DP IB Maths: AA HL

Your notes

5.7 Basic Limits & Continuity

Contents

5.7.1 Basic Limits & Continuity

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Limits

What are limits in mathematics?

- When we consider a **limit** in mathematics we look at the tendency of a mathematical process as it approaches, but never quite reaches, an 'end point' of some sort
- We use a special limit notation to indicate this
  - For example $\lim_{x \to 3} f(x)$ denotes 'the limit of the function *f*(*x*) as *x* goes to (or approaches) 3'
    - I.e., what value (if any) *f*(*x*) gets closer and closer to as *x* takes on values closer and closer to 3
    - We are not concerned here with what value (if any) *f*(*x*) takes when *x* is equal to 3 – only with the behaviour of *f*(*x*) as *x* gets close to 3
- The sum of an infinite geometric sequence is a type of limit
  - When you calculate S_∞ for an infinite geometric sequence, what you are actually finding is $\lim_{n \to \infty} S_n$
    - I.e., what value (if any) the sum of the first *n* terms of the sequence gets closer and closer to as the number of terms (*n*) goes to infinity
    - The sum never actually reaches S_∞, but as more and more terms are included in the sum it gets closer and closer to that value
- In this section of the IB course we will be considering the limits of functions
  - This may include finding the limit at a point where the function is undefined
    - For example, $f(x) = \frac{\sin x}{x}$ is undefined when *x* = 0, but we might want to know how the function behaves as *x* gets closer and closer to zero
  - Or it may include finding the limit of a function *f*(*x*) as *x* gets infinitely big in the positive or negative direction
    - For this type of limit we write $\lim_{x \to \infty} f(x)$ or $\lim_{x \to -\infty} f(x)$ (the first one can also be written as $\lim_{x \to +\infty} f(x)$ to distinguish it from the second one)
  - These sorts of limits are often used to find the **asymptotes** of the graph of a function

How do I find a simple limit?

- STEP 1: To find $\lim_{x \to a} f(x)$ begin by substituting *a* into the function *f*(*x*)
  - If *f*(*a*) exists with a well-defined value, then that is also the value of the limit
  - For example, for $f(x) = \frac{x-1}{x}$ we may find the limit as *x* approaches 3 like this:

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$$\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x-1}{x} = \frac{3-1}{3} = \frac{2}{3}$$

- In this case, $\lim_{x \to 3} f(x)$ is simply equal to *f*(3)

- STEP 2: If *f*(*a*) does not exist, it may be possible to use algebra to simplify *f*(*x*) so that substituting *a* into the simplified function gives a well-defined value
  - In that case, the well-defined value at $x = a$ of the simplified version of the function is also the value of the limit of the function as *x* goes to *a*
  - For example, $f(x) = \frac{x^2}{x}$ is not defined at $x = 0$, but we may use algebra to find the limit as *x* approaches zero:

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{x^2}{x} = \lim_{x \to 0} \frac{x}{1} \text{ (cancelling the } x\text{'s)} = \frac{0}{1} = 0$$

    - Note that $f(x) = \frac{x^2}{x}$ and $g(x) = x$ are **not** the same function!
    - They are equal for all values of *x* except zero
    - But for $x = 0$, $g(0) = 0$ while $f(0)$ is undefined
    - However *f*(*x*) gets closer and closer to zero as *x* gets closer and closer to zero
- If neither of these steps gives a well-defined value for the limit you may need to consider more advanced techniques to evaluate the limit
  - For example **l'Hôpital's Rule** or using **Maclaurin series**

How do I find a limit to infinity?

- As *x* goes to $+\infty$ or $-\infty$, a typical function *f*(*x*) may **converge** to a well-defined value, or it may **diverge** to $+\infty$ or $-\infty$
  - Other behaviours are possible – for example $\lim_{x \to \infty} \sin x$ is simply undefined, because sin *x* continues to oscillate between 1 and -1 as *x* gets larger and larger
- There are two key results to be used here:
  - $\lim_{x \to \pm\infty} \frac{k}{x^n}$ **converges** to 0 for all $n > 0$ and all $k \in \mathbb{R}$
  - $\lim_{x \to +\infty} x^n$ **diverges** to $+\infty$ for all $n > 0$
    - $\lim_{x \to -\infty} x^n$ for $n > 0$ will need to be considered on a case-by-case basis, because of the differing behaviour of x^n for different values of *n* when *x* is negative
- STEP 1: If necessary, use algebra to rearrange the function into a form where one or the other of the key results above may be applied

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- STEP 2: Use the key results above to evaluate your limit
- For example:

$$\lim_{x\to\infty}\frac{3x^2-2x+1}{4x^2-x+2}=\lim_{x\to\infty}\frac{3-\frac{2}{x}+\frac{1}{x^2}}{4-\frac{1}{x}+\frac{2}{x^2}}=\frac{3-0+0}{4-0+0}=\frac{3}{4}$$

- Or:

$$\lim_{x\to+\infty}\frac{x^2+5x-2}{32x+3}=\lim_{x\to+\infty}\frac{x+5-\frac{2}{x}}{32+\frac{3}{x}}=\frac{(+\infty)+5-0}{32+0}=+\infty$$

  - I.e., the limit diverges to $+\infty$ (because $\frac{x^2+5x-2}{32x+3}$ it gets bigger and bigger without limit as *x* gets bigger and bigger)
- Remember that neither $\frac{0}{0}$ nor $\frac{\pm\infty}{\pm\infty}$ has a well-defined value!
  - If you attempt to evaluate a limit and get one of these two forms, you will need to try another strategy
  - This may just mean different or additional algebraic rearrangement
  - But it may also mean that you need to consider using **l'Hôpital's Rule** or **Maclaurin series** to evaluate the limit
- It is also worth remembering that if $\lim_{x\to\infty}f(x)=\infty$, then $\lim_{x\to\infty}\frac{k}{f(x)}=0$ for any non-zero $k\in\mathbb{R}$
  - This can be useful for example when evaluating the limits of functions containing exponentials
    - $\lim_{x\to\infty}e^{px}=\infty$ for any *p* > 0, so we immediately have $\lim_{x\to\infty}e^{-px}=\lim_{x\to\infty}\frac{1}{e^{px}}=0$ for *p* > 0
    - See the worked example below for a more involved version of this

Do limits ever have 'directions'?

- Yes they do!
- The notation $\lim_{x\to a^+}f(x)$ means 'the limit of *f*(*x*) as *x* **approaches *a* from above**'
  - I.e., this is the limit as *x* comes 'down' towards *a*, only considering the function's behaviour for values of *x* that are greater than *a*
- The notation $\lim_{x\to a^-}f(x)$ means 'the limit of *f*(*x*) as *x* **approaches *a* from below**'
  - I.e., this is the limit as *x* comes 'up' towards *a*, only considering the function's behaviour for values of *x* that are less than *a*
- One place these sorts of limits appear is for functions defined piecewise

- In this case the limits 'from above' and 'from below' may well be different for values of *x* at which the different 'pieces' of the function are joined
- But also be aware of a situation like the following, where the limits from above and below may also be different:
    - $\lim_{x \to 0^+} \frac{1}{x} = +\infty$ (because $\frac{1}{x} > 0$ for $x > 0$, with $\frac{1}{x}$ becoming bigger and bigger in the positive direction as *x* gets closer and closer to zero 'from above')
    - $\lim_{x \to 0^-} \frac{1}{x} = +\infty$ (because $\frac{1}{x} < 0$ for $x < 0$, with $\frac{1}{x}$ becoming bigger and bigger in the negative direction as *x* gets closer and closer to zero 'from below')
    - The graph of $y = \frac{1}{x}$ shows this limiting behaviour as *x* approaches zero from the two different directions

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Worked example

a) Consider the function

$$f(x) = \frac{3 - 4x - 5x^4}{2x^4 + x^3 + 7},$$

find $\lim_{x \to \infty} f(x)$.

$$\frac{3 - 4x - 5x^4}{2x^4 + x^3 + 7} \cdot \frac{1/x^4}{1/x^4} = \frac{\frac{3}{x^4} - \frac{4}{x^3} - 5}{2 + \frac{1}{x} + \frac{7}{x^4}}$$

$$\lim_{x \to \infty} \frac{3 - 4x - 5x^4}{2x^4 + x^3 + 7} = \lim_{x \to \infty} \frac{\frac{3}{x^4} - \frac{4}{x^3} - 5}{2 + \frac{1}{x} + \frac{7}{x^4}}$$

$$= \frac{0 - 0 - 5}{2 + 0 + 0} = \boxed{-\frac{5}{2}}$$

b) Consider the function

$$g(x) = \begin{cases} \dfrac{1 - 5x}{x^2}, & x < 5 \\ x^2 - 4x - 6, & x \geq 5 \end{cases}$$

find (i) $\lim_{x \to 5^-} g(x)$, and (ii) $\lim_{x \to 5^+} g(x)$.

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(i) For $\lim_{x \to 5^-}$, we only consider $x < 5$

$$\lim_{x \to 5^-} g(x) = \lim_{x \to 5} \frac{1-5x}{x^2}$$

$$= \frac{1-5(5)}{(5)^2} = -\frac{24}{25}$$

(ii) For $\lim_{x \to 5^+}$, we only consider $x > 5$

$$\lim_{x \to 5^+} g(x) = \lim_{x \to 5} (x^2 - 4x - 6)$$

$$= (5)^2 - 4(5) - 6 = -1$$

c) Consider the function

$$h(x) = \frac{2e^{3x} - 3}{4 - 5e^{3x}}$$

find $\lim_{x \to \infty} h(x)$.

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$$\frac{2e^{3x}-3}{4-5e^{3x}} \cdot \frac{1/e^{3x}}{1/e^{3x}} = \frac{2-\frac{3}{e^{3x}}}{\frac{4}{e^{3x}}-5}$$

$$\lim_{x\to\infty} \frac{2e^{3x}-3}{4-5e^{3x}} = \lim_{x\to\infty} \frac{2-\frac{3}{e^{3x}}}{\frac{4}{e^{3x}}-5}$$

$$= \frac{2-0}{0-5} = \boxed{-\frac{2}{5}}$$

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Continuity & Differentiability

What does it mean for a function to be continuous at a point?

- If a function is **continuous** at a point then the graph of the function does not have any 'holes' or any sudden 'leaps' or 'jumps' at that point
  - One way to think about this is to imagine sketching the graph
    - So long as you can sketch the graph without lifting your pencil from the paper, then the function is continuous at all the points that your sketch goes through
    - But if you would have to lift your pencil off the paper at some point and continue drawing the graph from another point, then the function is not continuous at any such points where the function 'jumps'

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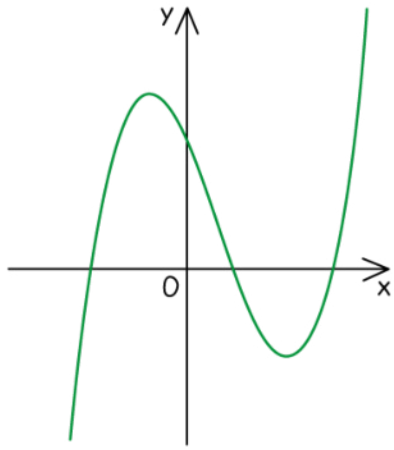
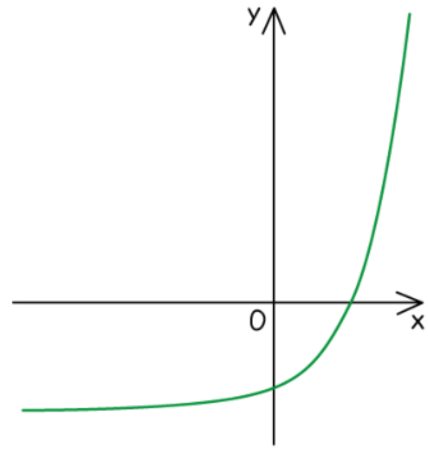
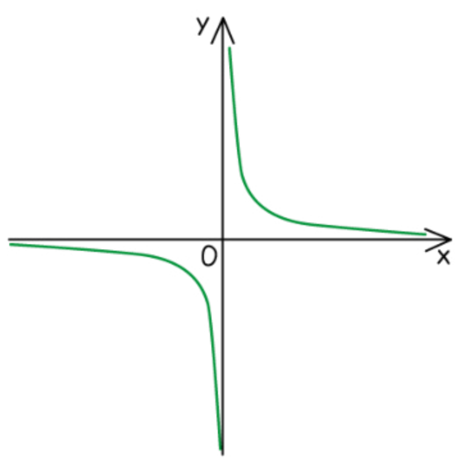
CONTINUOUS AT ALL POINTS

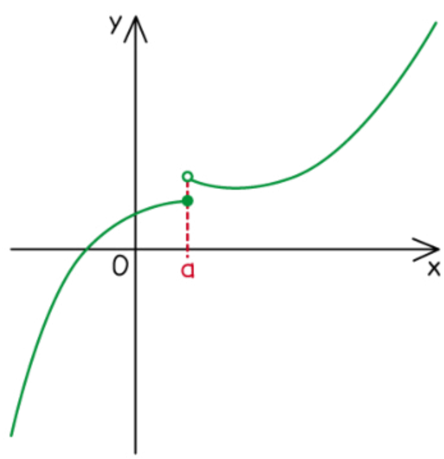




NOT CONTINUOUS AT x = 0



NOT CONTINUOUS AT x = a

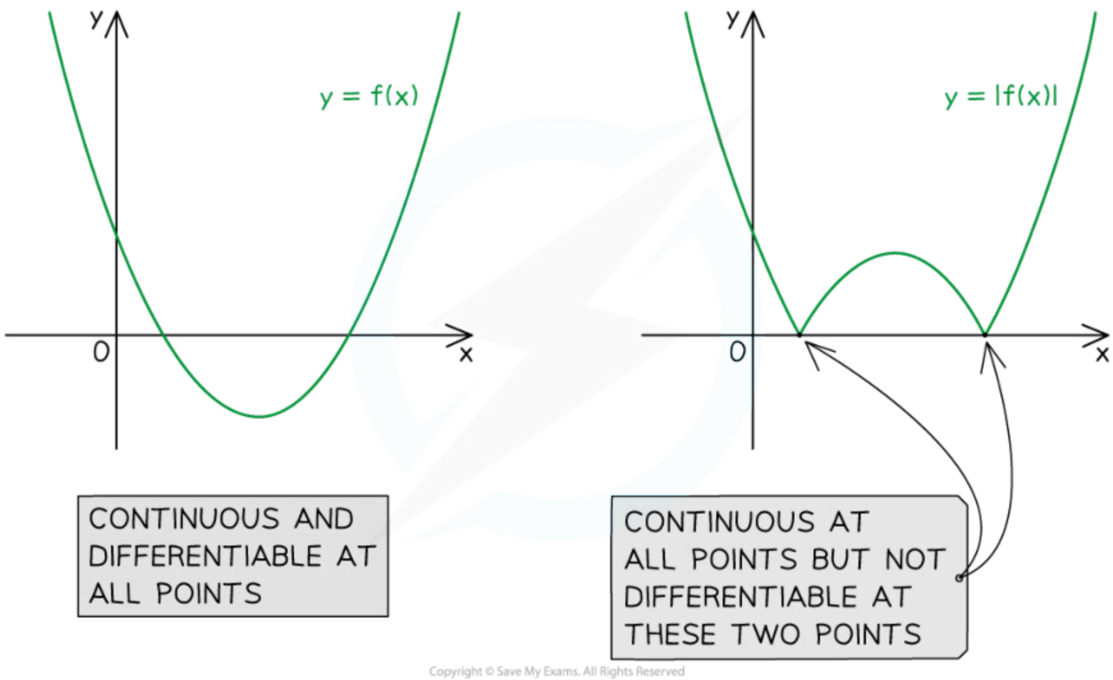


- There are two main ways a function can fail to be continuous at a point:
  - If the function is not defined for a particular value of *x* then it is not continuous at that value of *x*
    - For example, $f(x) = \frac{1}{x}$ is not continuous at $x = 0$
  - If the function is defined for a particular value of *x*, but then the value of the function 'jumps' as *x* moves away from that x value in the positive or negative direction, then the function is not continuous at that value of *x*

    - This type of discontinuity can occur in a piecewise function, for example, where the different pieces of the function's graph don't 'join up'
- You can use limits to show that a function is continuous at a point
  - Let $f(x)$ be a function defined at $x = a$, such that $f(a) = b$
    - If $\lim_{x \to a^-} f(x) = b$ and $\lim_{x \to a^+} f(x) = b$, then $f(x)$ is continuous at $x = a$
    - If either of those limits is not equal to b, then $f(x)$ is not continuous at $x = a$
  - This is a slightly more formal way of expressing the 'you don't have to lift your pencil from the paper' idea!

What does it mean for a function to be differentiable at a point?

- We say that a function $f(x)$ is **differentiable** at a point with x-coordinate x_0, if the derivative $f'(x)$ exists and has a well-defined value $f'(x_0)$ at that point
- To be differentiable at a point a function has to be continuous at that point
  - So if a function is not continuous at a point, then it is also not differentiable at that point
- But continuity by itself does not guarantee differentiability
  - This means that differentiability is a stronger condition than continuity
  - If a function is differentiable at a point, then the function is also continuous at that point
  - But a function may be continuous at a point without being differentiable at that point
  - This means there are functions that are continuous everywhere but are not differentiable everywhere
- In addition to being continuous a point, differentiability also requires that the function be **smooth** at that point
  - 'Smooth' means that the graph of the function does not have any 'corners' or sudden changes of direction at the point
  - An obvious example of a function that is not smooth at certain points is a modulus function $|f(x)|$ at any values of x where $f(x)$ changes sign from positive to negative
    - At any such point a modulus function will not be differentiable



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Examiner Tip

- On the exam you will not usually be asked to test a function for **continuity** at a point
  - You should however be familiar with the basic ideas about continuity outlined above
- On the exam you will not be asked to test a function for **differentiability** at a point
  - You should however be familiar with the basic ideas about differentiability and its relationship with continuity as outlined above

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Worked example

Consider the function f defined by

$$f(x)=\begin{cases} x^2-2x-1, & x<3 \\ 2 & x=3 \\ \dfrac{x+2}{2}, & x>3 \end{cases}$$

a) use limits to show that f is not continuous at $x=3$.

$f(3) = 2$

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3} (x^2-2x-1) = (3)^2 - 2(3) - 1 = 2$

↑ The limit 'from below'

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3} \left(\frac{x+2}{2}\right) = \frac{3+2}{2} = \frac{5}{2}$

↑ The limit 'from above'

$\lim_{x\to 3^+} f(x) \neq f(3)$, therefore f is not continuous at $x = 3$.

b) Hence explain why f cannot be differentiable at $x=3$.

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In order to be differentiable at a point, a function must be continuous at that point.

f is not continuous at $x = 3$, therefore it cannot be differentiable at $x = 3$.