DP IB Maths: AI HL

Your notes

3.8 Vector Equations of Lines

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3.8.1 Vector Equations of Lines

Equation of a Line in Vector Form

How do I find the vector equation of a line?

- The formula for finding the **vector equation** of a line is
  - $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$
    - Where ***r*** is the **position vector** of any point on the line
    - ***a*** is the **position vector** of a known point on the line
    - ***b*** is a **direction** (displacement) **vector**
    - λ is a scalar
  - This is **given in the formula booklet**
  - This equation can be used for vectors in both 2– and 3– dimensions
- This formula is similar to a regular equation of a straight line in the form $y = mx + c$ but with a vector to show both a point on the line and the direction (or gradient) of the line
  - In 2D the gradient can be found from the direction vector
  - In 3D a numerical value for the direction cannot be found, it is given as a vector
- As ***a*** could be the position vector of **any** point on the line and ***b*** could be **any scalar multiple** of the direction vector there are infinite vector equations for a single line
- Given any two points on a line with position vectors ***a*** and ***b*** the **displacement** vector can be written as ***b*** - ***a***
  - So the formula **r = a** + λ(**b** - **a)** can be used to find the vector equation of the line
  - This is **not given in the formula booklet**

How do I determine whether a point lies on a line?

- Given the equation of a line $\boldsymbol{r} = \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{pmatrix} + \lambda \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \boldsymbol{b}_3 \end{pmatrix}$ the point **c** with position vector $\begin{pmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \\ \boldsymbol{c}_3 \end{pmatrix}$ is on the line if there exists a value of λ such that
  - $\begin{pmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \\ \boldsymbol{c}_3 \end{pmatrix} = \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{pmatrix} + \lambda \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \boldsymbol{b}_3 \end{pmatrix}$
  - This means that there exists a single value of λ that satisfies the three equations:
    - $c_1 = a_1 + \lambda b_1$
    - $c_2 = a_2 + \lambda b_2$
    - $c_3 = a_3 + \lambda b_3$

- A GDC can be used to solve this system of linear equations for
  - The point only lies on the line if a single value of λ exists for all three equations
- Solve one of the equations first to find a value of λ that satisfies the first equation and then check that this value also satisfies the other two equations
- If the value of λ does not satisfy all three equations, then the point **c** does not lie on the line

Your notes

Examiner Tip

- Remember that the vector equation of a line can take many different forms
  - This means that the answer you derive might look different from the answer in a mark scheme
- You can choose whether to write your vector equations of lines using unit vectors or as column vectors
  - Use the form that you prefer, however column vectors is generally easier to work with

Your notes

Worked example

a) Find a vector equation of a straight line through the points with position vectors **a** = 4**i** – 5**k** and **b** = 3**i** - 3**k**

Use the position vectors to find the displacement vector between them.

$$\vec{OA} = \begin{pmatrix}4\\0\\-5\end{pmatrix}, \quad \vec{OB} = \begin{pmatrix}3\\0\\-3\end{pmatrix} \implies \vec{AB} = \begin{pmatrix}3\\0\\-3\end{pmatrix} - \begin{pmatrix}4\\0\\-5\end{pmatrix} = \begin{pmatrix}-1\\0\\2\end{pmatrix}$$

Vector equation of a line	$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$

$$r = \underbrace{\begin{pmatrix}4\\0\\-5\end{pmatrix}}_{\text{position vector of point a}} + \lambda \underbrace{\begin{pmatrix}-1\\0\\2\end{pmatrix}}_{\text{direction vector}} \quad \text{or} \quad r = \underbrace{\begin{pmatrix}3\\0\\-3\end{pmatrix}}_{\text{position vector of point b}} + \lambda \underbrace{\begin{pmatrix}-1\\0\\2\end{pmatrix}}_{\text{direction vector}}$$

$$\boxed{r = \begin{pmatrix}4\\0\\-5\end{pmatrix} + \lambda \begin{pmatrix}-1\\0\\2\end{pmatrix}}$$

b) Determine whether the point C with coordinate (2, 0, -1) lies on this line.

Let $c = \begin{pmatrix}2\\0\\-1\end{pmatrix}$, then check to see if there exists a value of λ such that

$$\begin{pmatrix}2\\0\\-1\end{pmatrix} = \begin{pmatrix}4\\0\\-5\end{pmatrix} + \lambda \begin{pmatrix}-1\\0\\2\end{pmatrix}$$

From the 'i' component : $4 - \lambda = 2$ ①

From the 'j' component : $0 + 0\lambda = 0$ ② (✓) Works for all λ

From the 'k' component : $-5 + 2\lambda = -1$ ③

① $\implies \lambda = 2$ sub into ③ $\implies -5 + (2 \times 2) = -5 + 4 = -1$ ✓

$$\boxed{\text{Point C lies on the line}}$$

Your notes

Equation of a Line in Parametric Form

How do I find the vector equation of a line in parametric form?

- By considering the three separate components of a vector in the *x*, *y* and *z* directions it is possible to write the **vector equation** of a line as **three separate equations**
  - Letting $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$ becomes
  - $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix}$
    - Where $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ is a position vector and $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$ is a direction vector
  - This vector equation can then be split into its three separate component forms:
    - $x = x_0 + \lambda l$
    - $y = y_0 + \lambda m$
    - $z = z_0 + \lambda n$
  - These are **given in the formula booklet**

Your notes

Worked example

Write the parametric form of the equation of the line which passes through the point (-2, 1, 0) with direction vector $\begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}$.

Parametric form of the equation of a line	$x = x_0 + \lambda l,\ y = y_0 + \lambda m,\ z = z_0 + \lambda n$

Use $r = a + \lambda b$ to write the equation in vector form first:

$$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}$$

position vector of a point

direction vector

Separate the components into their 3 separate equations.

$$\boxed{\begin{aligned} x &= -2 + 3\lambda \\ y &= 1 + \lambda \\ z &= -4\lambda \end{aligned}}$$

Your notes

Angle Between Two Lines

How do we find the angle between two lines?

- The angle between two lines is equal to the angle between their **direction vectors**
  - It can be found using the **scalar product** of their direction vectors
- Given two lines in the form $\boldsymbol{r} = \boldsymbol{a}_1 + \lambda \boldsymbol{b}_1$ and $\boldsymbol{r} = \boldsymbol{a}_2 + \lambda \boldsymbol{b}_2$ use the formula
  - $\theta = \cos^{-1}\left(\frac{\boldsymbol{b}_1 \bullet \boldsymbol{b}_2}{|\boldsymbol{b}_1||\boldsymbol{b}_2|}\right)$
- If you are given the equations of the lines in a different form or two points on a line you will need to find their direction vectors first
- To find the angle ABC the vectors BA and BC would be used, both starting from the point B
- The intersection of two lines will always create **two angles**, an acute one and an obtuse one
  - A **positive scalar product** will result in the **acute angle** and a **negative scalar product** will result in the **obtuse angle**
    - Using the **absolute value** of the scalar product will **always result in the acute angle**

Examiner Tip

- In your exam read the question carefully to see if you need to find the acute or obtuse angle
  - When revising, get into the practice of double checking at the end of a question whether your angle is acute or obtuse and whether this fits the question

Your notes

Worked example

Find the acute angle, in radians between the two lines defined by the equations:

$$l_1:\ \boldsymbol{a} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix} \text{ and } l_2:\ \boldsymbol{b} = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 2 \\ 5 \end{pmatrix}$$

STEP 1: Find the scalar product of the direction vectors:

$$\begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 2 \\ 5 \end{pmatrix} = (1\times -3) + (-4\times 2) + (-3\times 5) = -3 + (-8) + (-15) = -26$$

negative, so the angle will be the obtuse angle.

STEP 2: Find the magnitudes of the direction vectors:

$$\sqrt{(1)^2 + (-4)^2 + (-3)^2} = \sqrt{26} \qquad \sqrt{(-3)^2 + (2)^2 + (5)^2} = \sqrt{38}$$

STEP 3: Find the angle: $\cos\theta = \dfrac{|-26|}{\sqrt{26}\sqrt{38}}$

Using the absolute value will result in the acute angle

$$\theta = \cos^{-1}\left(\frac{26}{\sqrt{26}\sqrt{38}}\right)$$

$$\boxed{\theta = 0.597 \text{ radians (3sf)}}$$

Your notes

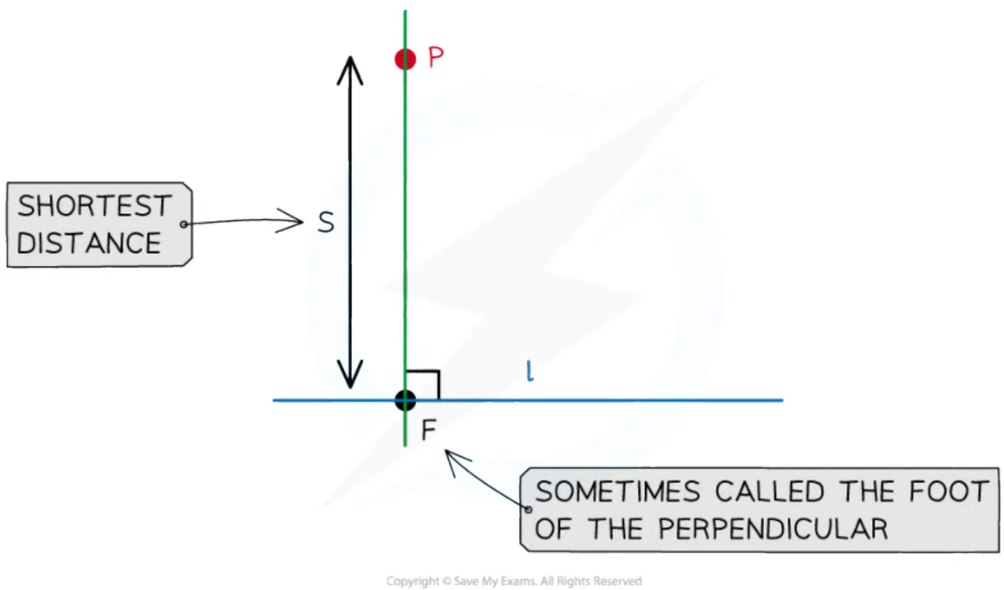
3.8.2 Shortest Distances with Lines

Shortest Distance Between a Point and a Line

How do I find the shortest distance from a point to a line?

- The shortest distance from any point to a line will always be the **perpendicular** distance
  - Given a line *l* with equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and a point *P* not on *l*
  - The **scalar product** of the direction vector, **b**, and the vector in the direction of the **shortest distance** will be zero
- The shortest distance can be found using the following steps:
  - STEP 1: Let the vector equation of the line be ***r*** and the point not on the line be *P,* then the point on the line closest to *P* will be the point *F*
    - The point *F* is sometimes called the foot of the perpendicular
  - STEP 2: Sketch a diagram showing the line *l* and the points *P* and *F*
    - The vector $\overrightarrow{FP}$ will be **perpendicular** to the line *l*
  - STEP 3: Use the equation of the line to find the position vector of the point *F* in terms of λ
  - STEP 4: Use this to find the displacement vector $\overrightarrow{FP}$ in terms of λ
  - STEP 5: The scalar product of the direction vector of the line *l* and the displacement vector $\overrightarrow{FP}$ will be zero
    - Form an equation $\overrightarrow{FP} \cdot \mathbf{b} = 0$ and solve to find λ
  - STEP 6: Substitute λ into $\overrightarrow{FP}$ and find the magnitude $\left|\overrightarrow{FP}\right|$
    - The shortest distance from the point to the line will be the magnitude of $\overrightarrow{FP}$
- Note that the shortest distance between the point and the line is sometimes referred to as the **length of the perpendicular**

Your notes



How do we use the vector product to find the shortest distance from a point to a line?

- The vector product can be used to find the shortest distance from any point to a line on a 2-dimensional plane
- Given a point, P, and a line $\boldsymbol{r} = a + \lambda b$
  - The shortest distance from P to the line will be $\dfrac{|\overrightarrow{AP} \times b|}{|b|}$
  - Where A is a point on the line
  - This is **not** given in the formula booklet

Examiner Tip

- Column vectors can be easier and clearer to work with when dealing with scalar products.

Your notes

Worked example

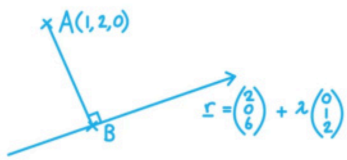
Point A has coordinates (1, 2, 0) and the line l has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.

Point B lies on the l such that $[AB]$ is perpendicular to l.

Find the shortest distance from A to the line l.

B is on l so can be written in terms of λ:

$$\overrightarrow{OB} = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ \lambda \\ 6+2\lambda \end{pmatrix}$$



Find $\overrightarrow{AB}$ using $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ \lambda \\ 6+2\lambda \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ \lambda-2 \\ 6+2\lambda \end{pmatrix}$$

$\overrightarrow{AB}$ is perpendicular to l: $\overrightarrow{AB} \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = 0$

$$\begin{pmatrix} 1 \\ \lambda-2 \\ 6+2\lambda \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = 0$$

$$\lambda - 2 + 2(6+2\lambda) = 0$$

$$5\lambda + 10 = 0$$

$$\lambda = -2$$

Substitute back into $\overrightarrow{AB}$ and find the magnitude:

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ -2-2 \\ 6+2(-2) \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix}$$

$$|\overrightarrow{AB}| = \sqrt{1^2 + (-4)^2 + 2^2} = \sqrt{21}$$

$$\boxed{\text{Shortest distance } = \sqrt{21} \text{ units}}$$

Your notes

Shortest Distance Between Two Lines

How do we find the shortest distance between two parallel lines?

- Two **parallel** lines will never intersect
- The shortest distance between two **parallel lines** will be the **perpendicular distance** between them
- Given a line l_1 with equation $\mathbf{r} = \mathbf{a}_1 + \lambda \mathbf{d}_1$ and a line l_2 with equation $\mathbf{r} = \mathbf{a}_2 + \mu \mathbf{d}_2$ then the shortest distance between them can be found using the following steps:
  - STEP 1: Find the vector between $\mathbf{a}_1$ and a general coordinate from l_2 in terms of μ
  - STEP 2: Set the scalar product of the vector found in STEP 1 and the direction vector $\mathbf{d}_1$ equal to zero
    - Remember the direction vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ are scalar multiples of each other and so either can be used here
  - STEP 3: Form and solve an equation to find the value of μ
  - STEP 4: Substitute the value of μ back into the equation for l_2 to find the coordinate on l_2 closest to l_1
  - STEP 5: Find the distance between $\mathbf{a}_1$ and the coordinate found in STEP 4
- Alternatively, the formula $\dfrac{\left|\overrightarrow{AB} \times \mathbf{d}\right|}{|\mathbf{d}|}$ can be used
  - Where $\overrightarrow{AB}$ is the vector connecting the two given coordinates $\mathbf{a}_1$ and $\mathbf{a}_2$
  - **d** is the simplified vector in the direction of $\mathbf{d}_1$ and $\mathbf{d}_2$
  - This is **not** given in the formula booklet

How do we find the shortest distance from a given point on a line to another line?

- The shortest distance from any point on a line to another line will be the **perpendicular** distance from the point to the line
- If the angle between the two lines is known or can be found then right-angled trigonometry can be used to find the perpendicular distance
  - The formula $\dfrac{\left|\overrightarrow{AB} \times \mathbf{d}\right|}{|\mathbf{d}|}$ given above is derived using this method and can be used
- Alternatively, the equation of the line can be used to find a general coordinate and the steps above can be followed to find the shortest distance

How do we find the shortest distance between two skew lines?

- Two **skew** lines are not parallel but will never intersect
- The shortest distance between two **skew lines** will be perpendicular to **both** of the lines
  - This will be at the point where the two lines pass each other with the perpendicular distance where the point of intersection would be

Your notes

- The **vector product** of the two direction vectors can be used to find a vector in the direction of the shortest distance
- The shortest distance will be a vector **parallel** to the vector product

- To find the shortest distance between two skew lines with equations $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{d}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{d}_2$,
  - STEP 1: Find the vector product of the direction vectors $\mathbf{d}_1$ and $\mathbf{d}_2$
    - $\mathbf{d} = \mathbf{d}_1 \times \mathbf{d}_2$
  - STEP 2: Find the vector in the direction of the line between the two general points on l_1 and l_2 in terms of λ and μ
    - $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$
  - STEP 3: Set the two vectors parallel to each other
    - $\mathbf{d} = k\overrightarrow{AB}$
  - STEP 4: Set up and solve a system of linear equations in the three unknowns, k, λ and μ

Examiner Tip

- Exam questions will often ask for the shortest, or minimum, distance within vector questions
- If you're unsure start by sketching a quick diagram
- Sometimes calculus can be used, however vector methods are usually required

Your notes

Worked example

A drone travels in a straight line and at a constant speed. It moves from an initial point (-5, 4, -8) in the direction of the vector $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$. At the same time as the drone begins moving a bird takes off from initial point (6, -4, 3) and moves in a straight line at a constant speed in the direction of the vector $\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$.

Find the minimum distance between the bird and the drone during this movement.

Your notes

Find the vector product of the direction vectors.

$$\begin{pmatrix}2\\-3\\4\end{pmatrix}\times\begin{pmatrix}-1\\2\\1\end{pmatrix}=\begin{pmatrix}(-3)(1)-(4)(2)\\(4)(-1)-(2)(1)\\(2)(2)-(-3)(-1)\end{pmatrix}=\begin{pmatrix}-11\\-6\\1\end{pmatrix}$$

Find the vector in the direction of the line between the general coordinates.

$$\overrightarrow{AB}=\underbrace{\begin{pmatrix}-5-\mu\\4+2\mu\\-8+\mu\end{pmatrix}}_{\text{A point on } l_2}-\underbrace{\begin{pmatrix}6+2\lambda\\-4-3\lambda\\3+4\lambda\end{pmatrix}}_{\text{A point on } l_1}=\begin{pmatrix}-11-\mu-2\lambda\\8+2\mu+3\lambda\\-11+\mu-4\lambda\end{pmatrix}$$

$$\begin{pmatrix}-11-\mu-2\lambda\\8+2\mu+3\lambda\\-11+\mu-4\lambda\end{pmatrix}=k\begin{pmatrix}-11\\-6\\1\end{pmatrix}$$

$\overrightarrow{AB}$ is parallel to $\begin{pmatrix}-11\\-6\\1\end{pmatrix}$ so $\overrightarrow{AB}=k\begin{pmatrix}-11\\-6\\1\end{pmatrix}$

Set up and solve a system of equations.

$$\left.\begin{aligned}11k-2\lambda-\mu&=11\\6k+3\lambda+2\mu&=-8\\\mu-4\lambda-k&=11\end{aligned}\right\}$$

Solve using GDC: $k=\frac{31}{79}$ $\lambda=-\frac{238}{79}$ $\mu=-\frac{52}{79}$

Substitute back into the expression for $\overrightarrow{AB}$ and find the magnitude:

$$|\overrightarrow{AB}|=\left|\begin{pmatrix}-11-\left(-\frac{52}{79}\right)-2\left(-\frac{238}{79}\right)\\8+2\left(-\frac{52}{79}\right)+3\left(-\frac{238}{79}\right)\\-11+\left(-\frac{52}{79}\right)-4\left(-\frac{238}{79}\right)\end{pmatrix}\right|=\left|\begin{pmatrix}-\frac{341}{79}\\-\frac{186}{79}\\\frac{31}{79}\end{pmatrix}\right|=\sqrt{\left(-\frac{341}{79}\right)^2+\left(-\frac{186}{79}\right)^2+\left(\frac{31}{79}\right)^2}$$

$$\boxed{\text{Shortest distance} = 4.93 \text{ units } (3 \text{ s.f.})}$$