

DP IB Maths: AI HL



Your notes

1.2 Exponentials & Logs

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Your notes

1.2.1 Exponents

Laws of Indices

What are the laws of indices?

- Laws of indices (or index laws) allow you to simplify and manipulate expressions involving exponents
 - An exponent is a power that a number (called the base) is raised to
 - Laws of indices can be used when the numbers are written with the same base
- The index laws you need to know are:
 - $(xy)^m = x^m y^m$
 - $\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$
 - $x^m \times x^n = x^{m+n}$
 - $x^m \div x^n = x^{m-n}$
 - $(x^m)^n = x^{mn}$
 - $x^1 = x$
 - $x^0 = 1$
 - $\frac{1}{x^m} = x^{-m}$
 - $x^{\frac{1}{n}} = \sqrt[n]{x}$
 - $x^{\frac{m}{n}} = \sqrt[n]{x^m}$
- These laws are **not in the formula booklet** so you must remember them

How are laws of indices used?

- You will need to be able to carry out multiple calculations with the laws of indices
 - Take your time and apply each law individually
 - Work with numbers first and then with algebra
- Index laws only work with terms that have the same base, make sure you **change the base** of the term before using any of the index laws
 - Changing the base means rewriting the number as an exponent with the base you need
 - For example, $9^4 = (3^2)^4 = 3^2 \times 4 = 3^8$
 - Using the above can them help with problems like $9^4 \div 3^7 = 3^8 \div 3^7 = 3^1 = 3$

 **Examiner Tip**

- Index laws are rarely a question on their own in the exam but are often needed to help you solve other problems, especially when working with logarithms or polynomials
- Look out for times when the laws of indices can be applied to help you solve a problem algebraically



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Worked example

Simplify the following equations:

i)
$$\frac{(3x^2)(2x^3y^2)}{(6x^2y)}$$

Apply each law separately:

$$\begin{array}{l} \overset{3 \times 2 = 6}{\frac{(3x^2)(2x^3y^2)}{6x^2y}} \\ \xrightarrow{\text{expand numerator}} \frac{(6x^2)(x^3y^2)}{6x^2y} \\ \xrightarrow{\text{cancelling}} \frac{\cancel{6}x^5y^2}{\cancel{6}x^2y} \\ \xrightarrow{\begin{array}{l} x^5 \div x^2 = x^{5-2} = x^3 \\ y^2 \div y = y^{2-1} = y \end{array}} x^3y \end{array}$$

$$\boxed{\frac{(3x^2)(2x^3y^2)}{6x^2y} = x^3y}$$

ii) $(4x^2y^{-4})^3(2x^3y^{-1})^{-2}$



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$$(4x^2y^{-4})^3(2x^3y^{-1})^{-2}$$

Rewrite as a fraction

$$\frac{(4x^2y^{-4})^3}{(2x^3y^{-1})^2}$$

expand numerator and denominator

$$\frac{64x^6y^{-12}}{4x^6y^{-2}}$$

cancelling

$$\frac{\cancel{64}x^{\cancel{6}}y^2}{\cancel{4}x^{\cancel{6}}y^2}$$

The negative exponents can be rewritten as their reciprocals

$$16y^{-10}$$

$\frac{16}{y^{10}}$



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1.2.2 Logarithms

Introduction to Logarithms

What are logarithms?

- A logarithm is the inverse of an exponent
 - If $a^x = b$ then $\log_a(b) = x$ where $a > 0, b > 0, a \neq 1$
 - This is in the formula booklet
 - The number a is called the **base** of the logarithm
 - Your GDC will be able to use this function to solve equations involving exponents
- Try to get used to 'reading' logarithm statements to yourself
 - $\log_a(b) = x$ would be read as "the power that you raise a to, to get b , is x "
 - So $\log_5 125 = 3$ would be read as "the power that you raise 5 to, to get 125, is 3"
- Two important cases are:
 - $\ln x = \log_e(x)$
 - Where e is the mathematical constant 2.718...
 - This is called the **natural logarithm** and will have its own button on your GDC
 - $\log x = \log_{10}(x)$
 - Logarithms of **base 10** are used often and so abbreviated to **log x**

Why use logarithms?

- Logarithms allow us to solve equations where the exponent is the unknown value
 - We can solve some of these by inspection
 - For example, for the equation $2^x = 8$ we know that x must be 3
 - Logarithms allow use to solve more complicated problems
 - For example, the equation $2^x = 10$ does not have a clear answer
 - Instead, we can use our GDCs to find the value of $\log_2 10$

Examiner Tip

- Before going into the exam, make sure you are completely familiar with your GDC and know how to use its logarithm functions



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 **Worked example**

Solve the following equations:

i) $x = \log_3 27,$

$$x = \log_3 27 \iff 3^x = 27$$

We can see from inspection:

$$3^3 = 27 \iff x = 3$$

$$x = 3$$

OR: use GDC to find answer directly.

ii) $2^x = 21.4,$ giving your answer to 3 s.f.

$2^x = 21.4$ This cannot be seen
from inspection:

$$2^x = 21.4 \iff x = \log_2 21.4$$

use GDC to find answer directly.

$$\log_2 21.4 = 4.4195\dots$$

$$x = 4.42 \text{ (3 s.f.)}$$



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Laws of Logarithms

What are the laws of logarithms?

- Laws of logarithms allow you to simplify and manipulate expressions involving logarithms
 - The laws of logarithms are equivalent to the **laws of indices**
- The laws you need to know are, given $a, x, y > 0$:
 - $\log_a xy = \log_a x + \log_a y$
 - This relates to $a^x \times a^y = a^{x+y}$
 - $\log_a \frac{x}{y} = \log_a x - \log_a y$
 - This relates to $a^x \div a^y = a^{x-y}$
 - $\log_a x^m = m \log_a x$
 - This relates to $(a^x)^y = a^{xy}$
- These laws are **in the formula booklet** so you do not need to remember them
 - You must make sure you know how to use them

$$\log_a xy = \log_a x + \log_a y$$

$$\text{RELATES TO } a^x \times a^y = a^{x+y}$$

$$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\text{RELATES TO } \frac{a^x}{a^y} = a^{x-y}$$

$$\log_a x^k = k \log_a x$$

$$\text{RELATES TO } (a^x)^y = a^{xy}$$

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Useful results from the laws of logarithms

- Given $a > 0, a \neq 1$
 - $\log_a 1 = 0$
 - This is equivalent to $a^0 = 1$
- If we substitute b for a into the given identity in the formula booklet
 - $a^x = b \Leftrightarrow \log_a b = x$ where $a > 0, b > 0, a \neq 1$
 - $a^x = a \Leftrightarrow \log_a a = x$ gives $a^1 = a \Leftrightarrow \log_a a = 1$
 - This is an important and useful result



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- Substituting this into the third law gives the result
 - $\log_a a^k = k$
- Taking the inverse of its operation gives the result
 - $a^{\log_a x} = x$
- From the third law we can also conclude that
 - $\log_a \frac{1}{x} = -\log_a x$

$\log_a a = 1$	"THE POWER YOU RAISE a TO, TO GET a, IS 1"
$\log_a a^x = x$	$\log_a a^x = x \log_a a$ $= x$
$a^{\log_a x} = x$	AN OPERATION AND ITS INVERSE
$\log_a 1 = 0$	$a^0 = 1$
$\log_a \frac{1}{x} = -\log_a x$	$\log_a \frac{1}{x} = \log_a x^{-1}$ $= -\log_a x$

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- These useful results are **not in the formula booklet** but can be deduced from the laws that are
- Beware...
 - ... $\log_a(x + y) \neq \log_a x + \log_a y$
- These results apply to $\ln x$ ($\log_e x$) too
 - Two particularly useful results are
 - $\ln e^x = x$
 - $e^{\ln x} = x$
- Laws of logarithms can be used to ...
 - simplify expressions
 - solve logarithmic equations

- solve exponential equations

Examiner Tip

- Remember to check whether your solutions are valid
 - $\log(x+k)$ is only defined if $x > -k$
 - You will lose marks if you forget to reject invalid solutions



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 **Worked example**

a)

Write the expression $2 \log 4 - \log 2$ in the form $\log k$, where $k \in \mathbb{Z}$.Using the law $\log_a x^m = m \log_a x$

$$2 \log 4 = \log 4^2 = \log 16$$

$$\begin{aligned} 2 \log 4 - \log 2 &= \log 4^2 - \log 2 \\ &= \log 16 - \log 2 \end{aligned}$$

Using the law $\log_a \frac{x}{y} = \log_a x - \log_a y$

$$\log 16 - \log 2 = \log \frac{16}{2} = \log 8$$

$$\boxed{2 \log 4 - \log 2 = \log 8}$$

b) Hence, or otherwise, solve $2 \log 4 - \log 2 = -\log \frac{1}{x}$.



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To solve $2\log 4 - \log 2 = \log \frac{1}{x}$ rewrite as

$$\begin{array}{l} \text{from} \\ \text{part (a)} \end{array} \log 8 = -\log \frac{1}{x}$$

Use the index law $\frac{1}{x} = x^{-1}$

$$\log 8 = -\log x^{-1}$$

$$\log 8 = \log x \quad \leftarrow \log_a x^m = m \log_a x$$

$$8 = x$$

$$\boxed{x = 8}$$