DP IB Maths: AA HL

Your notes

1.3 Sequences & Series

Contents

1.3.1 Language of Sequences & Series

Your notes

Language of Sequences & Series

What is a sequence?

- A **sequence** is an ordered set of numbers with a rule for finding all of the numbers in the sequence
  - For example 1, 3, 5, 7, 9, … is a sequence with the rule *'start at one and add two to each number'*
- The numbers in a sequence are often called **terms**
- The terms of a sequence are often referred to by letters with a subscript
  - In IB this will be the letter u
  - So in the sequence above, $u_1 = 1$, $u_2 = 3$, $u_3 = 5$ and so on
- Each term in a sequence can be found by **substituting** the term number into **formula for the nth term**

What is a series?

- You get a **series** by summing up the terms in a sequence
  - E.g. For the sequence 1, 3, 5, 7, … the associated series is $1 + 3 + 5 + 7 + \ldots$
- We use the notation S_n to refer to the sum of the first n terms in the series
  - $S_n = u_1 + u_2 + u_3 + \ldots + u_n$
  - So for the series above $S_5 = 1 + 3 + 5 + 7 + 9 = 25$

Your notes

Worked example

Determine the first five terms and the value of S_5 in the sequence with terms defined by $u_n = 5 - 2n$.

$$u_n = 5 - 2n$$

find the term you want by replacing n with it's value.

term number

first term

$$u_1 = 5 - 2(1) = 3$$
$$u_2 = 5 - 2(2) = 1$$
$$u_3 = 5 - 2(3) = -1$$
$$u_4 = 5 - 2(4) = -3$$
$$u_5 = 5 - 2(5) = -5$$

recognise the pattern.

-2

-2 rule is subtract 2

'start with 3 and subtract 2 from each number'.

$$S_5 = 3 + 1 + (-1) + (-3) + (-5) = -5$$

the sum of the first 5 terms

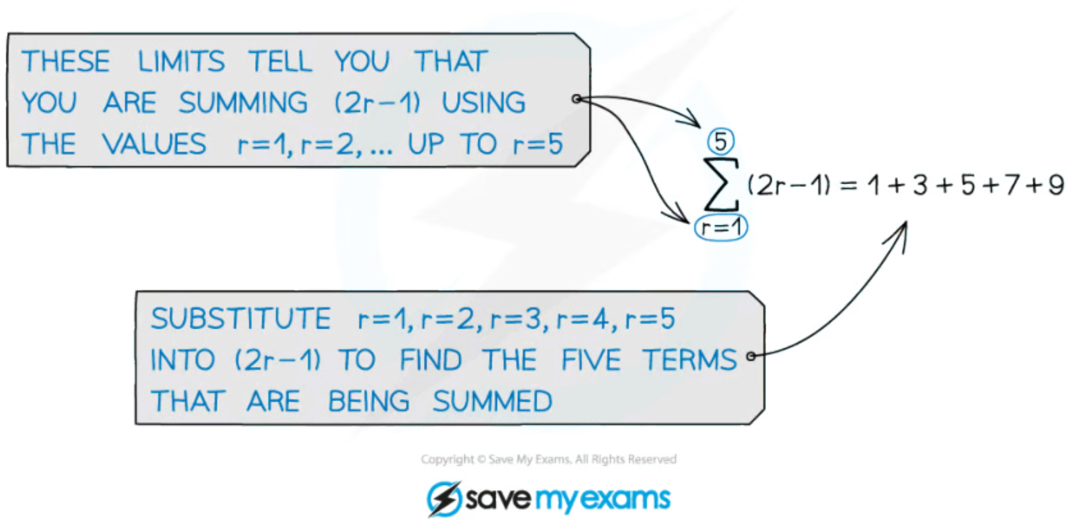
$$\boxed{\begin{gathered} 3,\ 1,\ -1,\ -3,\ -5 \\ S_5 = -5 \end{gathered}}$$

Your notes

Sigma Notation

What is sigma notation?

- Sigma notation is used to show the sum of a certain number of terms in a sequence
- The symbol Σ is the capital Greek letter sigma
- Σ stands for 'sum'
  - The expression to the right of the Σ tells you what is being summed, and the limits above and below tell you which terms you are summing



- Be careful, the limits don't have to start with 1
  - For example $\sum_{k=0}^{4}(2k+1)$ or $\sum_{k=7}^{14}(2k-13)$
  - r and k are commonly used variables within sigma notation

Examiner Tip

- Your GDC will be able to use sigma notation, familiarise yourself with it and practice using it to check your work

Your notes

Worked example

A sequence can be defined by $u_n = 2 \times 3^{n-1}$ for $n \in \mathbb{Z}^+$.

a) Write an expression for $u_1 + u_2 + u_3 + \ldots + u_6$ using sigma notation.

$u_n = 2 \times 3^{n-1}, \ n \in \mathbb{Z}^+$ ← n is the set of all positive integers

Using sigma notation

$$u_1 + u_2 + \ldots + u_6 = \sum_{k=1}^{6} u_k$$

$$\boxed{\sum_{k=1}^{6}(2 \times 3^{k-1})}$$

b) Write an expression for $u_7 + u_8 + u_9 + \ldots + u_{12}$ using sigma notation.

Your notes

$u_n = 2 \times 3^{n-1},\ n \in \mathbb{Z}^+$ ← n is the set of all positive integers

Using sigma notation

$$u_7 + u_8 + \ldots + u_{12} = \sum_{k=7}^{12} u_k$$

$$\boxed{\sum_{k=7}^{12} (2 \times 3^{k-1})}$$

1.3.2 Arithmetic Sequences & Series

Your notes

Arithmetic Sequences

What is an arithmetic sequence?

- In an **arithmetic sequence,** the difference between consecutive terms in the sequence is constant
- This **constant difference** is known as the **common difference, *d*,** of the sequence
  - For example, 1, 4, 7, 10, ... is an arithmetic sequence with the rule '*start at one and add three to each number*'
    - The **first term, u_1,** is 1
    - The **common difference, *d*,** is 3
  - An arithmetic sequence can be **increasing** (positive common difference) or **decreasing** (negative common difference)
  - Each term of an arithmetic sequence is referred to by the letter *u* with a subscript determining its place in the sequence

How do I find a term in an arithmetic sequence?

- The n^{th} term formula for an arithmetic sequence is given as

$$u_n = u_1 + (n-1)d$$

  - Where u_1 is the first term, and d is the common difference
  - This is given in the formula booklet, you do not need to know how to derive it
- Enter the information you have into the formula and use your GDC to find the value of the term
- Sometimes you will be given a term and asked to find the first term or the common difference
  - Substitute the information into the formula and solve the equation
    - You could use your GDC for this
- Sometimes you will be given two terms and asked to find both the first term and the common difference
  - Substitute the information into the formula and set up a **system of linear equations**
  - Solve the simultaneous equations
    - You could use your GDC for this

Examiner Tip

- Simultaneous equations are often needed within arithmetic sequence questions, make sure you are confident solving them with and without the GDC

Your notes

Worked example

The fourth term of an arithmetic sequence is 10 and the ninth term is 25, find the first term and the common difference of the sequence.

$u_4 = 10 \quad , \quad u_9 = 25$

Formula for n^{th} term of an arithmetic series:

$$u_n = u_1 + (n-1)d$$

Sub in $u_4 = 10$ and $u_9 = 25$

$$u_4 = u_1 + (4-1)d = u_1 + 3d = 10$$

$$u_9 = u_1 + (9-1)d = u_1 + 8d = 25$$

Solve using GDC:

let $u_1 = x$ and $d = y$

$$x + 3y = 10$$
$$x + 8y = 25$$

$$x = 1 \quad , \quad y = 3$$

$$\boxed{\begin{aligned} u_1 &= 1 \\ d &= 3 \end{aligned}}$$

Your notes

Arithmetic Series

How do I find the sum of an arithmetic series?

- An **arithmetic series** is the sum of the terms in an **arithmetic sequence**
  - For the arithmetic sequence 1, 4, 7, 10, ... the arithmetic series is 1 + 4 + 7 + 10 + ...
- Use the following formulae to find the sum of the first n terms of the arithmetic series:

$$S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right) \quad ; \quad S_n = \frac{n}{2}\left(u_1 + u_n\right)$$

    - u_1 is the first term
    - d is the common difference
    - u_n is the last term
  - Both formulae are given in the formula booklet, you do not need to know how to derive them
- You can use whichever formula is more convenient for a given question
  - If you know the first term and common difference use the first version
  - If you know the first and last term then the second version is easier to use
- A question will often give you the sum of a certain number of terms and ask you to find the value of the first term or the common difference
  - Substitute the information into the formula and solve the equation
    - You could use your GDC for this

Examiner Tip

- The formulae you need for arithmetic series are in the formula book, you do not need to remember them
  - Practice finding the formulae so that you can quickly locate them in the exam

Your notes

Worked example

The sum of the first 10 terms of an arithmetic sequence is 630.

a) Find the common difference, *d*, of the sequence if the first term is 18.

$S_{10} = 630$

Formula for the sum of an arithmetic series:

$S_n = \frac{n}{2}(2u_1 + (n-1)d)$

Sub in $S_{10} = 630$, $u_1 = 18$

$S_{10} = \frac{10}{2}(2(18) + (10-1)d) = 630$

$5(36 + 9d) = 630$

Solve: $36 + 9d = 126$

$9d = 90$

$d = 10$

$\boxed{d = 10}$

b) Find the first term of the sequence if the common difference, *d*, is 11.

Sub in $S_{10} = 630$, $d = 11$

$$S_{10} = \frac{10}{2}(2u_1 + (10-1)(11)) = 630$$

$$5(2u_1 + 99) = 630$$

Solve: $2u_1 + 99 = 126$

$$2u_1 = 27$$

$$\boxed{u_1 = 13.5}$$

Your notes

1.3.3 Geometric Sequences & Series

Your notes

Geometric Sequences

What is a geometric sequence?

- In a **geometric sequence**, there is a **common ratio, *r*,** between consecutive terms in the sequence
  - For example, 2, 6, 18, 54, 162, ... is a sequence with the rule *'start at two and multiply each number by three'*
    - The **first term, u_1,** is 2
    - The **common ratio, *r*,** is 3
- A geometric sequence can be **increasing** ($r > 1$) or **decreasing** ($0 < r < 1$)
- If the common ratio is a **negative number** the terms will alternate between positive and negative values
  - For example, 1, -4, 16, -64, 256, ... is a sequence with the rule *'start at one and multiply each number by negative four'*
    - The **first term, u_1,** is 1
    - The **common ratio, *r*,** is -4
- Each term of a geometric sequence is referred to by the letter *u* with a subscript determining its place in the sequence

How do I find a term in a geometric sequence?

- The n^{th} term formula for a geometric sequence is given as

$$u_n = u_1 r^{n-1}$$

  - Where u_1 is the first term, and r is the common ratio
  - This formula allows you to find **any term** in the geometric sequence
  - It is given in the formula booklet, you do not need to know how to derive it
- Enter the information you have into the formula and use your GDC to find the value of the term
- Sometimes you will be given a term and asked to find the first term or the common ratio
  - Substitute the information into the formula and solve the equation
    - You could use your GDC for this
- Sometimes you will be given two or more consecutive terms and asked to find both the first term and the common ratio
  - Find the common ratio by dividing a term by the one before it
  - Substitute this and one of the terms into the formula to find the first term
- Sometimes you may be given a term and the formula for the n^{th} term and asked to find the value of *n*
  - You can solve these using **logarithms** on your GDC

Examiner Tip

- You will sometimes need to use logarithms to answer geometric sequences questions
  - Make sure you are confident doing this
  - Practice using your GDC for different types of questions

Your notes

Your notes

Worked example

The sixth term, u_6, of a geometric sequence is 486 and the seventh term, u_7, is 1458.

Find,

i) the common ratio, r, of the sequence,

$u_6 = 486 \quad , \quad u_7 = 1458$

The common ratio, r, is given by

$$r = \frac{u_2}{u_1} = \frac{u_3}{u_2} = \ldots = \frac{u_{n+1}}{u_n}$$

Sub in $u_6 = 486$, $u_7 = 1458$

$$r = \frac{u_7}{u_6} = \frac{1458}{486} = 3$$

$$\boxed{r = 3}$$

ii) the first term of the sequence, u_1.

Your notes

Formula for n^{th} term of a geometric series:

$$u_n = u_1 r^{n-1}$$

Sub in $r = 3$ and either $u_6 = 486$ or $u_7 = 1458$

$$u_6 = u_1(3)^{6-1} = 486$$

Solve: $243 u_1 = 486$

$$u_1 = 2$$

$$\boxed{u_1 = 2}$$

Your notes

Geometric Series

How do I find the sum of a geometric series?

- A **geometric series** is the sum of a certain number of terms in a **geometric sequence**
  - For the geometric sequence 2, 6, 18, 54, ... the geometric series is 2 + 6 + 18 + 54 + ...
- The following formulae will let you find the sum of the first *n* terms of a geometric series:

$$S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}$$

    - u_1 is the first term
    - r is the common ratio
  - Both formulae are given in the formula booklet, you do not need to know how to derive them
- You can use whichever formula is more convenient for a given question
  - The first version of the formula is more convenient if $r > 1$ and the second is more convenient if $r < 1$
- A question will often give you the sum of a certain number of terms and ask you to find the value of the first term, the common ratio, or the number of terms within the sequence
  - Substitute the information into the formula and solve the equation
    - You could use your GDC for this

Examiner Tip

- The geometric series formulae are in the formulae booklet, you don't need to memorise them
  - Make sure you can locate them quickly in the formula booklet

Your notes

Worked example

A geometric sequence has $u_1 = 25$ and $r = 0.8$. Find the value of u_5 and S_5.

$u_1 = 25, \quad r = 0.8$

Formula for n^{th} term of a geometric series:

$$u_n = u_1 r^{n-1}$$

Sub in $u_1 = 25, \; r = 0.8$

$$u_5 = 25(0.8)^4 = 10.24$$

Formula for the sum of a geometric series:

$$S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}$$

$r < 1$ so this version is easier to use.

Sub in $u_1 = 25, \; r = 0.8$

$$S_5 = \frac{u_1(1 - r^5)}{1 - r} = \frac{25(1 - 0.8^5)}{1 - 0.8} = 84.04$$

$$\boxed{\begin{aligned} u_5 &= 10.24 \\ S_5 &= 84.04 \end{aligned}}$$

Your notes

Sum to Infinity

What is the sum to infinity of a geometric series?

- A geometric sequence will either increase or decrease away from zero or the terms will get progressively closer to zero
  - Terms will get closer to zero if the common ratio, *r*, is between 1 and -1
- If the terms are getting closer to zero then the series is said to **converge**
  - This means that the sum of the series will approach a limiting value
  - As the number of terms increase, the sum of the terms will get closer to the limiting value

How do we calculate the sum to infinity?

- If asked to find out if a geometric sequence converges find the value of *r*
  - If $|r| < 1$ then the sequence converges
  - If $|r| \geq 1$ then the sequence does not converge and the sum to infinity cannot be calculated
  - $|r| < 1$ means $-1 < r < 1$
- If $|r| < 1$, then the geometric series **converges** to a finite value given by the formula

$$S_\infty = \frac{u_1}{1-r}, \quad |r| < 1$$

  - u_1 is the first term
  - r is the common ratio
  - This is **in the formula book**, you do not need to remember it

Examiner Tip

- Learn and remember the conditions for when a sum to infinity can be calculated

Your notes

Worked example

The first three terms of a geometric sequence are $6\,,\ 2\,,\ \dfrac{2}{3}$. Explain why the series converges and find the sum to infinity.

$u_1 = 6, \quad u_2 = 2, \quad u_3 = \frac{2}{3}$

Find the value of r: $r = \frac{u_2}{u_1}$

$$r = \frac{u_2}{u_1} = \frac{2}{6} = \frac{1}{3}$$

$\boxed{|r| < 1 \text{ so the series converges}}$

Find the sum to infinity: $S_\infty = \frac{u_1}{1-r}$

$$S_\infty = \frac{u_1}{1-r} = \frac{6}{1-\frac{1}{3}} = \frac{6}{2/3} = 9$$

$$\boxed{S_\infty = 9}$$

1.3.4 Applications of Sequences & Series

Your notes

Applications of Arithmetic Sequences & Series

Many real-life situations can be modelled using sequences and series, including but not limited to: patterns made when tiling floors; seating people around a table; the rate of change of a population; the spread of a virus and many more.

What do I need to know about applications of arithmetic sequences and series?

- If a quantity is changing repeatedly by having a fixed amount **added to** or **subtracted from** it then the use of **arithmetic sequences** and **arithmetic series** is appropriate to **model** the situation
  - If a sequence seems to fit the pattern of an arithmetic sequence it can be said to be **modelled** by an arithmetic sequence
  - The scenario can be **modelled** using the given information and the formulae from the formula booklet
- A common application of arithmetic sequences and series is **simple interest**
  - Simple interest is when an initial investment is made and then a percentage of the initial investment is added to this amount on a regular basis (usually per year)
- Arithmetic sequences can be used to make estimations about how something will change in the future

Examiner Tip

- Exam questions won't always tell you to use sequences and series methods, practice spotting them by looking for clues in the question
- If a given amount is repeated periodically then it is likely the question is on arithmetic sequences or series

Your notes

Worked example

Jasper is saving for a new car. He puts USD $100 into his savings account and then each month he puts in USD $10 more than the month before. Jasper needs USD $1200 for the car. Assuming no interest is added, find,

i) the amount Jasper has saved after four months,

Identify the arithmetic sequence :

$u_1 = 100 \quad , \quad d = 10$

After 4 months Jasper will have saved:

$u_1 + u_2 + u_3 + u_4 = S_4$

Formula for the sum of an arithmetic series :

$S_n = \frac{n}{2}(2u_1 + (n-1)d)$

$S_4 = \frac{4}{2}(2u_1 + (4-1)d)$

Sub in $u_1 = 100$ and $d = 10$

$S_4 = \frac{4}{2}(2(100) + (4-1)(10))$

$= 2(200 + 30)$

$= 2(230)$

$\boxed{S_4 = \$460}$

ii) the month in which Jasper reaches his goal of USD $1200.

Your notes

Sub $S_n = 1200$, $u_1 = 100$, $d = 10$ into formula:

$$1200 = \frac{n}{2}(2(100) + (n-1)(10))$$

Solve using algebraic solver on GDC:

$n = 8.67...$ or $n = -27.67...$

disregard as n cannot be negative.

$\therefore \quad S_8 < 1200$

$S_9 > 1200$ reaches total in 9th month

Jasper will reach USD $1200 in the 9th month.

Applications of Geometric Sequences & Series

Your notes

What do I need to know about applications of geometric sequences and series?

- If a quantity is changing repeatedly by a fixed **percentage,** or by being **multiplied** repeatedly by a fixed amount, then the use of **geometric sequences** and **geometric series** is appropriate to **model** the situation
  - If a sequence seems to fit the pattern of a geometric sequence it can be said to be **modelled** by a geometric sequence
  - The scenario can be **modelled** using the given information and the formulae from the formula booklet
- A common application of geometric sequences and series is **compound interest**
  - Compound interest is when an initial investment is made and then interest is paid on the initial amount **and on the interest already earned** on a regular basis (usually every year)
- Geometric sequences can be used to make estimations about how something will change in the future
- The questions won't always tell you to use sequences and series methods, so be prepared to spot 'hidden' sequences and series questions
  - Look out for questions on savings accounts, salaries, sales commissions, profits, population growth and decay, spread of bacteria etc

Examiner Tip

- Exam questions won't always tell you to use sequences and series methods, practice spotting them by looking for clues in the question
- If a given amount is changing by a percentage or multiple then it is likely the question is on geometric sequences or series

Your notes

Worked example

A new virus is circulating on a remote island. On day one there were 10 people infected, with the number of new infections increasing at a rate of 40% per day.

a) Find the expected number of people newly infected on the 7^{th} day.

Identify the geometric sequence:

$u_1 = 10$, $r = 1.4$

40% increase so 140% of the day before

New infections : u_7

Formula for n^{th} term of a geometric series :

$$u_n = u_1 r^{n-1}$$

Sub in $u_1 = 10$, $r = 1.4$

$$u_7 = 10(1.4)^6 = 75.29...$$

$\boxed{\text{Expected number of new infections} = 75}$

b) Find the expected number of infected people after one week (7 days), assuming no one has recovered yet.

Your notes

Total infections : S_7

Formula for the sum of a geometric series :

$$S_n = \frac{u_1(r^n - 1)}{r - 1}$$ ← $r > 1$ so this version is easier to use.

Sub in $u_1 = 10$, $r = 1.4$

$$S_7 = \frac{10(1.4^7 - 1)}{1.4 - 1} = 238.53\ldots$$

$$\boxed{\text{Expected number of total infections} = 239}$$

1.3.5 Compound Interest & Depreciation

Your notes

Compound Interest

What is compound interest?

- Interest is a small percentage paid by a bank or company that is added on to an initial investment
  - Interest can also refer to an amount paid on a loan or debt, however IB compound interest questions will always refer to interest on **investments**
- **Compound interest** is where interest is paid on **both the initial investment** and any interest that has **already been paid**
  - Make sure you know the difference between compound interest and simple interest
    - Simple interest pays interest only on the initial investment
- The interest paid each time will increase as it is a percentage of a higher number
- Compound interest will be paid in instalments in a given timeframe
  - The interest rate, *r*, will be per annum (per year)
    - This could be written *r* % p.a.
  - Look out for phrases such as **compounding annually** (interest paid yearly) **or compounding monthly** (interest paid monthly)
    - If $\alpha\%$ p.a. (per annum) is paid compounding monthly, then $\frac{\alpha}{12}\%$ will be paid each month
    - The formula for compound interest allows for this so you do not have to compensate separately

How is compound interest calculated?

- The formula for calculating compound interest is:

$$FV = PV \times \left(1 + \frac{r}{100k}\right)^{kn}$$

  - Where
    - *FV* is the future value
    - *PV* is the present value
    - *n* is the number of years
    - *k* is the number of compounding periods per year
    - *r* % is the nominal annual rate of interest
  - This formula is **given in the formula booklet**, you do not have to remember it
- Be careful with the *k* value
  - Compounding annually means *k* = 1
  - Compounding half-yearly means *k* = 2
  - Compounding quarterly means *k* = 4
  - Compounding monthly means *k* = 12
- Your GDC will have a finance solver app on it which you can use to find the future value
  - This may also be called the TVM (time value of money) solver

  - You will have to enter the information from the question into your calculator
- Be aware that many questions will be set up such that you will have to use the formula
  - So for compound interest questions it is better to use the formula from your formula booklet than your GDC

Your notes

Examiner Tip

- Your GDC will be able to solve some compound interest problems so it is a good idea to make sure you are confident using it, however you must also familiarise yourself with the formula and make sure you can find it in the formula booklet

Your notes

Worked example

Kim invests MYR 2000 (Malaysian Ringgit) in an account that pays a nominal annual interest rate of 2.5% **compounded monthly**. Calculate the amount that Kim will have in her account after 5 years.

Compound interest formula:

$$FV = PV\left(1 + \frac{r}{100k}\right)^{kn}$$

- FV – future value
- PV – present value
- r – interest rate
- k – compounding periods
- n – number of years

Substitute values in:

$PV = 2000$ (initial investment)

$k = 12$ (compounding monthly)

$r = 2.5\%$

$n = 5$ (number of years)

$$FV = 2000\left(1 + \frac{2.5}{(100)(12)}\right)^{(12 \times 5)}$$

$$= 2266.002\ldots$$

$$\boxed{FV \approx \text{MYR } 2270 \text{ (3sf)}}$$

Depreciation

What is depreciation?

- Depreciation is when the **value** of something **falls** over time
- The most common examples of depreciation are the value of cars and technology
- If the depreciation is occurring at a **constant rate** then it is **compound depreciation**

How is compound depreciation calculated?

- The formula for calculating compound depreciation is:

$$FV = PV \times \left(1 - \frac{r}{100}\right)^{n}$$

  - Where
    - *FV* is the future value
    - *PV* is the present value
    - *n* is the number of years
    - *r%* is the rate of depreciation
  - This formula is **not** given in the formula booklet, however it is almost the same as the formula for compound interest but
    - with a **subtraction** instead of an addition
    - the value of *k* will always be 1
- Your GDC **could** again be used to solve some compound depreciation questions, but watch out for those which are set up such that you will have to use the formula

Your notes

Examiner Tip

- You can use your GDC's "Finance Solver" (TI) or "Compound Interest" (Casio) feature to solve most depreciation questions, by entering the interest rate as a negative value

Your notes

Worked example

Kyle buys a new car for AUD $14 999. The value of the car depreciates by 15% each year.

a) Find the value of the car after 5 years.

Depreciation formula :

$$FV = PV\left(1 - \frac{r}{100}\right)^n$$

rate of depreciation

$n \leftarrow$ number of years

future value

present value

Substitute values in :

$PV = 14999$ (initial cost)

$r = 15\%$

$n = 5$ (number of years)

$$FV = 14999\left(1 - \frac{15}{100}\right)^5$$

$$= 6655.13\ldots$$

$$\boxed{FV \approx \text{AUD } \$6660 \text{ (3sf)}}$$

b) Find the number of years and months it will take for the value of the car to be approximately AUD $9999.

Your notes

$$FV = PV\left(1 - \frac{r}{100}\right)^n$$

$FV \approx 9999$

$PV = 14999$

$r = 15\%$

Substitute values in:

$$9999 \approx 14999\left(1 - \frac{15}{100}\right)^n$$

Use GDC to solve:

$$n = 2.495\ldots$$

2 years   0.495th of a year

Convert to years and months:

2 years + 0.495... × 12 months

$\boxed{\approx \text{2 years and 6 months}}$