DP IB Maths: AI HL

Your notes

2.6 Further Modelling with Functions

Contents

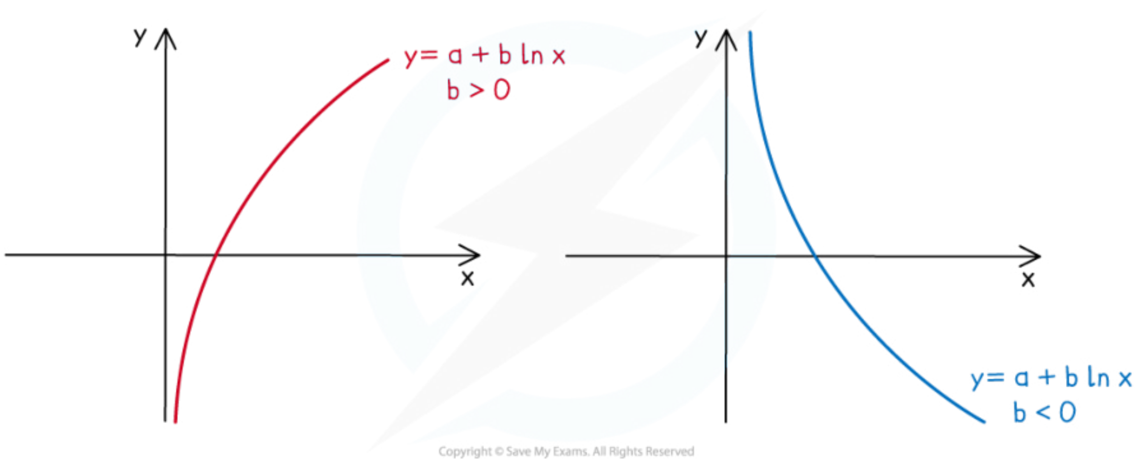
Your notes

2.6.1 Properties of Further Graphs

Logarithmic Functions & Graphs

What are the key features of logarithmic graphs?

- A **logarithmic function** is of the form $f(x) = a + b\ln x,\ x > 0$
- Remember the natural logarithmic function $\ln x \equiv \log_e(x)$
  - This is the inverse of $f(x) = e^x$
    - $\ln(e^x) = x$ and $e^{\ln x} = x$
  - The graphs **will always** pass through the point (1, *a*)
  - The graphs **do not have a *y*-intercept**
    - The graphs have a **vertical asymptote** at the *y*-axis:
  - The graphs have **one root** at $\left(e^{-\frac{a}{b}}, 0\right)$
    - This can be found using your GDC
  - The graphs **do not have any minimum or maximum points**
  - The value of *b* determines whether the graph is increasing or decreasing
    - If *b* is positive then the graph is increasing
    - If *b* is negative then the graph is decreasing

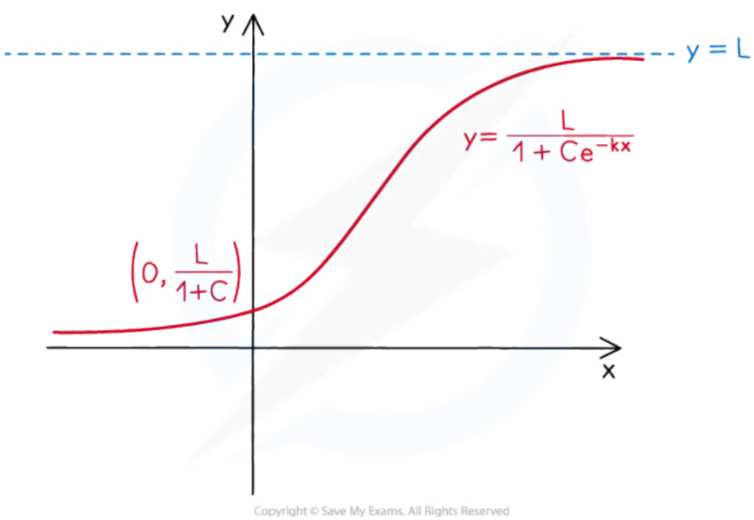


Your notes

Logistic Functions & Graphs

What are the key features of logistic graphs?

- A **logistic function** is of the form $f(x) = \frac{L}{1 + Ce^{-kx}}$
  - *L*, *C* & *k* are positive constants
- Its **domain** is the set of **all real values**
- Its **range** is the set of **real positive values less than *L***
- The *y*-intercept is at the point $\left(0, \frac{L}{1 + C}\right)$
- There are **no roots**
- There is a **horizontal asymptote** at *y* = *L*
  - This is called the carrying capacity
    - This is the upper limit of the function
    - For example: it could represent the limit of a population size
- There is a **horizontal asymptote** at *y* = 0
- The graph is **always increasing**



2.6.2 Natural Logarithmic Models

Your notes

Natural Logarithmic Models

What are the parameters of natural logarithmic models?

- A **natural logarithmic model** is of the form $f(x) = a + b \ln x$
- The a represents the value of the function when $x = 1$
- The b determines the rate of change of the function
  - A bigger absolute value of b leads to a faster rate of change

What can be modelled as a natural logarithmic model?

- A **natural logarithmic model** can be used when the variable increases rapidly for a period followed by a much slower rate of increase with no limiting value
  - $M(I)$ is the magnitude of an earthquake with an intensity of I
  - $d(I)$ is the decibels measured of a noise with an intensity of I

What are possible limitations a natural logarithmic model?

- A **natural logarithmic graph** is unbounded
  - However in real-life the variable might have a limiting value

Worked example

The sound intensity level, L, in decibels (dB) can be modelled by the function

$$L(I) = a + 8\ln I,$$

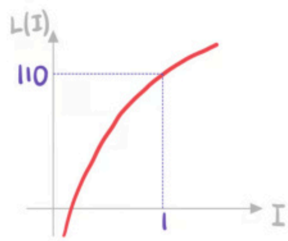
where I is the sound intensity, in watts per square metre (Wm^{-2}).

a) Given that a sound intensity of 1 Wm^{-2} produces a sound intensity level of 110 dB, write down the value of a.

Substitute $I = 1$ and $L = 110$

$110 = a + 8\ln 1$

$\boxed{a = 110}$



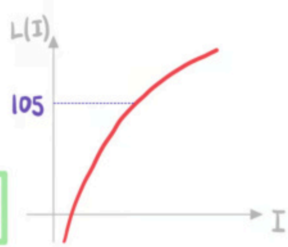
b) Find the sound intensity, in Wm^{-2}, of a car alarm that has a sound intensity level of 105 dB.

Use GDC to solve $L(I) = 105$

$110 + 8\ln I = 105$

$I = 0.535261...$

$\boxed{I = 0.535 \text{ Wm}^{-2} \text{ (3sf)}}$



Your notes

2.6.3 Logistic Models

Your notes

Logistic Models

What are the parameters of logistic models?

- A **logistic model** is of the form $f(x) = \dfrac{L}{1 + Ce^{-kx}}$
- The L represents the limiting capacity
  - This is the value that the model tends to as x gets large
- The C (along with the L) helps to determine the initial value of the model
  - The initial value is given by $\dfrac{L}{1 + C}$
  - Once L has been determined you can then determine C
- The k determines the rate of increase of the model

What can be modelled using a logistic model?

- A **logistic model** can be used when the variable initially increases exponentially and then tends towards a limit
  - $H(t)$ is the height of a giraffe t weeks after birth
  - $P(t)$ is the number of bacteria on an apple t seconds after removing from protective packaging
  - $P(t)$ is the population of rabbits in a woodlands area t weeks after releasing an initial amount into the area

What are possible limitations of a logistic model?

- A logistic graph is **bounded** by the limit L
  - However in real-life the variable might be unbounded
    - For example: the cumulative total number of births in a town over time
- A logistic graph is **always increasing**
  - However in real-life there could be periods where the variable decreased or fluctuates

Your notes

Worked example

The number of fish in a lake, F, can be modelled by the function

$$F(t) = \frac{800}{1 + Ce^{-0.6t}}$$

where t is the number of months after fish were introduced to the lake.

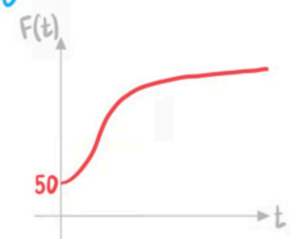
a) Initially, 50 fish were introduced to the lake. Find the value of C.

Substitute $t = 0$ and $F = 50$

$$50 = \frac{800}{1 + Ce^0}$$

$$50 = \frac{800}{1 + C}$$

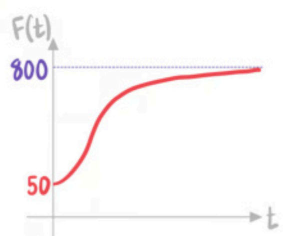
$$\boxed{C = 15}$$



b) Write down the limiting capacity for the number of fish in the lake.

Find the horizontal asymptote

$\boxed{\text{Limiting capacity is } 800}$



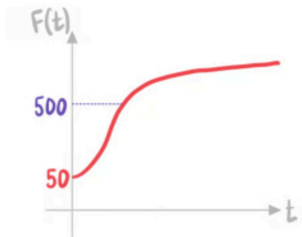
c) Calculate the number of months it takes until there are 500 fish in the lake.

Solve $F(t) = 500$

$$\frac{800}{1+15e^{-0.6t}} = 500$$

$$t = 5.3647...$$

$$\boxed{5.36 \text{ months}}$$



2.6.4 Piecewise Models

Your notes

Linear Piecewise Models

What are the parameters of a piecewise linear model?

- A **piecewise linear model** is made up of multiple linear models $f_i(x) = m_i x + c_i$
- For each linear model there will be
  - The rate of change for that interval m_i
  - The value if the independent variable was not present c_i

What can be modelled as a piecewise linear model?

- Piecewise linear models can be used when the rate of change of a function changes for different intervals
  - These commonly apply when there are different tariffs or levels of charges
- Anything with a constant rate of change for set intervals
  - *C*(*d*) is the taxi charge for a journey of *d* km
    - The charge might double after midnight
  - *R*(*d*) is the rental fee for a car used for *d* days
    - The daily fee might triple if the car is rented over bank holidays
  - *s*(*t*) is the speed of a car travelling for *t* seconds with constant acceleration
    - The car might reach a maximum speed

What are possible limitations of a piecewise linear model?

- Piecewise linear models have a constant rate of change (represented by a straight line) in each interval
  - In real-life this might not be the case
  - The data in some intervals might have a continuously variable rate of change (represented by a curve) rather than a constant rate
  - Or the transition from one constant rate of change to another may be gradual- i.e. a curve rather than a sudden change in gradient

Examiner Tip

- Make sure that you know how to plot a piecewise model on your GDC

Your notes

Worked example

The total monthly charge, £ C, of phone bill can be modelled by the function

$$C(m) = \begin{cases} 10 + 0.02m & 0 \le m \le 100 \\ 9 + 0.03m & m > 100 \end{cases},$$

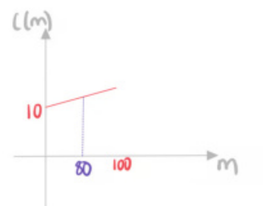
where m is the number of minutes used.

a) Find the total monthly charge if 80 minutes have been used.

Substitute $m = 80$ into the first function

$C(80) = 10 + 0.02(80)$

$\boxed{£11.60}$



b) Given that the total monthly charge is £16.59, find the number of minutes that were used.

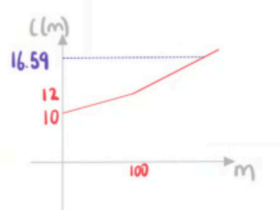
Substitute $C = 16.59$ into the second function

$16.59 = 9 + 0.03m$

$0.03m = 7.59$

$m = \dfrac{7.59}{0.03}$

$\boxed{253 \text{ minutes}}$



Your notes

Non-Linear Piecewise Models

What are the parameters of non-linear piecewise models?

- A **non-linear piecewise model** is made up of multiple functions $f_i(x)$
  - Each function will be defined for a range of values of *x*
- The individual functions can contain **any function**
  - For example: quadratic, cubic, exponential, etc
- When graphed the individual functions should join to make a continuous graph
  - This fact can be used to find unknown parameters
    - If $f(x) = \begin{cases} f_1(x) & a \leq x < b \\ f_2(x) & b \leq x < c \end{cases}$ then $f_1(b) = f_2(b)$

What can be modelled as a non-linear piecewise model?

- Piecewise models can be used when different functions are needed to represent the output for different intervals of the variable
  - *S*(*x*) is the standardised score on a test with *x* raw marks
    - For small values of *x* there might be a quadratic model
    - For large values of *x* there might be a linear model
  - *H*(*t*) is the height of water in a bathtub with after *t* minutes
    - Initially a cubic model might be a appropriate if the bottom of the bathtub is curved
    - Then a linear model might be a appropriate if the sides of top of the bathtub has the shape of a prism

What are possible limitations a non-linear piecewise model?

- Piecewise models can be used to model real-life accurately
- Piecewise models can be difficult to analyse or apply mathematical techniques to

Examiner Tip

- Read and re-read the question carefully, try to get involved in the context of the question!
- Pay particular attention to the domain of each section, if it is not given think carefully about any restrictions there may be as a result of the context of the question
- If sketching a piecewise function, make sure to include the coordinates of all key points including the point at which two sections of the piecewise model meet

Your notes

Worked example

Jamie is running a race. His distance from the start, x metres, can be modelled by the function

$$x(t) = \begin{cases} 3t & 0 \le t < 5 \\ 125 - a(t-15)^2 & 5 \le t < 15 \end{cases}$$

where t is the time, in seconds, elapsed since the start of the race.

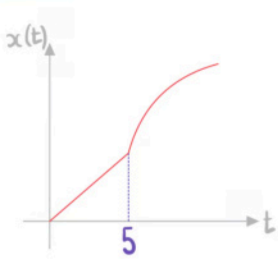
a) Find the value of a.

The function is continuous at $t = 5$

$3(5) = 125 - a(5 - 15)^2$

$15 = 125 - 100a$

$\boxed{a = 1.1}$



b) Find the time taken for Jamie to reach 100 metres from the start.

Decide which function to use

$x(5) = 15$

$100 > 15$

Solve $x(t) = 100$

$125 - 1.1(t - 15)^2 = 100$

$t = 10.23...$ or $t = 19.76...$

Reject as $5 \le t < 15$

$\boxed{t = 10.2 \text{ seconds}}$