DP IB Maths: AA HL

Your notes

5.10 Differential Equations

Contents

5.10.1 Numerical Solutions to Differential Equations

First Order Differential Equations

What is a differential equation?

- A **differential equation** is simply an equation that contains derivatives
  - For example $\frac{dy}{dx} = 12xy^2$ is a differential equation
  - And so is $\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 7x = 5\sin t$

What is a first order differential equation?

- A **first order differential equation** is a differential equation that contains first derivatives but no second (or higher) derivatives
  - For example $\frac{dy}{dx} = 12xy^2$ is a first order differential equation
  - But $\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 7x = 5\sin t$ is **not** a first order differential equation, because it contains the second derivative $\frac{d^2x}{dt^2}$

Wait – haven't I seen first order differential equations before?

- Yes you have!
  - For example $\frac{dy}{dx} = 3x^2$ is also a first order differential equation, because it contains a first derivative and no second (or higher) derivatives
  - But for that equation you can just integrate to find the solution $y = x^3 + c$ (where c is a constant of integration)
- In this section of the course you learn how to solve differential equations that can't just be solved right away by integrating

Euler's Method: First Order

What is Euler's method?

- **Euler's method** is a numerical method for finding approximate solutions to differential equations
- It treats the derivatives in the equation as being constant over short 'steps'
- The accuracy of the Euler's Method approximation can be improved by making the step sizes smaller

How do I use Euler's method with a first order differential equation?

- STEP 1: Make sure your differential equation is in $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$ form
- STEP 2: Write down the recursion equations using the formulae $y_{n+1} = y_n + h \times f(x_n, y_n)$ and $x_{n+1} = x_n + h$ from the exam formula booklet
  - *h* in those equations is the **step size**
  - the exam question will usually tell you the correct value of *h* to use
- STEP 3: Use the recursion feature on your GDC to calculate the Euler's method approximation over the correct number of steps
  - the values for x_0 and y_0 will come from the boundary conditions given in the question

Examiner Tip

- Be careful with letters – in the equations in the exam, and in your GDC's recursion calculator, the variables may not be *x* and *y*
- If an exam question asks you how to improve an Euler's method approximation, the answer will almost always have to do with decreasing the step size!

Worked example

Consider the differential equation $\frac{dy}{dx} + y = x + 1$ with the boundary condition $y(0) = 0.5$.

a) Apply Euler's method with a step size of $h = 0.2$ to approximate the solution to the differential equation at $x = 1$.

Euler's method	$y_{n+1} = y_n + h \times f(x_n, y_n)$; $x_{n+1} = x_n + h$	h is a constant (step length)

from formula booklet

STEP 1: $\frac{dy}{dx} = \underbrace{x - y + 1}_{f(x,y)}$

STEP 2: $y_{n+1} = y_n + \underbrace{0.2}_{h \text{ (from question)}} \times \underbrace{(x_n - y_n + 1)}_{f(x_n,\, y_n)}$ $\quad x_{n+1} = x_n + 0.2$

STEP 3: We need to get x from 0 to 1, so we will need $\frac{1-0}{0.2} = 5$ steps.

$y(0) = 0.5$

n	x_n	y_n
0	0	0.5
1	0.2	0.6
2	0.4	0.72
3	0.6	0.856
4	0.8	1.0048
5	1	1.16384

from GDC

$$\boxed{y(1) = 1.16 \text{ (3 s.f.)}}$$

b) Explain how the accuracy of the approximation in part (a) could be improved.

Make the step size smaller.

5.10.2 Analytical Solutions to Differential Equations

Separation of Variables

What is separation of variables?

- **Separation of variables** can be used to solve certain types of first order differential equations
- Look out for equations of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)h(y)$
  - i.e. $\frac{\mathrm{d}y}{\mathrm{d}x}$ is a function of x multiplied by a function of y
  - be careful – the 'function of x' $g(x)$ may just be a constant!
    - For example in $\frac{\mathrm{d}y}{\mathrm{d}x} = 6y$, $g(x) = 6$ and $h(y) = y$
- If the equation is in that form you can use separation of variables to try to solve it
- If the equation is not in that form you will need to use another solution method

How do I solve a differential equation using separation of variables?

- STEP 1: Rearrange the equation into the form $\left(\frac{1}{h(y)}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)$
- STEP 2: Take the integral of both sides to change the equation into the form

$$\int \frac{1}{h(y)}\ \mathrm{d}y = \int g(x)\ \mathrm{d}x$$

  - You can think of this step as 'multiplying the $\mathrm{d}x$ across and integrating both sides'
    - Mathematically that's not *quite* what is actually happening, but it will get you the right answer here!
- STEP 3: Work out the integrals on both sides of the equation to find the **general solution** to the differential equation
  - Don't forget to include a constant of integration
    - Although there are two integrals, you only need to include one constant of integration
  - Look out for integrals that require you to use **partial fractions** to solve them
    - See 'Integrating with Partial Fractions' in 5.9 Advanced Integration
- STEP 4: Use any boundary or initial conditions in the question to work out the value of the integration constant
- STEP 5: If necessary, rearrange the solution into the form required by the question

Your notes

Examiner Tip

- Be careful with letters – the equation on an exam may not use x and y as the variables
- Unless the question asks for it, you don't have to change your solution into $y = f(x)$ form – sometimes it might be more convenient to leave your solution in another form

Worked example

For each of the following differential equations, either (i) solve the equation by using separation of variables giving your answer in the form $y = f(x)$, or (ii) state why the equation may not be solved using separation of variables.

a) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{e}^x + 4x}{3y^2}$.

STEP 1 : $3y^2\frac{dy}{dx} = e^x + 4x$ $\quad g(x) = e^x + 4x \quad h(y) = \frac{1}{3y^2}$

STEP 2 : $\int 3y^2\,dy = \int (e^x + 4x)\,dx$

STEP 3 : $y^3 = e^x + 2x^2 + c$ ← Don't forget constant of integration

STEP 4 : No boundary conditions given, so skip step

STEP 5 : $\boxed{y = \sqrt[3]{e^x + 2x^2 + c}}$ $\quad y = f(x)$

b) $\frac{\mathrm{d}y}{\mathrm{d}x} = 4xy - 2\ln x$.

$4xy - 2\ln x$ is not of the form $g(x)h(y)$, so it may not be solved using separation of variables.

c) $\frac{\mathrm{d}y}{\mathrm{d}x} = 2y^2 + 2y$, given that $y = 2$ when $x = 0$.

STEP 1: $\frac{1}{y^2+y}\frac{dy}{dx} = 2$ $\quad g(x) = 2 \quad h(y) = y^2 + y$

STEP 2: $\int \frac{1}{y^2+y}\,dy = \int 2\,dx$

STEP 3: $\int \frac{1}{y^2+y}\,dy = \int \overbrace{\left(\frac{1}{y} - \frac{1}{y+1}\right)}^{\text{partial fractions}} dy = \ln\left|\frac{y}{y+1}\right| + c$

$\ln\left|\frac{y}{y+1}\right| = 2x + c$ ← Don't forget constant of integration

← $y = 2$ when $x = 0$

STEP 4: $\ln\left|\frac{2}{2+1}\right| = 2(0) + c \implies c = \ln\left(\frac{2}{3}\right)$

STEP 5: For the boundary condition $y = 2$, $\frac{y}{y+1} > 0$.
Therefore we can drop the modulus sign from $\left|\frac{y}{y+1}\right|$.

$$\frac{y}{y+1} = e^{2x + \ln(2/3)} = \left(e^{2x}\right)\left(e^{\ln(2/3)}\right) = \frac{2}{3}e^{2x}$$

$$\implies \boxed{y = \frac{2e^{2x}}{3 - 2e^{2x}}} \quad y = f(x)$$

Your notes

Homogeneous Differential Equations

What is a homogeneous first order differential equation?

- If a first order differential equation can be written in the form $\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right)$ then it is said to be **homogeneous**

How do I solve a homogeneous first order differential equation?

- These equations can be solved using the substitution $v = \frac{y}{x} \Leftrightarrow y = vx$
- STEP 1: If necessary, rearrange the equation into the form $\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right)$
- STEP 2: Replace all instances of $\frac{y}{x}$ in your equation with *v*
- STEP 3: Use the product rule and implicit differentiation to replace $\frac{\mathrm{d}y}{\mathrm{d}x}$ in your equation with $v + x\frac{\mathrm{d}v}{\mathrm{d}x}$
  - This is because $y = vx \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(vx) = v\frac{\mathrm{d}}{\mathrm{d}x}(x) + x\frac{\mathrm{d}}{\mathrm{d}x}(v) = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$
- STEP 4: Solve your new differential equation to find the solution in terms of *v* and *x*
  - You may need to use other methods for differential equations, such as **separation of variables**, at this stage
- STEP 5: Substitute $v = \frac{y}{x}$ into the solution from Step 4, in order to find the solution in terms of *y* and *x*

What else should I know about solving homogeneous first order differential equations?

- After finding the solution in terms of *y* and *x* you may be asked to do other things with the solution
  - For example you may be asked to find the solution corresponding to certain initial or boundary conditions
  - Or you may be asked to express your answer in a particular form, such as *y* = *f*(*x*)
- It is sometimes possible to solve differential equations that are *not* homogeneous by using the substitution $v = \frac{y}{x}$
  - For such a situation in an exam question, you would be told explicitly to use the substitution
  - You would not be expected to know that you could use the substitution in a case where the differential equation was not homogeneous

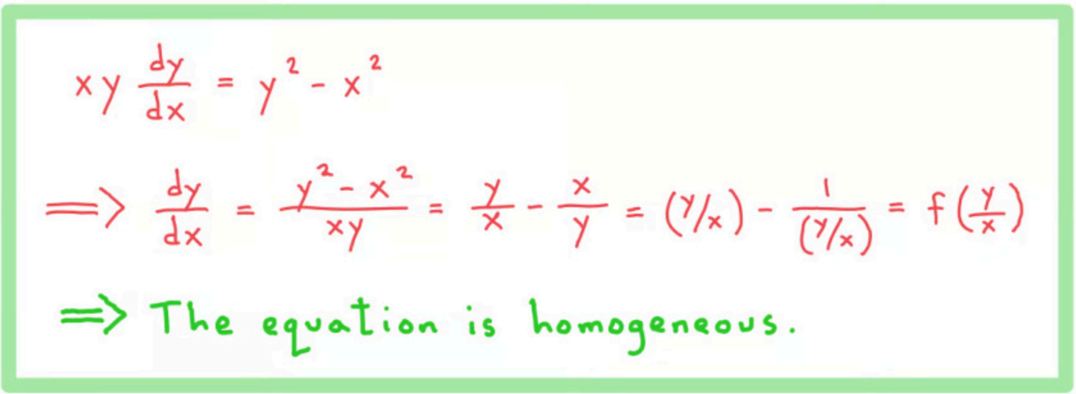
Examiner Tip

- Unless the question asks for it, you don't have to change your solution into $y = f(x)$ form – sometimes it might be more convenient to leave your solution in another form

Worked example

Consider the differential equation $xy\dfrac{\mathrm{d}y}{\mathrm{d}x} = y^2 - x^2$ where $y = 3$ when $x = 1$.

a) Show that the differential equation is homogeneous.



$$xy\frac{dy}{dx} = y^2 - x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2 - x^2}{xy} = \frac{y}{x} - \frac{x}{y} = (y/x) - \frac{1}{(y/x)} = f\left(\frac{y}{x}\right)$$

$\Rightarrow$ The equation is homogeneous.

b) Use the substitution $v = \dfrac{y}{x}$ to solve the differential equation with the given boundary condition.

STEP 1: $\frac{dy}{dx} = (y/x) - \frac{1}{(y/x)}$ } from part (a)

STEP 2: Let $v = \frac{y}{x}$. Then $\frac{dy}{dx} = v - \frac{1}{v}$.

STEP 3: And $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

So $v + x\frac{dv}{dx} = v - \frac{1}{v} \Rightarrow x\frac{dv}{dx} = -\frac{1}{v}$

STEP 4: $\int 2v\,dv = \int \frac{-2}{x}\,dx$ Separation of variables

$v^2 = -2\ln|x| + c = c - \ln(x^2)$ Don't forget constant of integration

STEP 5: $\left(\frac{y}{x}\right)^2 = c - \ln(x^2) \Rightarrow y^2 = x^2\left(c - \ln(x^2)\right)$

Now use the boundary condition

And $y(1) = 3$, so $(3)^2 = (1)^2\left(c - \ln(1^2)\right) \Rightarrow c = 9$

$$\boxed{y^2 = x^2\left(9 - \ln(x^2)\right)}$$

Question doesn't ask for solution to be in $y = f(x)$ form, so it's easiest just to keep it like this.

Integrating Factor

What is an integrating factor?

- An **integrating factor** can be used to solve a differential equation that can be written in the **standard form** $\frac{\mathrm{d}y}{\mathrm{d}x}+p(x)y=q(x)$
  - Be careful – the 'functions of *x*' $p(x)$ and $q(x)$ may just be constants!
    - For example in $\frac{\mathrm{d}y}{\mathrm{d}x}+6y=\mathrm{e}^{-2x}$, $p(x) = 6$ and $q(x) = \mathrm{e}^{-2x}$
    - While in $\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{y}{2x}=12$, $p(x)=\frac{1}{2x}$ and $q(x) = 12$
- For an equation in standard form, the integrating factor is $\mathrm{e}^{\int p(x)\ \mathrm{d}x}$

How do I use an integrating factor to solve a differential equation?

- STEP 1: If necessary, rearrange the differential equation into standard form
- STEP 2: Find the integrating factor
  - Note that you don't need to include a constant of integration here when you integrate $\int p(x)\,\mathrm{d}x$
- STEP 3: Multiply both sides of the differential equation by the integrating factor
- This will turn the equation into an **exact differential equation** of the form $\frac{\mathrm{d}}{\mathrm{d}x}\left(y\mathrm{e}^{\int p(x)\ \mathrm{d}x}\right)=q(x)\mathrm{e}^{\int p(x)\ \mathrm{d}x}$
- STEP 4: Integrate both sides of the equation with respect to *x*
  - The left side will automatically integrate to $y\mathrm{e}^{\int p(x)\ dx}$
  - For the right side, integrate $\int q(x)\mathrm{e}^{\int p(x)\mathrm{d}x}\mathrm{d}x$ using your usual techniques for integration
  - Don't forget to include a constant of integration
    - Although there are two integrals, you only need to include one constant of integration
- STEP 5: Rearrange your solution to get it in the form $y = f(x)$

What else should I know about using an integrating factor to solve differential equations?

- After finding the general solution using the steps above you may be asked to do other things with the solution
  - For example you may be asked to find the solution corresponding to certain initial or boundary conditions

Your notes

Your notes

Worked example

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 2xy + 5\mathrm{e}^{x^2}$ where $y = 7$ when $x = 0$.

Use an integrating factor to find the solution to the differential equation with the given boundary condition.

Integrating factor for $y' + P(x)y = Q(x)$	$e^{\int P(x)\mathrm{d}x}$

} from formula booklet

STEP 1: $\frac{dy}{dx} - 2xy = 5e^{x^2}$ $\quad p(x) = -2x \quad q(x) = 5e^{x^2}$

STEP 2: $e^{\int -2x\,dx} = e^{-x^2}$

STEP 3: $\left(\frac{dy}{dx} - 2xy\right)e^{-x^2} = \left(5e^{x^2}\right)e^{-x^2}$

$$e^{-x^2}\frac{dy}{dx} - 2xye^{-x^2} = 5$$

$$\frac{d}{dx}\left(ye^{-x^2}\right) = 5$$

Exact differential equation

STEP 4: $ye^{-x^2} = \int 5\,dx = 5x + c$

Don't forget constant of integration

STEP 5: $y = e^{x^2}(5x + c)$

Now use the boundary condition

$$7 = e^0(5(0) + c) \implies c = 7$$

$$\boxed{y = e^{x^2}(5x + 7)}$$

5.10.3 Modelling with Differential Equations

Modelling with Differential Equations

Why are differential equations used to model real-world situations?

- A **differential equation** is an equation that contains one or more derivatives
- Derivatives deal with rates of change, and with the way that variables change with respect to one another
- Therefore differential equations are a natural way to model real-world situations involving change
  - Most frequently in real-world situations we are interested in how things change over time, so the derivatives used will usually be with respect to time t

How do I set up a differential equation to model a situation?

- An exam question may require you to create a differential equation from information provided
- The question will provide a context from which the differential equation is to be created
- Most often this will involve the rate of change of a variable being proportional to some function of the variable
    - For example, the rate of change of a population of bacteria, P, at a particular time may be proportional to the size of the population at that time
- The expression 'rate of' ('rate of change of…', 'rate of growth of…', etc.) in a modelling question is a strong hint that a differential equation is needed, involving derivatives with respect to time t
    - So with the bacteria example above, the equation will involve the derivative $\frac{\mathrm{d}P}{\mathrm{d}t}$
- Recall the basic equation of proportionality
  - If y is proportional to x, then $y = kx$ for some **constant of proportionality** k
    - So for the bacteria example above the differential equation needed would be $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$
  - The precise value of k will generally not be known at the start, but will need to be found as part of the process of solving the differential equation
  - It can often be useful to assume that $k > 0$ when setting up your equation
    - In this case, $-k$ will be used in the differential equation in situations where the rate of change is expected to be negative
    - So in the bacteria example, if it were known that the population of bacteria was decreasing, then the equation could instead be written $\frac{\mathrm{d}P}{\mathrm{d}t} = -kP$

Worked example

a) In a particular pond, the rate of change of the area covered by algae, A, at any time t is directly proportional to the square root of the area covered by algae at that time. Write down a differential equation to model this situation.

$\frac{dA}{dt} = k\sqrt{A}$ (where k is a constant of proportionality)

b) Newton's Law of Cooling states that the rate of change of the temperature of an object, T, at any time t is proportional to the difference between the temperature of the object and the ambient temperature of its surroundings, T_a, at that time. Assuming that the object starts off warmer than its surroundings, write down the differential equation implied by Newton's Law of Cooling.

The object is assumed to be warmer than its surroundings, so $T - T_a > 0$

$$\frac{dT}{dt} = -k(T - T_a)$$

(where $k>0$ is a constant of proportionality)

We expect the temperature to be decreasing, so $-k$ in the equation combined with $k>0$ assures that $\frac{dT}{dt}$ is negative.

The Logistic Equation

What is the logistic equation?

- The differential equation $\frac{\mathrm{d}N}{\mathrm{d}t} = kN$ is a very simple example of a model in which the rate of change of a population at any moment in time is dependent on the size of the population (*N*) at that time
  - The solution is $N = A\mathrm{e}^{kt}$ (where $A > 0$ is a constant)
  - If $k > 0$, this represents unlimited exponential growth of the variable *N*
- In many real-world contexts (for example when considering populations of living organisms), unlimited growth is not a realistic modelling assumption
  - For reproducing populations it is logical to assume that the rate of change of the population will be dependent on the size of the population (more rabbits means more production of baby rabbits!)
  - But there are generally limiting factors on populations that prevent them from growing without limits
    - For example, availability of food or other resources, or the presence of predators or other threats, may limit the population that can exist in a given area
- A **logistic equation** incorporates such limiting factors into the model, and therefore can provide a more realistic model for real-world populations
- The **standard logistic equation** is of the form

$$\frac{\mathrm{d}N}{\mathrm{d}t} = kN(a - N)$$

  - *t* represents the time (since the moment defined as $t = 0$) that the population has been growing
  - *N* represents the size of the population at time *t*
  - $k \in \mathbb{R}$ is a constant determining the relative rate of population growth
    - For the models dealt with here it is most common to have $k > 0$, with a larger value of *k* representing a faster rate of change
  - $a \in \mathbb{R}$ is a constant that places a limit on the maximum size to which the population *N* can grow
    - For a population model it can be assumed that $a > 0$
    - For $k > 0$ and an initial population N_0 such that $0 < N_0 < a$, the population *N* will grow and will converge to the value *a* as time *t* increases
    - For $k < 0$ and an initial population N_0 such that $0 < N_0 < a$, the population *N* will shrink and will converge to the value *a* as time *t* increases
- There are other forms of logistic equation
  - The exact form of the logistic equation you are to use will always be given in an exam question

How do I solve problems that involve a logistic equation model?

- Solving the differential equation will generally involve the technique of **separation of variables**
  - Usually this will also involve rearranging one of the integrals using **partial fractions** (see the worked example below for an example)
- You will usually be given 'boundary conditions' specific to the context of the problem
  - For example, you may be told the initial population at time $t = 0$

  - These conditions will allow you to work out the exact value of any integrating constants that occur while solving the differential equation
- You will need to take account of the context of the question in answering the question or in commenting on the model used

Worked example

A group of ecologists are studying a population of rabbits on a particular island. The population of rabbits, *N*, on the island is modelled by the logistic equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = 0.0012N(1500 - N)$$

where *t* represents the time in years since the ecologists began their study. At the time the study begins there are 300 rabbits on the island.

a)

Show that the population of rabbits at time *t* years is given by $N = \dfrac{1500\mathrm{e}^{1.8t}}{4 + \mathrm{e}^{1.8t}}$.

Your notes

$$\int \frac{1}{N(1500-N)}\,dN = 0.0012\int dt$$

Separation of variables

$$\int \left(\frac{1}{N} - \frac{1}{N-1500}\right) dN = 1.8\int dt$$

Partial fractions

$$\ln\left|\frac{N}{N-1500}\right| = 1.8t + c$$

Don't forget constant of integration

$$A = e^{c}$$

$$\frac{N}{1500-N} = Ae^{1.8t}$$

$$0 < N < 1500, \text{ so } \left|\frac{N}{N-1500}\right| = -\frac{N}{N-1500} = \frac{N}{1500-N}$$

$$N(0) = 300, \text{ so } \frac{300}{1500-300} = Ae^{1.8(0)} \Rightarrow A = \frac{1}{4}$$

Use initial condition

$$\Rightarrow \frac{N}{1500-N} = \frac{1}{4}e^{1.8t} \Rightarrow \frac{4N}{1500-N} = e^{1.8t}$$

$$\Rightarrow \boxed{N = \frac{1500e^{1.8t}}{4+e^{1.8t}}}$$

Rearrange

b) Find the population of rabbits that the model predicts will be on the island two years after the beginning of the study.

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$$N(2) = \frac{1500e^{1.8(2)}}{4+e^{1.8(2)}} = \frac{1500e^{3.6}}{4+e^{3.6}} = 1352.210\ldots$$

$$\boxed{1352 \text{ rabbits}}$$ Round to nearest rabbit

c) Determine the maximum size that the model predicts the population of rabbits can grow to. Justify your answer by an appropriate analysis of the equation in part (a).

Take limit to determine long-term behaviour :

$$\lim_{t\to\infty} \frac{1500e^{1.8t}}{4+e^{1.8t}} = \lim_{t\to\infty} \frac{1500}{\frac{4}{e^{1.8t}}+1} = \frac{1500}{0+1} = 1500$$

$$\boxed{\text{1500 rabbits is the maximum population predicted by the model.}}$$