DP IB Maths: AA HL

Your notes

1.1 Number & Algebra Toolkit

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1.1.1 Standard Form

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Standard Form

Standard form (sometimes called **scientific notation** or **standard index form**) gives us a way of writing very big and very small numbers using powers of 10.

Why use standard form?

- Some numbers are too big or too small to write easily or for your calculator to display at all
  - Imagine the number 50^{50}, the answer would take 84 digits to write out
  - Try typing 50^{50} into your calculator, you will see it displayed in **standard form**
- Writing very big or very small numbers in standard form allows us to:
  - Write them more neatly
  - Compare them more easily
  - Carry out calculations more easily
- Exam questions could ask for your answer to be written in standard form

How is standard form written?

- In standard form numbers are always written in the form $a \times 10^k$ where a and k satisfy the following conditions:
  - $1 \le a < 10$
    - So there is one non – zero digit before the decimal point
  - $k \in \mathbb{Z}$
    - So k must be an integer
  - $k > 0$ for large numbers
    - How many times a is multiplied by 10
  - $k < 0$ for small numbers
    - How many times a is divided by 10

How are calculations carried out with standard form?

- Your GDC will display large and small numbers in standard form when it is in normal mode
  - Your GDC may display standard form as *aEn*
    - For example, 2.1×10^{-5} will be displayed as $2.1\text{E}-5$
    - If so, be careful to **rewrite the answer given in the correct form**, you will not get marks for copying directly from your GDC
- Your GDC will be able to carry out calculations in standard form
  - If you put your GDC into scientific mode it will automatically convert numbers into standard form
    - Beware that your GDC may have more than one mode when in scientific mode

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        - This relates to the number of significant figures the answer will be displayed in
        - Your GDC may add extra zeros to fill spaces if working with a high number of significant figures, you do not need to write these in your answer
- To add or subtract numbers written in the form $a \times 10^k$ without your GDC you will need to write them in full form first
    - Alternatively you can use 'matching powers of 10', because if the powers of 10 are the same, then the 'number parts' at the start can just be added or subtracted normally
        - For example
          $$(6.3 \times 10^{14}) + (4.9 \times 10^{13}) = (6.3 \times 10^{14}) + (0.49 \times 10^{14}) = 6.79 \times 10^{14}$$
        - Or
          $$(7.93 \times 10^{-11}) - (5.2 \times 10^{-12}) = (7.93 \times 10^{-11}) - (0.52 \times 10^{-11}) = 7.41 \times 10^{-11}$$
- To multiply or divide numbers written in the form $a \times 10^k$ without your GDC you can either write them in full form first or use the laws of indices

Examiner Tip

- Your GDC will give very big or very small answers in standard form and will have a setting which will allow you to carry out calculations in scientific notation
- Make sure you are familiar with the form that your GDC gives answers in as it may be different to the form you are required to use in the exam

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Worked example

Calculate the following, giving your answer in the form $a \times 10^k$, where $1 \le a < 10$ and $k \in \mathbb{Z}$.

i) 3780×200

Using GDC: Choose scientific mode.

Input directly into GDC as ordinary numbers.

$3780 \times 200 = 7.56 \times 10^5$

GDC will automatically give answer in standard form.

Without GDC:

Calculate the value:

$3780 \times 200 = 756000$

Convert to standard form:

$756000 = 7.56 \times 10^5$

7.56×10^5

ii) $(7 \times 10^5) - (5 \times 10^4)$

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Using GDC: Choose scientific mode.
Input directly into GDC

$7 \times 10^5 - 5 \times 10^4 = 6.5 \times 10^5$

This may be displayed as 6.5E5

Without GDC:
Convert to ordinary numbers:

$7 \times 10^5 = 700000$

$5 \times 10^4 = 50000$

Carry out the calculation:

$700000 - 50000 = 650000$

Convert to standard form:

$650000 = 6.5 \times 10^5$

6.5×10^5

iii) $(3.6 \times 10^{-3})(1.1 \times 10^{-5})$

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Input directly into GDC:
(Choose scientific mode).

$(3.6 \times 10^{-3})(1.1 \times 10^{-5}) = 3.96 \times 10^{-8}$

3.96×10^{-8}

Note:

$10^{-3} \times 10^{-5} = 10^{-8}$

$(3.6 \times 10^{-3})(1.1 \times 10^{-5}) = 3.96 \times 10^{-8}$

$3.6 \times 1.1 = 3.96$

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1.1.2 Laws of Indices

Laws of Indices

What are the laws of indices?

- Laws of indices (or index laws) allow you to simplify and manipulate expressions involving exponents
  - An exponent is a power that a number (called the base) is raised to
  - Laws of indices can be used when the numbers are written with the same base
- The index laws you need to know are:
  - $(xy)^m = x^m y^m$
  - $\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$
  - $x^m \times x^n = x^{m+n}$
  - $x^m \div x^n = x^{m-n}$
  - $(x^m)^n = x^{mn}$
  - $x^1 = x$
  - $x^0 = 1$
  - $\frac{1}{x^m} = x^{-m}$
  - $x^{\frac{1}{n}} = \sqrt[n]{x}$
  - $x^{\frac{m}{n}} = \sqrt[n]{x^m}$
- These laws are **not in the formula booklet** so you must remember them

How are laws of indices used?

- You will need to be able to carry out multiple calculations with the laws of indices
  - Take your time and apply each law individually
  - Work with numbers first and then with algebra
- Index laws only work with terms that have the same base, make sure you **change the base** of the term before using any of the index laws
  - Changing the base means rewriting the number as an exponent with the base you need
  - For example, $9^4 = (3^2)^4 = 3^{2\times 4} = 3^8$
  - Using the above can them help with problems like $9^4 \div 3^7 = 3^8 \div 3^7 = 3^1 = 3$

Examiner Tip

- Index laws are rarely a question on their own in the exam but are often needed to help you solve other problems, especially when working with logarithms or polynomials
- Look out for times when the laws of indices can be applied to help you solve a problem algebraically

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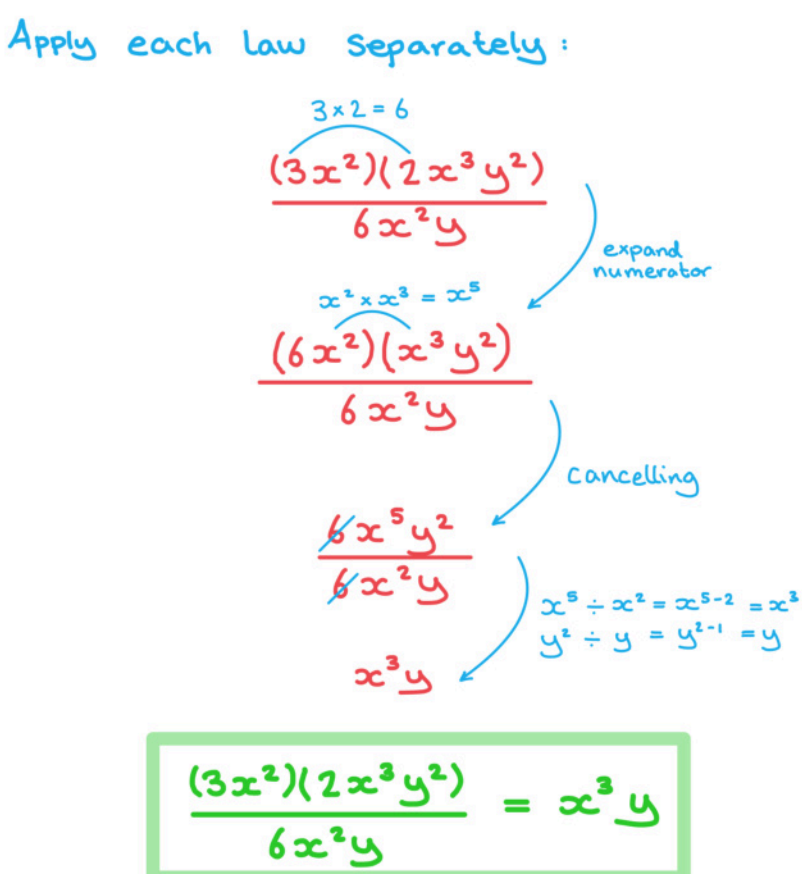
Worked example

Simplify the following equations:

i) $\dfrac{(3x^2)(2x^3y^2)}{(6x^2y)}$.



ii) $(4x^2y^{-4})^3(2x^3y^{-1})^{-2}$.

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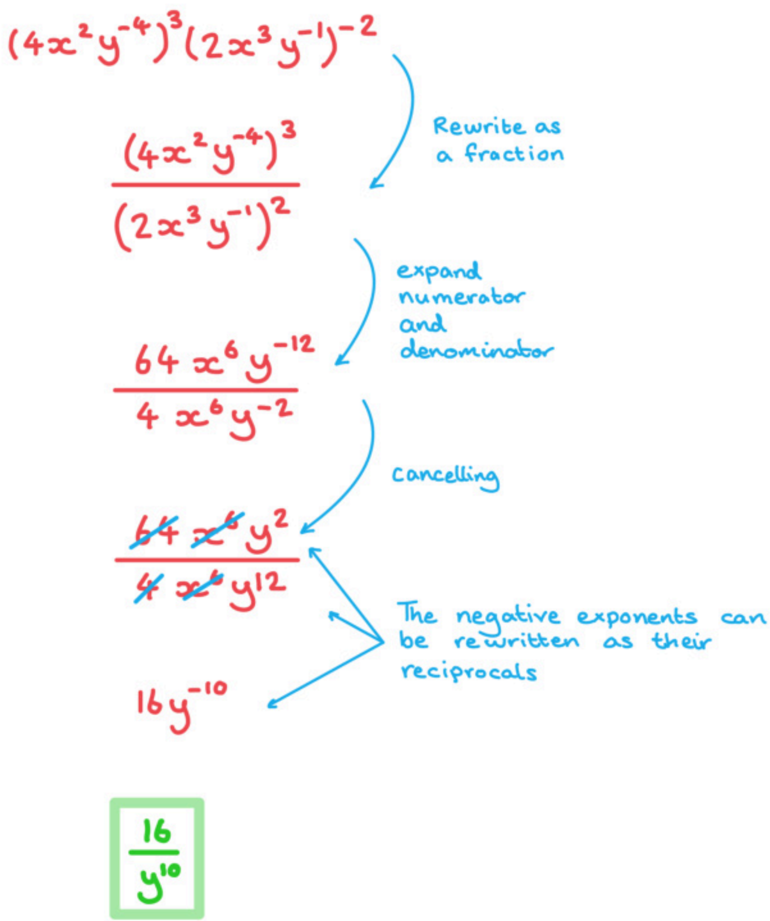
$(4x^2y^{-4})^3(2x^3y^{-1})^{-2}$
Rewrite as a fraction
$\frac{(4x^2y^{-4})^3}{(2x^3y^{-1})^2}$
expand numerator and denominator
$\frac{64x^6y^{-12}}{4x^6y^{-2}}$
Cancelling
$\frac{64x^6y^2}{4x^6y^{12}}$
The negative exponents can be rewritten as their reciprocals
$16y^{-10}$
$\frac{16}{y^{10}}$

1.1.3 Partial Fractions

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Partial Fractions

What are partial fractions?

- Partial fractions allow us to simplify rational expressions into the sum of two or more fractions with constant numerators and linear denominators
  - This allows for integration of rational functions
- The method of partial fractions is essentially the reverse of adding or subtracting fractions
  - When adding fractions, a common denominator is required
  - In partial fractions the common denominator is split into parts (factors)
- If we have a rational function with a quadratic on the denominator partial fractions can be used to rewrite it as the sum of two rational functions with linear denominators
  - This works if the non-linear denominator can be **factorised** into two distinct factors
    - For example: $\frac{ax+b}{(cx+d)(ex+f)} = \frac{A}{cx+d} + \frac{B}{ex+f}$
- If we have a rational function with a linear numerator and denominator partial fractions can be used to rewrite it as the sum of a constant and a fraction with a linear denominator
  - The linear denominator does not need to be factorised
  - For example: $\frac{ax+b}{cx+d} = A + \frac{B}{cx+d}$

How do I find partial fractions if the denominator is a quadratic?

- STEP 1
  Factorise the denominator into the product of two linear factors
  - Check the numerator and cancel out any common factors
    - e.g. $\frac{5x+5}{x^2+x-6} = \frac{5x+5}{(x+3)(x-2)}$
- STEP 2
  Split the fraction into a **sum** of two fractions with single **linear denominators** each having unknown **constant numerators**
  - Use A and B to represent the unknown numerators
    - e.g. $\frac{5x+5}{(x+3)(x-2)} \equiv \frac{A}{x+3} + \frac{B}{x-2}$
- STEP 3
  Multiply through by the denominator to eliminate fractions
  - Eliminate fractions by cancelling all common expressions
    - e.g. $5x+5 \equiv A(x-2) + B(x+3)$
- STEP 4
  Substitute values into the identity and solve for the unknown constants

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- Use the root of each **linear factor** as a value of to find the unknowns
    - e.g. Let $x = 2$: $5(2) + 5 \equiv A((2) - 2) + B((2) + 3)$ etc
- An alternative method is **comparing coefficients**
    - e.g. $5x + 5 \equiv (A + B)x + (-2A + 3B)$
- STEP 5
  Write the **original** as partial fractions
    - Substitute the values you found for *A* and *B* into your expression from STEP 2
        - e.g. $\dfrac{5x + 5}{x^2 + x - 6} = \dfrac{2}{x + 3} + \dfrac{3}{x - 2}$

How do I find partial fractions if the numerator and denominator are both linear?

- If the denominator is not a quadratic expression you will be given the form in which the partial fractions should be expressed
- For example express $\dfrac{12x - 2}{3x - 1}$ in the form $A + \dfrac{B}{3x - 1}$
- STEP 1
  Multiply through by the denominator to eliminate fractions
    - e.g. $12x - 2 \equiv A(3x - 1) + B$
- STEP 2
  Expand the expression on the right-hand side and **compare coefficients**
    - Compare the coefficients of *x* and solve for the first unknown
        - e.g. $12x = 3Ax$
        - therefore $A = 4$
    - Compare the constant coefficients and solve for the second unknown
        - e.g. $-2 = -A + B = -4 + B$
        - therefore $B = 2$
- STEP 3
  Write the **original** as partial fractions
    - $\dfrac{12x - 2}{3x - 1} = 4 + \dfrac{2}{3x - 1}$

How do I find partial fractions if the denominator has a squared linear term?

- A **squared linear factor** in the denominator actually represents two factors rather than one
- This must be taken into account when the rational function is split into partial fractions
    - For the squared linear denominator $(ax + b)^2$ there will be two factors: $(ax + b)$ and $(ax + b)^2$
    - So the rational expression $\dfrac{p}{(ax + b)^2}$ becomes $\dfrac{A}{ax + b} + \dfrac{B}{(ax + b)^2}$
- In IB you will be given the form into which you should split the partial fractions
    - Put the rational expression equal to the given form and then continue with the steps above
- There is more than one way of finding the missing values when working with partial fractions

- Substituting values is usually quickest, however you should look at the number of times a bracket is repeated to help you decide which method to use

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Examiner Tip

- An exam question will often have partial fractions as part (a) and then integration or using the binomial theorem as part (b)
  - Make sure you use your partial fractions found in part (a) to answer the next part of the question

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Worked example

a) Express $\dfrac{2x - 13}{x^2 - x - 2}$ in partial fractions.

$$\frac{2x-13}{x^2-x-2} = \frac{2x-13}{(x+1)(x-2)}$$

The denominator is a quadratic so factorise first.

$$\frac{2x-13}{(x+1)(x-2)} \equiv \frac{A}{x+1} + \frac{B}{x-2}$$

Multiply through by the denominator to eliminate fractions

$$2x - 13 \equiv A(x-2) + B(x+1)$$

Choose values of x to substitute into the identity that will eliminate each constant:

Let $x = 2$: $2(2) - 13 = A((2)-2) + B((2)+1)$

$x - 2 = 0$, $x = 2$

$-9 = 3B$

$B = -3$

Let $x = -1$: $2(-1) - 13 = A((-1)-2) + B((-1)+1)$

$x + 1 = 0$, $x = -1$

$-15 = -3A$

$A = 5$

$$\frac{2x-13}{x^2-x-2} = \frac{5}{x+1} - \frac{3}{x-2}$$

b) Express $\dfrac{x(3x - 13)}{(x + 1)(x - 3)^2}$ in the form $\dfrac{A}{(x + 1)} + \dfrac{B}{x - 3} + \dfrac{C}{(x - 3)^2}$.

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Multiply through by the denominator:

$$\frac{x(3x-13)}{(x+1)(x-3)^2} = \frac{A(x-3)^2 + B(x+1)(x-3) + C(x+1)}{(x+1)(x-3)^2}$$

Eliminate fractions and expand:

$$x(3x-13) = A(x^2-6x+9) + B(x^2-2x-3) + Cx + C$$

$$3x^2 - 13x = (A+B)x^2 + (-6A-2B+C)x + 9A - 3B + C$$

$(A+B)$ ← coefficient of x^2; $(-6A-2B+C)$ ← coefficient of x

Compare coefficients:

$A + B = 3$ ① (coefficients of x^2)

$-6A - 2B + C = -13$ ② (coefficients of x)

$9A - 3B + C = 0$ ③ (constant terms)

Rearrange ① and substitute into ② and ③

$$A = 3 - B \Rightarrow -6(3-B) - 2B + C = -13$$
$$-18 + 6B - 2B + C = -13$$
$$4B + C = 5 \quad ②$$

$$\Rightarrow 9(3-B) - 3B + C = 0$$
$$27 - 9B - 3B + C = 0$$
$$12B - C = 27 \quad ③$$

Solving ② and ③: $4B + C = 5$

$12B - C = 27$

$B = 2,\ C = -3$

Substitute into ①: $A = 3 - B = 3 - 2 = 1$

$$\frac{x(3x-13)}{(x+1)(x-3)^2} = \frac{1}{(x+1)} + \frac{2}{(x-3)} - \frac{3}{(x-3)^2}$$