Therefore  $ax + by + cz = a_1 a + a_2 b + a_3 c$
- This simplifies to the form  $ax + by + cz = d$

### How do I find the equation of a plane in Cartesian form given the vector form?

- The **Cartesian** equation of a plane can be found if you know
  - the **normal vector** and
  - a **point** on the plane
- The **vector equation of a plane** can be used to find the **normal vector** by finding the **vector product** of the two direction vectors
  - A vector product is always perpendicular to the two vectors from which it was calculated
- The vector  $\mathbf{a}$  given in the vector equation of a plane is a **known point** on the plane

- Once you have found the normal vector then the point  $\mathbf{a}$  can be used in the formula  $\mathbf{n} \cdot \mathbf{r} = \mathbf{a} \cdot \mathbf{n}$  to find the equation in Cartesian form
- To find  $ax + by + cz = d$  given  $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$  :
  - Let  $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{b} \times \mathbf{c}$  then  $d = \mathbf{n} \cdot \mathbf{a}$

 **Examiner Tip**

- In an exam, using whichever form of the equation of the plane to write down a normal vector to the plane is always a good starting point



Your notes



Your notes

### Worked example

A plane  $\Pi$  contains the point  $A(2, 6, -3)$  and has a normal vector  $\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$ .

- a) Find the equation of the plane in its Cartesian form.

The components of the normal vector are the  $x$ -,  $y$ - and  $z$ -coefficients of the Cartesian form:

$$3x - y + 4z = d$$

The point  $(2, 6, -3)$  is on the plane so

$$d = 3(2) - (6) + 4(-3) = 6 - 6 - 12 = -12$$

Therefore

$$3x - y + 4z = -12$$

- b) Determine whether point B with coordinates  $(-1, 0, -2)$  lies on the same plane.

Test by putting the coordinates into the equation:

$$3(-1) - (0) + 4(-2) = -3 - 8 = -11 \neq -12$$

The point with coordinates  $(-1, 0, 2)$  does not lie on the plane





Your notes

## 3.11.2 Intersections of Lines & Planes

### Intersection of Line & Plane

#### How do I tell if a line is parallel to a plane?

- A line is parallel to a plane if its **direction vector** is **perpendicular** to the plane's **normal vector**
- If you know the Cartesian equation of the plane in the form  $ax + by + cz = d$  then the values of  $a$ ,  $b$ , and  $c$  are the individual components of a normal vector to the plane
- The **scalar product** can be used to check if the direction vector and the normal vector are perpendicular
  - If two vectors are perpendicular their scalar product will be zero

#### How do I tell if the line lies inside the plane?

- If the line is parallel to the plane then it will either **never intersect** or it will lie inside the plane
  - Check to see if they have a common point
- If a line is parallel to a plane and they share **any point**, then the line lies inside the plane

#### How do I find the point of intersection of a line and a plane?

- If a line is **not parallel** to a plane it will **intersect** it at a single point
- If both the **vector equation of the line** and the **Cartesian equation of the plane** is known then this can be found by:
- STEP 1: Set the position vector of the point you are looking for to have the individual components  $x$ ,  $y$ , and  $z$  and substitute into the vector equation of the line

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + \lambda \begin{bmatrix} l \\ m \\ n \end{bmatrix}$$

- STEP 2: Find the parametric equations in terms of  $x$ ,  $y$ , and  $z$ 
  - $x = x_0 + \lambda l$
  - $y = y_0 + \lambda m$
  - $z = z_0 + \lambda n$
- STEP 3: Substitute these parametric equations into the Cartesian equation of the plane and solve to find  $\lambda$ 
  - $a(x_0 + \lambda l) + b(y_0 + \lambda m) + c(z_0 + \lambda n) = d$
- STEP 4: Substitute this value of  $\lambda$  back into the vector equation of the line and use it to find the position vector of the point of intersection

- STEP 5: Check this value in the Cartesian equation of the plane to make sure you have the correct answer



Your notes

### Worked example

Find the point of intersection of the line  $r = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$  with the plane  $3x - 4y + z = 8$ .

Find the parametric form of the equation of the line:

$$\text{Let } r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \text{ then } \begin{array}{l} x = 1 + 2\lambda \\ y = -3 - \lambda \\ z = 2 - \lambda \end{array}$$

Substitute into the equation of the plane:

$$3(1 + 2\lambda) - 4(-3 - \lambda) + (2 - \lambda) = 8$$

Solve to find  $\lambda$ :

$$3 + 6\lambda + 12 + 4\lambda + 2 - \lambda = 8$$

$$\lambda = -1$$

Substitute  $\lambda = -1$  into the vector equation of the line:

$$r = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + (-1) \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 - 2 \\ -3 + 1 \\ 2 + 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}$$

**$(-1, -2, 3)$**



Your notes

## Intersection of Planes

### How do we find the line of intersection of two planes?

- Two planes will either be **parallel** or they will intersect along a **line**
  - Consider the point where a wall meets a floor or a ceiling
  - You will need to find the **equation of the line** of intersection
- If you have the Cartesian forms of the two planes then the equation of the line of intersection can be found by solving the two equations simultaneously
  - As the solution is a vector equation of a line rather than a unique point you will see below how the equation of the line can be found by part solving the equations
  - For example:
    - $2x - y + 3z = 7$  (1)
    - $x - 3y + 4z = 11$  (2)
- STEP 1: Choose one variable and substitute this variable for  $\lambda$  in both equations
  - For example, letting  $x = \lambda$  gives:
    - $2\lambda - y + 3z = 7$  (1)
    - $\lambda - 3y + 4z = 11$  (2)
- STEP 2: Rearrange the two equations to bring  $\lambda$  to one side
  - Equations (1) and (2) become
    - $y - 3z = 2\lambda - 7$  (1)
    - $3y - 4z = \lambda - 11$  (2)
- STEP 3: Solve the equations simultaneously to find the two variables in terms of  $\lambda$ 
  - $3(1) - (2)$  Gives
    - $z = 2 - \lambda$
  - Substituting this into (1) gives
    - $y = -1 - \lambda$
- STEP 4: Write the three parametric equations for  $x, y,$  and  $z$  in terms of  $\lambda$  and convert into the vector

equation of a line in the form 
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix}$$

- The parametric equations
  - $x = \lambda$
  - $y = -1 - \lambda$
  - $z = 2 - \lambda$
- Become
  - $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$



Your notes

- If you have fractions in your direction vector you can change its magnitude by multiplying each one by their common denominator
  - The magnitude of the direction vector can be changed without changing the equation of a line
- An alternative method is to find two points on both planes by setting either  $x$ ,  $y$ , or  $z$  to zero and solving the system of equations using your GDC or row reduction
  - Repeat this twice to get two points on both planes
  - These two points can then be used to find the vector equation of the line between them
  - This will be the line of intersection of the planes
  - This method relies on the line of intersection having points where the chosen variables are equal to zero

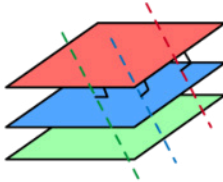
### How do we find the relationship between three planes?

- Three planes could either be **parallel**, intersect at one **point**, or intersect along a **line**
- If the three planes have a **unique point of intersection** this point can be found by using your GDC (or row reduction) to solve the three equations in their Cartesian form
  - Make sure you know how to use your GDC to solve a **system of linear equations**
  - Enter all three equations in for the three variables  $x$ ,  $y$ , and  $z$
  - Your GDC will give you the unique solution which will be the coordinates of the point of intersection
- If the three planes do not intersect at a unique point you will not be able to use your GDC to solve the equations
  - If there are no solutions to the system of Cartesian equations then there is no unique point of intersection
- If the three planes are all **parallel** their **normal vectors** will be parallel to each other
  - Show that the normal vectors all have equivalent **direction vectors**
  - These direction vectors may be **scalar multiples** of each other
- If the three planes have **no point of intersection** and are **not all parallel** they may have a relationship such as:
  - Each plane intersects two other planes such that they form a **prism** (none are parallel)
  - Two planes are parallel with the third plane intersecting each of them
  - Check the normal vectors to see if any two of the planes are parallel to decide which relationship they have
- If the three planes intersect along a line there will not be a unique solution to the three equations but there will be a **vector equation of a line** that will satisfy the three equations
- The system of equations will need to be solved by **elimination** or **row reduction**
  - Choose one variable to substitute for  $\lambda$
  - Solve two of the equations simultaneously to find the other two variables in terms of  $\lambda$
  - Write  $x$ ,  $y$ , and  $z$  in terms of  $\lambda$  in the parametric form of the equation of the line and convert into the vector form of the equation of a line



Your notes

3 PARALLEL PLANES



3 NORMALS ARE PARALLEL

NO POINT OF INTERSECTION

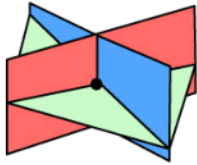
2 PARALLEL PLANES



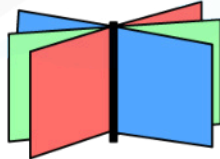
2 NORMALS ARE PARALLEL

2 LINES OF INTERSECTION

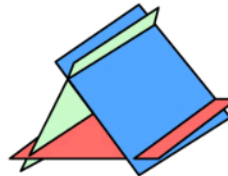
NO PARALLEL PLANES



A UNIQUE POINT OF INTERSECTION



ONE LINE OF INTERSECTION



EACH PLANE INTERSECTS WITH TWO OTHERS

Copyright © Save My Exams. All Rights Reserved

### Examiner Tip

- In an exam you may need to decide the relationship between three planes by using row reduction to determine the number of solutions
  - Make sure you are confident using row reduction to solve systems of linear equations
  - Make sure you remember the different forms three planes can take



Your notes

### Worked example

Two planes  $\Pi_1$  and  $\Pi_2$  are defined by the equations:

$$\Pi_1: 3x + 4y + 2z = 7$$

$$\Pi_2: x - 2y + 3z = 5$$

Find the vector equation of the line of intersection of the two planes.

STEP 1: Let  $z = \lambda$ , then  $3x + 4y + 2\lambda = 7$  ①

You can substitute any variable here, look at the equations to see which is easiest.  $x - 2y + 3\lambda = 5$  ②

STEP 2: ①:  $3x + 4y = 7 - 2\lambda$  Write the two equations as simultaneous equations for the two remaining constants.  
 ②:  $x - 2y = 5 - 3\lambda$

STEP 3: Find  $x$  and  $y$  in terms of  $\lambda$ :

$$\begin{array}{r} \text{①} - 2 \times \text{②}: (3x + 4y = 7 - 2\lambda) \\ + (2x - 4y = 10 - 6\lambda) \\ \hline 5x = 17 - 8\lambda \\ x = \frac{17 - 8\lambda}{5} \end{array}$$

sub into ②  $\frac{17 - 8\lambda}{5} - 2y + 3\lambda = 5$   
 $y = \frac{7\lambda}{10} - \frac{8}{10}$

STEP 4:  $\left. \begin{array}{l} x = \frac{17 - 8\lambda}{5} \\ y = \frac{7\lambda}{10} - \frac{4}{5} \\ z = \lambda \end{array} \right\} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{17}{5} \\ -\frac{4}{5} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -\frac{8}{10} \\ \frac{7}{10} \\ 1 \end{pmatrix}$

The components of the direction vector can be multiplied by a scalar without changing the direction.

$$\mathbf{r} = \begin{pmatrix} \frac{17}{5} \\ -\frac{4}{5} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \frac{16}{10} \\ \frac{7}{10} \\ 1 \end{pmatrix}$$



Your notes

### 3.11.3 Angles Between Lines & Planes

## Angle Between Line & Plane

### What is meant by the angle between a line and a plane?

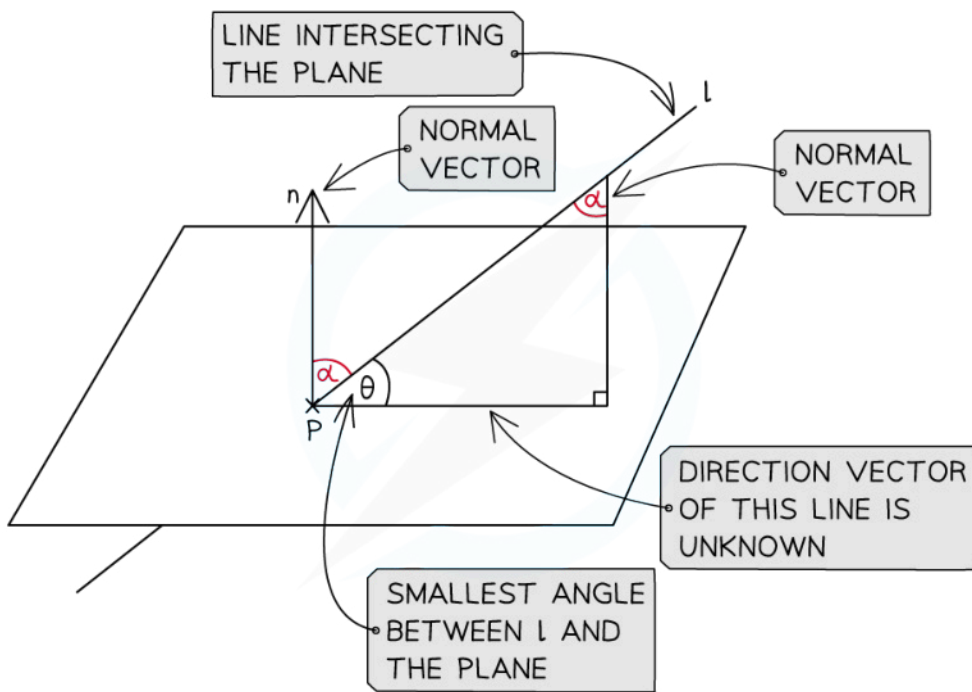
- When you find the angle between a line and a plane you will be finding the angle between the line itself and the line on the plane that creates the smallest angle with it
  - This means the line on the plane directly under the line as it joins the plane
- It is easiest to think of these two lines making a right-triangle with the normal vector to the plane
  - The line joining the plane will be the **hypotenuse**
  - The line on the plane will be **adjacent** to the angle
  - The normal will be the **opposite** the angle

### How do I find the angle between a line and a plane?

- You need to know:
  - A **direction vector** for the **line** ( $\mathbf{b}$ )
    - This can easily be identified if the equation of the line is in the form  $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$
  - A **normal vector** to the **plane** ( $\mathbf{n}$ )
    - This can easily be identified if the equation of the plane is in the form  $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$
- Find the **acute angle** between the **direction of the line** and the **normal to the plane**
  - Use the formula  $\cos \alpha = \frac{|\mathbf{b} \cdot \mathbf{n}|}{|\mathbf{b}| |\mathbf{n}|}$ 
    - The **absolute value** of the **scalar product** ensures that the angle is **acute**
- Subtract** this angle from  $90^\circ$  to find the **acute angle** between the line and the plane
  - Subtract the angle from  $\frac{\pi}{2}$  if working in **radians**



Your notes



Copyright © Save My Exams. All Rights Reserved

 **Examiner Tip**

- Remember that if the scalar product is negative your answer will result in an obtuse angle
  - Taking the absolute value of the scalar product will ensure that you get the acute angle as your answer





Your notes

### Worked example

Find the angle in radians between the line  $L$  with vector equation

$\mathbf{r} = (2 - \lambda)\mathbf{i} + (\lambda + 1)\mathbf{j} + (1 - 2\lambda)\mathbf{k}$  and the plane  $\Pi$  with Cartesian equation  $x - 3y + 2z = 5$ .

Rewrite line equation in standard vector form:

$$\mathbf{r} = \begin{pmatrix} 2 - \lambda \\ 1 + \lambda \\ 1 - 2\lambda \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$$

↖ direction vector of the line

Find the normal vector of the plane:

$$x - 3y + 2z = 5 \Rightarrow \text{normal vector} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$$

components of the normal vector

Find the angle between the direction vector and the normal vector,  $\alpha$ :

Angle between two vectors	$\cos \theta = \frac{v_1 w_1 + v_2 w_2 + v_3 w_3}{ \mathbf{v}   \mathbf{w} }$
---------------------------	---

$$\cos \alpha = \frac{\left| \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \right|}{\sqrt{(-1)^2 + 1^2 + (-2)^2} \times \sqrt{1^2 + (-3)^2 + 2^2}} = \frac{|(-1)(1) + (1)(-3) + (-2)(2)|}{\sqrt{6} \sqrt{14}}$$

$$\theta = \frac{\pi}{2} - \cos^{-1} \alpha$$

$$\theta = \frac{\pi}{2} - \cos^{-1} \left( \frac{|-8|}{\sqrt{6} \sqrt{14}} \right)$$

Using the absolute value ensures we find the acute angle.

$$\theta = 1.06 \text{ radians (3s.f.)}$$

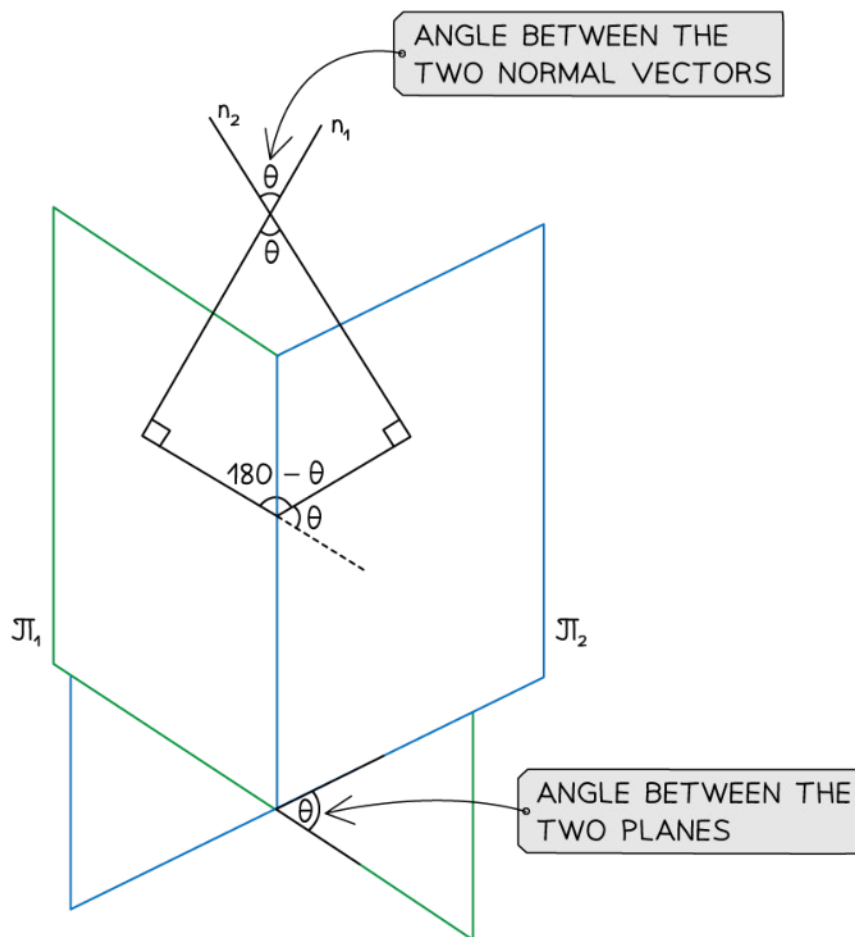


Your notes

## Angle Between Two Planes

### How do I find the angle between two planes?

- The angle between two planes is equal to the angle between their **normal vectors**
  - It can be found using the **scalar product** of their normal vectors
- $$\cos \theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|}$$
- If two planes  $\Pi_1$  and  $\Pi_2$  with normal vectors  $\mathbf{n}_1$  and  $\mathbf{n}_2$  meet at an angle then the two planes and the two normal vectors will form a quadrilateral
  - The angles between the planes and the normal will both be  $90^\circ$
  - The angle between the two planes and the angle opposite it (between the two normal vectors) will add up to  $180^\circ$





Your notes

### 💡 Examiner Tip

- In your exam read the question carefully to see if you need to find the acute or obtuse angle
  - When revising, get into the practice of double checking at the end of a question whether your angle is acute or obtuse and whether this fits the question

### 🟢 Worked example

Find the acute angle between the two planes which can be defined by equations

$$\Pi_1: 2x - y + 3z = 7 \text{ and } \Pi_2: x + 2y - z = 20.$$

Find the normal vectors of each of the planes:

$$\Pi_1: 2x - y + 3z = 7 \Rightarrow \text{normal vector, } n_1 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$$\Pi_2: x + 2y - z = 20 \Rightarrow \text{normal vector, } n_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

Find the angle between the two normal vectors:

Angle between two vectors	$\cos \theta = \frac{v_1 w_1 + v_2 w_2 + v_3 w_3}{ v   w }$
---------------------------	---

$$\cos \theta = \frac{n_1 \cdot n_2}{|n_1| |n_2|} = \frac{|(2)(1) + (-1)(2) + (3)(-1)|}{\sqrt{2^2 + (-1)^2 + 3^2} \times \sqrt{1^2 + 2^2 + (-1)^2}} = \frac{|-3|}{\sqrt{14} \times \sqrt{6}}$$

$$\theta = \cos^{-1}\left(\frac{3}{2\sqrt{21}}\right) \quad \text{Using the absolute value ensures we find the acute angle.}$$

$$\theta = 1.24 \text{ radians (3 s.f.)}$$



Your notes

### 3.11.4 Shortest Distances with Planes

## Shortest Distance Between a Line and a Plane

### How do I find the shortest distance between a point and a plane?

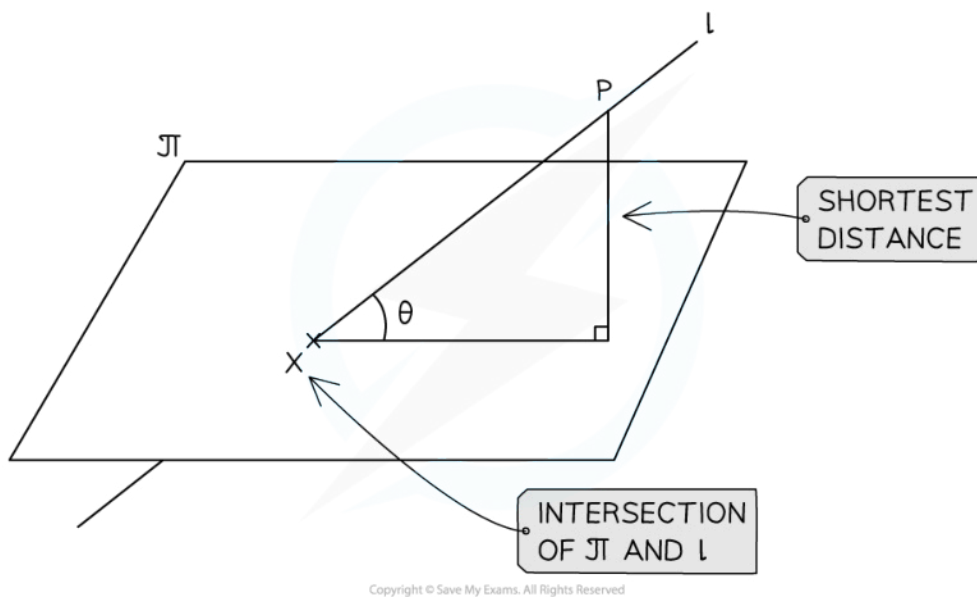
- The shortest distance from any point to a plane will always be the **perpendicular** distance from the point to the plane
- Given a point,  $P$  with position vector  $\mathbf{p}$  and a plane  $\Pi$  with equation  $\mathbf{r} \cdot \mathbf{n} = d$ 
  - STEP 1: Find the **vector equation of the line** perpendicular to the plane that goes through the point,  $P$ 
    - This will have the position vector of the point,  $P$ , and the direction vector  $\mathbf{n}$
    - $\mathbf{r} = \mathbf{p} + \lambda \mathbf{n}$
  - STEP 2: Find the value of  $\lambda$  at the **point of intersection** of this line with  $\Pi$  by substituting the equation of the line into the equation of the plane
  - STEP 3: Find the **distance** between the point and the point of intersection
    - Substitute  $\lambda$  into the equation of the line to find the coordinates of the point on the plane closest to point  $P$
    - Find the distance between this point and point  $P$
    - As a shortcut, this distance will be equal to  $|\lambda \mathbf{n}|$

### How do I find the shortest distance between a given point on a line and a plane?

- The shortest distance from any point on a line to a plane will always be the **perpendicular** distance from the point to the plane
- You can follow the same **steps above**
- A question may provide the acute angle between the line and the plane
  - Use right-angled trigonometry to find the perpendicular distance between the point on the line and the plane
    - Drawing a clear diagram will help



Your notes

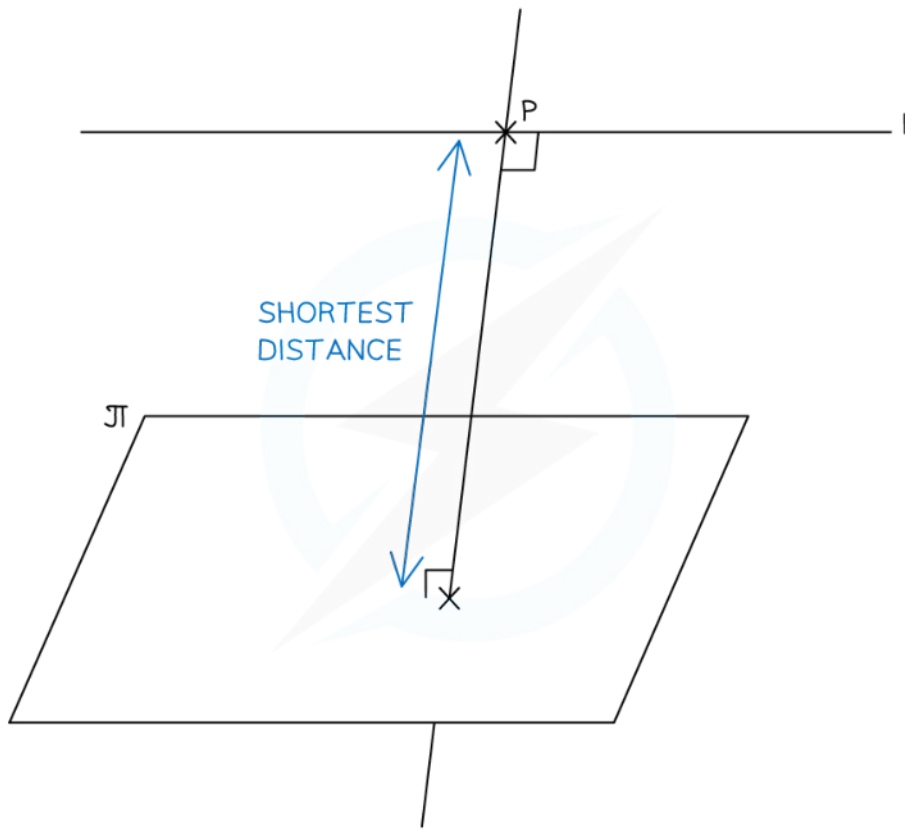


### How do I find the shortest distance between a plane and a line parallel to the plane?

- The shortest distance between a line and a plane that are parallel to each other will be the **perpendicular** distance from the line to the plane
- Given a line  $l_1$  with equation  $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$  and a plane  $\Pi$  parallel to  $l_1$  with equation  $\mathbf{r} \cdot \mathbf{n} = d$ 
  - Where  $\mathbf{n}$  is the **normal vector** to the plane
  - STEP 1: Find the equation of the line  $l_2$  perpendicular to  $l_1$  and  $\Pi$  going through the point  $\mathbf{a}$  in the form  $\mathbf{r} = \mathbf{a} + \mu \mathbf{n}$
  - STEP 2: Find the point of intersection of the line  $l_2$  and  $\Pi$
  - STEP 3: Find the distance between the point of intersection and the point,



Your notes



Copyright © Save My Exams. All Rights Reserved

 **Examiner Tip**

- Vector planes questions can be tricky to visualise, read the question carefully and sketch a very simple diagram to help you get started



Your notes

 **Worked example**

The plane  $\Pi$  has equation  $\mathbf{r} \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 6$ .

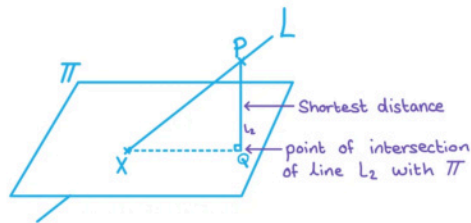
The line  $L$  has equation  $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$ .

The point  $P(-2, 11, -15)$  lies on the line  $L$ .

Find the shortest distance between the point  $P$  and the plane  $\Pi$ .



Your notes



STEP 1: Use the given point,  $P$  and the known normal to the plane,  $\underline{n}$  to write an equation for the line perpendicular to  $\pi$ ,  $L_2$ .

$$\underline{r} = \begin{pmatrix} -2 \\ 11 \\ -15 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$\underbrace{\hspace{1.5cm}}_P \qquad \qquad \qquad \underbrace{\hspace{1.5cm}}_{\underline{n}}$

STEP 2: Find the point of intersection,  $Q$ , of the new line,  $L_2$ , with  $\pi$ .

$$\begin{aligned} \left( \begin{pmatrix} -2 \\ 11 \\ -15 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right) \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} &= 6 \\ 2(-2 + 2\lambda) - (11 - \lambda) + (\lambda - 15) &= 6 \\ -4 + 4\lambda - 11 + \lambda + \lambda - 15 &= 6 \\ 6\lambda - 30 &= 6 \\ \lambda = 6 &\Rightarrow \vec{OQ} = \begin{pmatrix} -2 \\ 11 \\ -15 \end{pmatrix} + 6 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \\ -9 \end{pmatrix} \end{aligned}$$

STEP 3: Find the distance between  $P$  and  $Q$ .

$$|\vec{PQ}| = \sqrt{(10 - -2)^2 + (5 - 11)^2 + (-9 - -15)^2} = 6\sqrt{6} \text{ units}$$

Shortest distance =  $6\sqrt{6}$  units





Your notes

## Shortest Distance Between Two Planes

### How do I find the shortest distance between two parallel planes?

- Two **parallel** planes will never intersect
- The shortest distance between two **parallel planes** will be the **perpendicular distance** between them
- Given a plane  $\Pi_1$  with equation  $\mathbf{r} \cdot \mathbf{n} = d$  and a plane  $\Pi_2$  with equation  $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b} + \mu \mathbf{c}$  then the shortest distance between them can be found
  - STEP 1: The equation of the line perpendicular to both planes and through the point  $\mathbf{a}$  can be written in the form  $\mathbf{r} = \mathbf{a} + s\mathbf{n}$
  - STEP 2: Substitute the equation of the line into  $\mathbf{r} \cdot \mathbf{n} = d$  to find the coordinates of the point where the line meets  $\Pi_1$
  - STEP 3: Find the distance between the two points of intersection of the line with the two planes

### How do I find the shortest distance from a given point on a plane to another plane?

- The shortest distance from any point,  $P$  on a plane,  $\Pi_1$ , to another plane,  $\Pi_2$  will be the **perpendicular distance** from the point to  $\Pi_2$ 
  - STEP 1: Use the given coordinates of the point  $P$  on  $\Pi_1$  and the normal to the plane  $\Pi_2$  to find the vector equation of the line through  $P$  that is perpendicular to  $\Pi_2$
  - STEP 2: Find the point of intersection of this line with the plane  $\Pi_2$
  - STEP 3: Find the distance between the two points of intersection

#### Examiner Tip

- There are a lot of steps when answering these questions so set your methods out clearly in the exam



Your notes

**Worked example**

Consider the parallel planes defined by the equations:

$$\Pi_1 : \mathbf{r} \cdot \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = 44,$$

$$\Pi_2 : \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Find the shortest distance between the two planes  $\Pi_1$  and  $\Pi_2$ .



Your notes

Find the equation of the line perpendicular to the planes through the point  $(0,0,3)$

$$L: r = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + s \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}$$

position vector of  $\pi_2$  Normal vector of  $\pi_1$

Substitute the equation of  $L$  into the equation of  $\pi_1$ :

$$\begin{pmatrix} 3s \\ -5s \\ 3+2s \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = 44$$

$$3(3s) - 5(-5s) + 2(3+2s) = 44$$

$$38s + 6 = 44$$

$$s = 1$$

Substitute  $s = 1$  back into the equation of  $L$ :

$$r = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ 5 \end{pmatrix}$$

Find the distance between  $(0,0,3)$  and  $(3,-5,5)$

$$d = \sqrt{3^2 + (-5)^2 + (5-3)^2}$$

$$= \sqrt{38}$$

Shortest distance =  $\sqrt{38}$  units