

# DP IB Maths: AA SL



## 2.4 Further Functions & Graphs

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Your notes

## 2.4.1 Reciprocal & Rational Functions

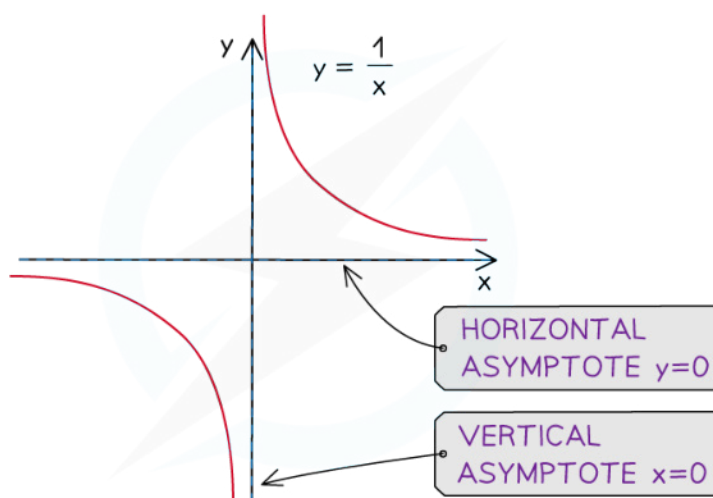
### Reciprocal Functions & Graphs

#### What is the reciprocal function?

- The **reciprocal function** is defined by  $f(x) = \frac{1}{x}$ ,  $x \neq 0$
- Its **domain** is the set of **all real values except 0**
- Its **range** is the set of **all real values except 0**
- The reciprocal function has a **self-inverse** nature
  - $f^{-1}(x) = f(x)$
  - $(f \circ f)(x) = x$

#### What are the key features of the reciprocal graph?

- The graph **does not have a y-intercept**
- The graph **does not have any roots**
- The graph has **two asymptotes**
  - A **horizontal** asymptote at the x-axis:  $y = 0$ 
    - This is the **limiting value** when the absolute value of x gets very large
  - A **vertical** asymptote at the y-axis:  $x = 0$ 
    - This is the value that causes the **denominator to be zero**
- The graph has **two axes of symmetry**
  - $y = x$
  - $y = -x$
- The graph **does not have any minimum or maximum points**



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Your notes

## Linear Rational Functions & Graphs

### What is a rational function?

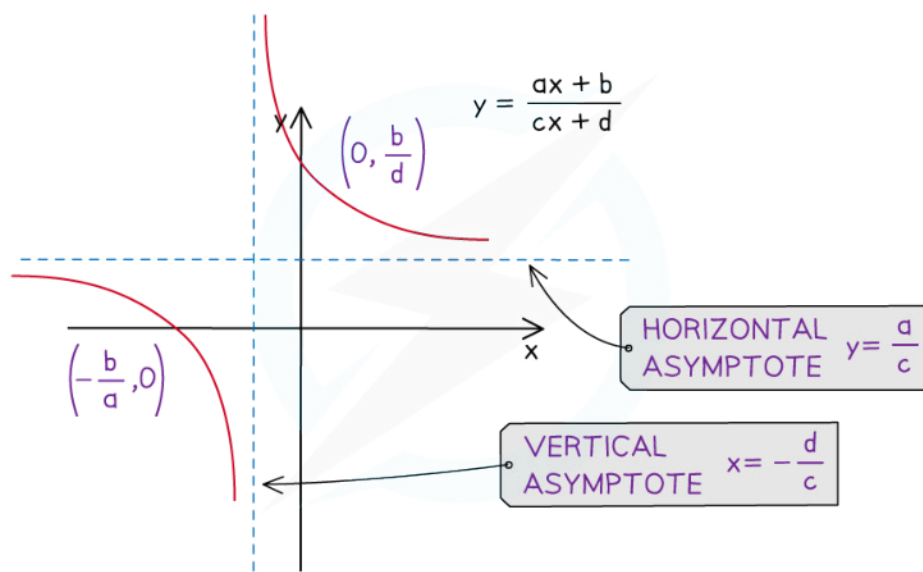
- A **rational function** is of the form  $f(x) = \frac{ax + b}{cx + d}$ ,  $x \neq -\frac{d}{c}$
- Its **domain** is the set of **all real values except**  $-\frac{d}{c}$
- Its **range** is the set of **all real values except**  $\frac{a}{c}$
- The **reciprocal function** is a **special case** of a rational function

### What are the key features of rational graphs?

- The graph has a **y-intercept** at  $\left(0, \frac{b}{d}\right)$  provided  $d \neq 0$
- The graph has **one root** at  $\left(-\frac{b}{a}, 0\right)$  provided  $a \neq 0$
- The graph has **two asymptotes**
  - A **horizontal asymptote**:  $y = \frac{a}{c}$ 
    - This is the **limiting value** when the absolute value of  $x$  gets very large
  - A **vertical asymptote**:  $x = -\frac{d}{c}$ 
    - This is the value that causes the **denominator to be zero**
- The graph **does not have any minimum or maximum points**
- If you are asked to **sketch or draw** a rational graph:
  - Give the **coordinates** of any **intercepts** with the axes
  - Give the **equations** of the **asymptotes**



Your notes



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 **Examiner Tip**

- If you draw a horizontal line anywhere it should only intersect this type of graph once at most
- The only horizontal line that should not intersect the graph is the horizontal asymptote
  - This can be used to check your sketch in an exam



Your notes

### Worked example

The function  $f$  is defined by  $f(x) = \frac{10 - 5x}{x + 2}$  for  $x \neq -2$ .

- a) Write down the equation of
- the vertical asymptote of the graph of  $f$ ,
  - the horizontal asymptote of the graph of  $f$ .

(i) Vertical asymptote is when denominator equals zero

$$x + 2 = 0 \quad \boxed{x = -2}$$

(ii) Horizontal asymptote is limiting value as  $x$  gets large

$$\lim_{x \rightarrow \infty} \frac{10 - 5x}{x + 2} = \lim_{x \rightarrow \infty} \frac{-5x}{x} \quad \boxed{y = -5}$$

- b) Find the coordinates of the intercepts of the graph of  $f$  with the axes.

$y$ -intercept occurs when  $x = 0$

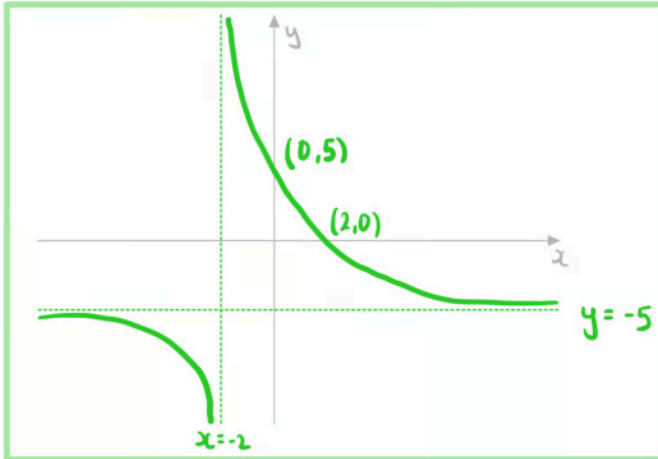
$$y = \frac{10 - 5(0)}{0 + 2} = 5 \quad \boxed{(0, 5)}$$

$x$ -intercept occurs when  $y = 0$

$$\frac{10 - 5x}{x + 2} = 0 \Rightarrow 10 - 5x = 0 \Rightarrow x = 2 \quad \boxed{(2, 0)}$$

- c) Sketch the graph of  $f$ .

Include asymptotes and intercepts



Your notes

## 2.4.2 Exponential & Logarithmic Functions



Your notes

### Exponential Functions & Graphs

#### What is an exponential function?

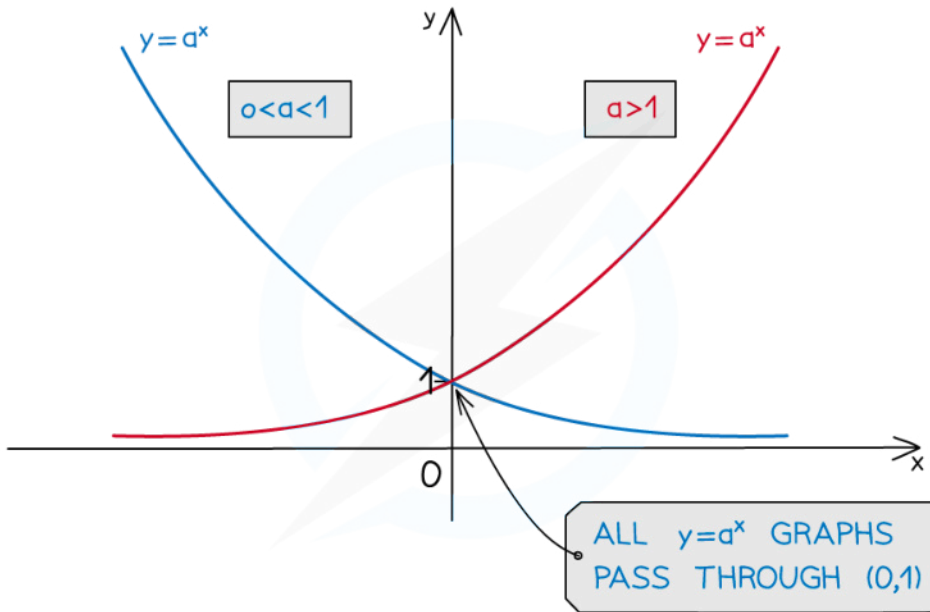
- An **exponential function** is defined by  $f(x) = a^x$ ,  $a > 0$
- Its **domain** is the set of **all real values**
- Its **range** is the set of **all positive real values**
- An important exponential function is  $f(x) = e^x$ 
  - Where  $e$  is the mathematical constant 2.718...
- Any exponential function can be written using  $e$ 
  - $a^x = e^{x \ln a}$ 
    - This is given in the **formula booklet**

#### What are the key features of exponential graphs?

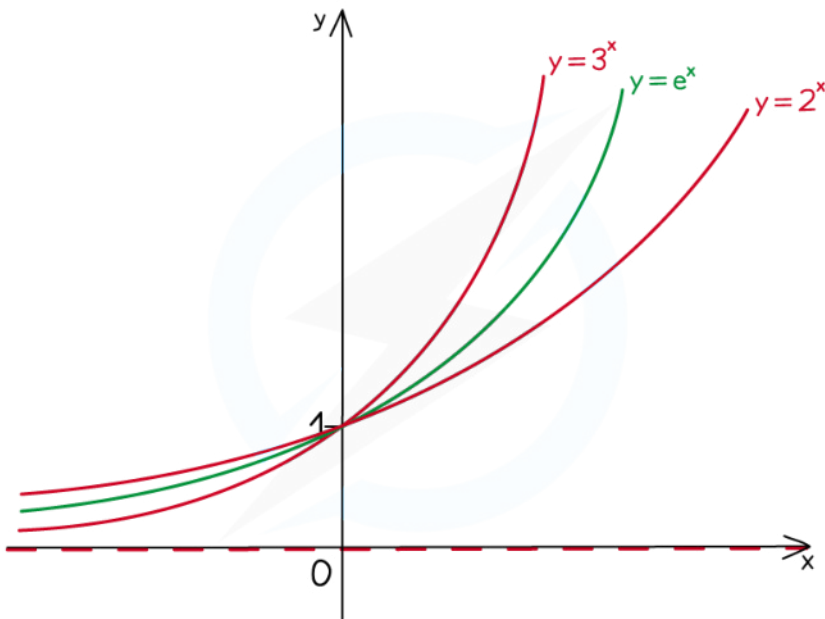
- The graphs have a **y-intercept** at  $(0, 1)$
- The graph **will always** pass through the **point**  $(1, a)$
- The graphs **do not have any roots**
- The graphs have a **horizontal asymptote** at the  $x$ -axis:  $y = 0$ 
  - For  $a > 1$  this is the **limiting value** when  $x$  tends to **negative infinity**
  - For  $0 < a < 1$  this is the **limiting value** when  $x$  tends to **positive infinity**
- The graphs **do not have any minimum or maximum points**



Your notes



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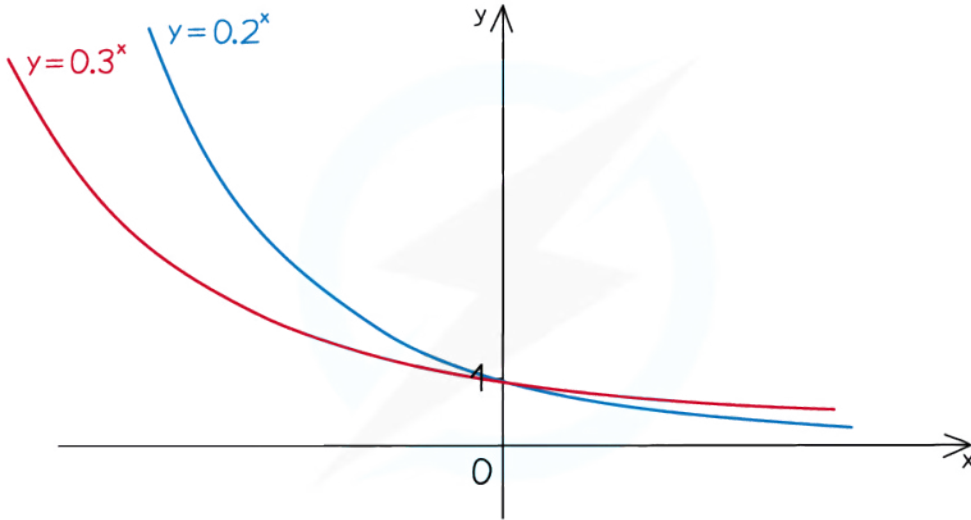


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## Logarithmic Functions & Graphs

### What is a logarithmic function?

- A **logarithmic function** is of the form  $f(x) = \log_a x, x > 0$
- Its **domain** is the set of all **positive real values**
  - You can't take a log of zero or a negative number
- Its **range** is set of **all real values**
- $\log_a x$  and  $a^x$  are **inverse** functions
- An important logarithmic function is  $f(x) = \ln x$ 
  - This is the natural logarithmic function  $\ln x \equiv \log_e x$
  - This is the inverse of  $e^x$ 
    - $\ln e^x = x$  and  $e^{\ln x} = x$
- Any logarithmic function can be written using  $\ln$ 
  - $\log_a x = \frac{\ln x}{\ln a}$  using the change of base formula

### What are the key features of logarithmic graphs?

- The graphs **do not have a y-intercept**
- The graphs have **one root** at (1, 0)
- The graphs **will always** pass through the point (a, 1)
- The graphs have a **vertical asymptote** at the y-axis:  $x = 0$
- The graphs **do not have any minimum or maximum points**



Your notes

### Worked example

The function  $f$  is defined by  $f(x) = \log_5 x$  for  $x > 0$ .

- a) Write down the inverse of  $f$ . Give your answer in the form  $e^{g(x)}$ .

Formula booklet

Exponents & logarithms	$a^x = b \Leftrightarrow x = \log_a b$	$a > 0, b > 0, a \neq 1$
------------------------	--	--------------------------

$$x = \log_5 y \Leftrightarrow y = 5^x$$

Formula booklet

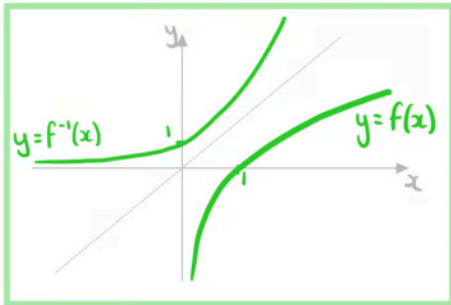
Exponential & logarithmic functions	$a^x = e^{x \ln a}$
-------------------------------------	---------------------

$$5^x = e^{x \ln 5}$$

$$f^{-1}(x) = e^{x \ln 5}$$

- b) Sketch the graphs of  $f$  and its inverse on the same set of axes.

$f$  and  $f^{-1}$  are reflections in line  $y=x$





Your notes

## 2.4.3 Solving Equations

### Solving Equations Analytically

How can I solve equations analytically where the unknown appears only once?

- These equations can be **solved by rearranging**
- For **one-to-one functions** you can just apply the **inverse**
  - Addition and subtraction are inverses
    - $y = x + k \Leftrightarrow x = y - k$
  - Multiplication and division are inverses
    - $y = kx \Leftrightarrow x = \frac{y}{k}$
  - Taking the reciprocal is a self-inverse
    - $y = \frac{1}{x} \Leftrightarrow x = \frac{1}{y}$
  - Odd powers and roots are inverses
    - $y = x^n \Leftrightarrow x = \sqrt[n]{y}$
    - $y = x^n \Leftrightarrow x = y^{\frac{1}{n}}$
  - Exponentials and logarithms are inverses
    - $y = a^x \Leftrightarrow x = \log_a y$
    - $y = e^x \Leftrightarrow x = \ln y$
- For **many-to-one functions** you will need to use your knowledge of the functions to find the **other solutions**
  - Even powers lead to positive and negative solutions
    - $y = x^n \Leftrightarrow x = \pm \sqrt[n]{y}$
  - Modulus functions lead to positive and negative solutions
    - $y = |x| \Leftrightarrow x = \pm y$
  - Trigonometric functions lead to infinite solutions using their symmetries
    - $y = \sin x \Leftrightarrow x = 2k\pi + \sin^{-1}y$  or  $x = (1 + 2k)\pi - \sin^{-1}y$
    - $y = \cos x \Leftrightarrow x = 2k\pi \pm \cos^{-1}y$
    - $y = \tan x \Leftrightarrow x = k\pi + \tan^{-1}y$
- Take care when you apply **many-to-one functions** to **both sides** of an equation as this can create **additional solutions** which are incorrect
  - For example: squaring both sides
    - $x + 1 = 3$  has one solution  $x = 2$
    - $(x + 1)^2 = 3^2$  has two solutions  $x = 2$  and  $x = -4$
- Always **check your solutions** by substituting back into the **original equation**



Your notes

### How can I solve equations analytically where the unknown appears more than once?

- Sometimes it is possible to **simplify expressions** to make the **unknown appear only once**
- **Collect all terms** involving  $x$  on **one side** and try to simplify into one term
  - For **exponents** use
    - $a^{f(x)} \times a^{g(x)} = a^{f(x) + g(x)}$
    - $\frac{a^{f(x)}}{a^{g(x)}} = a^{f(x) - g(x)}$
    - $(a^{f(x)})^{g(x)} = a^{f(x) \times g(x)}$
    - $a^{f(x)} = e^{f(x) \ln a}$
  - For **logarithms** use
    - $\log_a f(x) + \log_a g(x) = \log_a (f(x) \times g(x))$
    - $\log_a f(x) - \log_a g(x) = \log_a \left( \frac{f(x)}{g(x)} \right)$
    - $n \log_a f(x) = \log_a (f(x))^n$

### How can I solve equations analytically when the equation can't be simplified?

- Sometimes it is **not possible to simplify** equations
- Most of these equations **cannot be solved analytically**
- A **special case** that can be solved is where the equation can be **transformed into a quadratic** using a substitution
  - These will have **three terms** and involve the same type of function
- **Identify the suitable substitution** by considering which **function is a square of another**
  - For example: the following can be transformed into  $2y^2 + 3y - 4 = 0$ 
    - $2x^4 + 3x^2 - 4 = 0$  using  $y = x^2$
    - $2x + 3\sqrt{x} - 4 = 0$  using  $y = \sqrt{x}$
    - $\frac{2}{x^6} + \frac{3}{x^3} - 4 = 0$  using  $y = \frac{1}{x^3}$
    - $2e^{2x} + 3e^x - 4 = 0$  using  $y = e^x$
    - $2 \times 25^x + 3 \times 5^x - 4 = 0$  using  $y = 5^x$
    - $2^{2x+1} + 3 \times 2^x - 4 = 0$  using  $y = 2^x$
    - $2(x^3 - 1)^2 + 3(x^3 - 1) - 4 = 0$  using  $y = x^3 - 1$
- **To solve:**
  - Make the **substitution**  $y = f(x)$
  - **Solve the quadratic equation**  $ay^2 + by + c = 0$  to get  $y_1$  &  $y_2$
  - **Solve**  $f(x) = y_1$  and  $f(x) = y_2$ 
    - Note that some equations might have **zero or several solutions**



Your notes

### Can I divide both sides of an equation by an expression?

- When dividing by an expression you must consider whether the **expression could be zero**
- Dividing by an expression that could be zero could result in you **losing solutions to the original equation**
  - For example:  $(x + 1)(2x - 1) = 3(x + 1)$ 
    - If you divide both sides by  $(x + 1)$  you get  $2x - 1 = 3$  which gives  $x = 2$
    - However  $x = -1$  is also a solution to the original equation
- To ensure you **do not lose solutions** you can:
  - **Split the equation into two equations**
    - One where the dividing expression equals zero:  $x + 1 = 0$
    - One where the equation has been divided by the expression:  $2x - 1 = 3$
  - **Make the equation equal zero and factorise**
    - $(x + 1)(2x - 1) - 3(x + 1) = 0$
    - $(x + 1)(2x - 1 - 3) = 0$  which gives  $(x + 1)(2x - 4) = 0$
    - Set each factor equal to zero and solve:  $x + 1 = 0$  and  $2x - 4 = 0$

#### Examiner Tip

- A common mistake that students make in exams is applying functions to each term rather than to each side
  - For example: Starting with the equation  $\ln x + \ln(x - 1) = 5$  it would be incorrect to write  $e^{\ln x} + e^{\ln(x - 1)} = e^5$  or  $x + (x - 1) = e^5$
  - Instead it would be correct to write  $e^{\ln x + \ln(x - 1)} = e^5$  and then simplify from there



Your notes

### Worked example

Find the exact solutions for the following equations:

a)  $5 - 2\log_4 x = 0$ .

Rearrange using inverse functions

$$5 - 2\log_4 x = 0$$

$$2\log_4 x = 5$$

$$\log_4 x = \frac{5}{2}$$

$$x = 4^{\frac{5}{2}}$$

$$x = (\sqrt{4})^5$$

$$x = 32$$

$y = x - k \Leftrightarrow x = y + k$   
 $y = kx \Leftrightarrow x = \frac{y}{k}$   
 $y = \log_a x \Leftrightarrow x = a^y$   
 $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$

b)  $x = \sqrt{x+2}$ .

Square both sides (Many-to-one function)

$$x^2 = x + 2 \Rightarrow x^2 - x - 2 = 0$$

$$(x - 2)(x + 1) = 0 \Rightarrow x = 2 \text{ or } x = -1$$

Check whether each solution is valid

$x = 2$ : LHS = 2 RHS =  $\sqrt{2+2} = 2$  ✓

$x = -1$ : LHS = -1 RHS =  $\sqrt{-1+2} = 1$  ✗

$$x = 2$$

c)  $e^{2x} - 4e^x - 5 = 0$ .



Your notes

Notice  $e^{2x} = (e^x)^2$ , let  $y = e^x$

$$y^2 - 4y - 5 = 0 \Rightarrow (y+1)(y-5) = 0$$

$$y = -1 \text{ or } y = 5$$

Solve using  $y = e^x$

$e^x = -1$  has no solutions as  $e^x > 0$

$$e^x = 5 \quad \therefore x = \ln 5$$

$$x = \ln 5$$





Your notes

## Solving Equations Graphically

### How can I solve equations graphically?

- To solve  $f(x) = g(x)$ 
  - One method is to **draw the graphs**  $y = f(x)$  and  $y = g(x)$ 
    - The **solutions** are the **x-coordinates** of the points of **intersection**
  - Another method is to **draw the graph**  $y = f(x) - g(x)$  or  $y = g(x) - f(x)$ 
    - The **solutions** are the **roots (zeros)** of this graph
      - This method is sometimes quicker as it involves **drawing only one graph**

### Why do I need to solve equations graphically?

- Some equations **cannot be solved analytically**
  - **Polynomials** of degree higher than 4
    - $x^5 - x + 1 = 0$
  - Equations involving **different types of functions**
    - $e^x = x^2$

#### Examiner Tip

- On a calculator paper you are allowed to solve equations using your GDC unless the question asks for an algebraic method
- If your answer needs to be an exact value then you might need to solve analytically to get the exact value

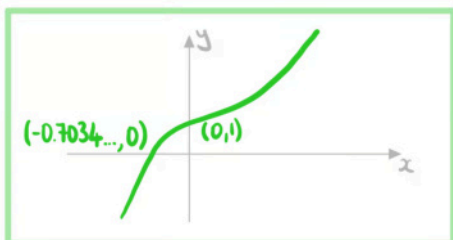


Your notes

 **Worked example**

- a) Sketch the graph  $y = e^x - x^2$ .

Sketch using GDC



- b) Hence find the solution to  $e^x = x^2$ .

$$e^x = x^2 \text{ when } e^x - x^2 = 0$$

Solution is the x-intercept of  $y = e^x - x^2$

$$x = -0.703 \text{ (3sf)}$$



Your notes

## 2.4.4 Modelling with Functions

### Modelling with Functions

#### What is a mathematical model?

- A **mathematical model** simplifies a real-world situation so it can be described using mathematics
  - The model can then be used to make predictions
- **Assumptions** about the situation are made in order to simplify the mathematics
- Models can be **refined** (improved) if further information is available or if the model is compared to real-world data

#### How do I set up the model?

- The question could:
  - give you the equation of the model
  - tell you about the relationship
    - It might say the relationship is linear, quadratic, etc
  - ask you to suggest a **suitable model**
    - Use your knowledge of each model
    - E.g. if it is compound interest then an exponential model is the most appropriate
- You may have to determine a **reasonable domain**
  - Consider real-life context
    - E.g. if dealing with hours in a day then  $0 \leq t < 24$
    - E.g. if dealing with physical quantities (such as length) then,  $x > 0$
  - Consider the **possible ranges**
    - If the outcome cannot be negative then you want to choose a domain which corresponds to a range with no negative values
    - **Sketching the graph** is helpful to determine a suitable domain

#### Which models might I need to use?

- You could be given any model and be expected to use it
- Common models include:
  - **Linear**
    - Arithmetic sequences
    - Linear regression
  - **Quadratic**
    - Projectile motion
    - The height of a cable supporting a bridge
    - Profit
  - **Exponential**
    - Geometric sequences
    - Exponential growth and decay
    - Compound interest

- **Logarithmic**
  - Richter scale for the magnitude of earthquakes
- **Rational**
  - Temperature of a cup of coffee
- **Trigonometric**
  - The depth of a tide



Your notes

### How do I use a model?

- You can use a model by substituting in values for the variable to **estimate outputs**
  - For example: Let  $h(t)$  be the height of a football  $t$  seconds after being kicked
    - $h(3)$  will be an estimate for the height of the ball 3 seconds after being kicked
- Given an **output** you can **form an equation** with the model to **estimate the input**
  - For example: Let  $P(n)$  be the profit made by selling  $n$  items
    - Solving  $P(n) = 100$  will give you an estimate for the number of items needing to be sold to make a profit of 100
- If your variable is **time** then substituting  $t = 0$  will give you the **initial value** according to the model
- Fully understand the **units for the variables**
  - If the units of  $P$  are measured in **thousand dollars** then  $P = 3$  represents \$3000
- Look out for **key words** such as:
  - Initially
  - Minimum/maximum
  - Limiting value

### What do I do if some of the parameters are unknown?

- A general method is to **form equations** by substituting in given values
  - You can form **multiple equations** and **solve them simultaneously** using your GDC
  - This method **works for all models**
- The **initial value** is the value of the function when the variable is 0
  - This is **normally one of the parameters** in the equation of the model



Your notes

### Worked example

The temperature,  $T^{\circ}\text{C}$ , of a cup of coffee is monitored. Initially the temperature is  $80^{\circ}\text{C}$  and 5 minutes later it is  $40^{\circ}\text{C}$ . It is suggested that the temperature follows the model:

$$T(t) = Ae^{kt} + 16, t \geq 0.$$

where  $t$  is the time, in minutes, after the coffee has been made.

- a) State the value of  $A$ .

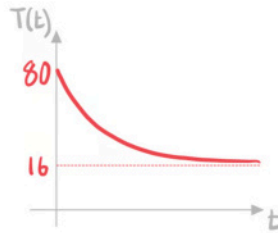
Initially temperature is  $80^{\circ}\text{C}$

$$T(0) = 80$$

$$Ae^{-k(0)} + 16 = 80$$

$$A + 16 = 80$$

$$A = 64$$



- b) Find the exact value of  $k$ .

$$t = 5, T = 40$$

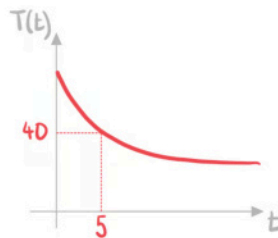
$$40 = 64e^{5k} + 16$$

$$64e^{5k} = 24$$

$$e^{5k} = \frac{3}{8}$$

$$5k = \ln \frac{3}{8}$$

$$k = \frac{1}{5} \ln \frac{3}{8}$$



- c) Find the time taken for the temperature of the coffee to reach  $30^{\circ}\text{C}$ .



Your notes

Find  $t$  such that  $T(t) = 30$

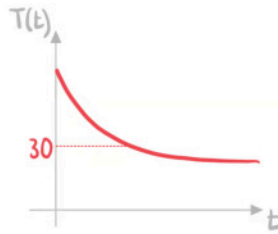
$$30 = 64e^{kt} + 16 \quad \text{Leave as } k \text{ until the end to save}$$

$$64e^{kt} = 14 \quad \text{writing } \frac{1}{5} \ln \frac{3}{8} \text{ each time}$$

$$e^{kt} = \frac{7}{32}$$

$$kt = \ln \frac{7}{32}$$

$$t = \frac{\ln \frac{7}{32}}{k} = \frac{\ln \frac{7}{32}}{\frac{1}{5} \ln \frac{3}{8}} = 7.7476..$$



**7.75 minutes (3sf)**