

# DP IB Maths: AA SL



## 3.6 Trigonometric Equations & Identities

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Your notes

## 3.6.1 Simple Identities

### Simple Identities

#### What is a trigonometric identity?

- Trigonometric identities are statements that are true for all values of  $x$  or  $\theta$
- They are used to help simplify trigonometric equations before solving them
- Sometimes you may see identities written with the symbol  $\equiv$ 
  - This means 'identical to'

#### What trigonometric identities do I need to know?

- The two trigonometric identities you must know are
  - $\tan \theta = \frac{\sin \theta}{\cos \theta}$ 
    - This is the identity for  $\tan \theta$
  - $\sin^2 \theta + \cos^2 \theta = 1$ 
    - This is the Pythagorean identity
    - Note that the notation  $\sin^2 \theta$  is the same as  $(\sin \theta)^2$
- Both identities can be found **in the formula booklet**
- Rearranging the second identity often makes it easier to work with
  - $\sin^2 \theta = 1 - \cos^2 \theta$
  - $\cos^2 \theta = 1 - \sin^2 \theta$

#### Where do the trigonometric identities come from?

- You do not need to know the proof for these identities but it is a good idea to know where they come from
- From SOHCAHTOA we know that
  - $\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{O}{H}$
  - $\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{A}{H}$
  - $\tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{O}{A}$
- The identity for  $\tan \theta$  can be seen by dividing  $\sin \theta$  by  $\cos \theta$ 
  - $\frac{\sin \theta}{\cos \theta} = \frac{\frac{O}{H}}{\frac{A}{H}} = \frac{O}{A} = \tan \theta$
- This can also be seen from the unit circle by considering a right-triangle with a hypotenuse of 1

$$\tan \theta = \frac{O}{A} = \frac{\sin \theta}{\cos \theta}$$

- The Pythagorean identity can be seen by considering a right-triangle with a hypotenuse of 1
  - Then  $(\text{opposite})^2 + (\text{adjacent})^2 = 1$
  - Therefore  $\sin^2 \theta + \cos^2 \theta = 1$
- Considering the equation of the unit circle also shows the Pythagorean identity
  - The equation of the unit circle is  $x^2 + y^2 = 1$
  - The coordinates on the unit circle are  $(\cos \theta, \sin \theta)$
  - Therefore the equation of the unit circle could be written  $\cos^2 \theta + \sin^2 \theta = 1$
- A third very useful identity is  $\sin \theta = \cos(90^\circ - \theta)$  or  $\sin \theta = \cos\left(\frac{\pi}{2} - \theta\right)$ 
  - This is not included in the formula booklet but is useful to remember

### How are the trigonometric identities used?

- Most commonly trigonometric identities are used to change an equation into a form that allows it to be solved
- They can also be used to prove further identities such as the **double angle formulae**

#### Examiner Tip

- If you are asked to show that one thing is identical ( $\equiv$ ) to another, look at what parts are missing – for example, if  $\tan x$  has gone it must have been substituted



Your notes



Your notes

### Worked example

Show that the equation  $2\sin^2 x - \cos x = 0$  can be written in the form  $a\cos^2 x + b\cos x + c = 0$ , where  $a$ ,  $b$  and  $c$  are integers to be found.

$$2\sin^2 x - \cos x = 0$$

Equation has both  $\sin x$  and  $\cos x$  so will need changing before it can be solved.

Use the identity  $\sin^2 x = 1 - \cos^2 x$

$$\text{Substitute: } 2(1 - \cos^2 x) - \cos x = 0$$

$$\text{Expand: } 2 - 2\cos^2 x - \cos x = 0$$

$$\text{Rearrange: } 2\cos^2 x + \cos x - 2 = 0$$

$$\boxed{a = 2, b = 1, c = -2}$$



Your notes

## 3.6.2 Double Angle Formulae

### Double Angle Formulae

#### What are the double angle formulae?

- The **double angle formulae** for **sine** and **cosine** are:
  - $\sin 2\theta = 2\sin \theta \cos \theta$
  - $\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2\cos^2 \theta - 1 = 1 - 2\sin^2 \theta$
- These can be found in the formula booklet
  - You do not need to know how to derive them

#### How are the double angle formulae used?

- Double angle formulae will often be used with...
  - ... trigonometry exact values
  - ... graphs of trigonometric functions
  - ... relationships between trigonometric ratios
- To help solve trigonometric equations which contain  $\sin \theta \cos \theta$ :
  - Substitute  $\frac{1}{2} \sin 2\theta$  for  $\sin \theta \cos \theta$
  - Solve for  $2\theta$ , finding all values in the range for  $2\theta$ 
    - The range will need adapting for  $2\theta$
  - Find the solutions for  $\theta$
- To help solve trigonometric equations which contain  $\sin 2\theta$  and  $\sin \theta$  or  $\cos \theta$ 
  - Substitute  $2\sin \theta \cos \theta$  for  $\sin 2\theta$
  - Isolate all terms in  $\theta$
  - Factorise or use another identity to write the equation in a form which can be solved
- To help solve trigonometric equations which contain  $\cos 2\theta$  and  $\sin \theta$  or  $\cos \theta$ 
  - Substitute either  $2\cos^2 \theta - 1$  or  $1 - 2\sin^2 \theta$  for  $\cos 2\theta$ 
    - Choose the trigonometric ratio that is already in the equation
  - Isolate all terms in  $\theta$
  - Solve
    - The equation will most likely be in the form of a quadratic
- Double angle formulae can be used in proving other trigonometric identities





Your notes

### 3.6.3 Relationship Between Trigonometric Ratios

## Relationship Between Trigonometric Ratios

### What relationships between trigonometric ratios should I know?

- If you know a value for one trig ratio you can often use this to work out the value for the others without needing to find  $\theta$
- If you know that  $\sin \theta = \frac{a}{b}$ , where  $a, b \in \mathbb{Z}^+$ , you can:
  - Sketch a right-triangle with  $a$  opposite  $\theta$  and  $b$  on the hypotenuse
  - Use Pythagoras' theorem to find the value of the adjacent side
  - Use SOHCAHTOA to find the values of  $\cos \theta$  and  $\tan \theta$
- If you know a value for  $\sin \theta$  or  $\cos \theta$  you can use the Pythagorean relationship
  - $\sin^2 \theta + \cos^2 \theta = 1$
  - to find the value of the other
- If you know a value for  $\sin \theta$  or  $\cos \theta$  you can use the double angle formulae to find the value of  $\sin 2\theta$  or  $\cos 2\theta$
- If you know two out of the three values for  $\sin \theta$ ,  $\cos \theta$  or  $\tan \theta$  you can use the identity in  $\tan$ 
  - $\tan \theta = \frac{\sin \theta}{\cos \theta}$
  - to find the value of the third ratio

### How do we determine whether a trigonometric ratio will be positive or negative?

- It is possible to determine whether a trigonometric ratio will be positive or negative by looking at the size of the angle and considering the **unit circle**
  - Angles in the range  $0^\circ < \theta < 90^\circ$  will be positive for all three ratios
  - Angles in the range  $90^\circ < \theta < 180^\circ$  will be positive for  $\sin$  and negative for  $\cos$  and  $\tan$
  - Angles in the range  $180^\circ < \theta < 270^\circ$  will be positive for  $\tan$  and negative for  $\sin$  and  $\cos$
  - Angles in the range  $270^\circ < \theta < 360^\circ$  will be positive for  $\cos$  and negative for  $\sin$  and  $\tan$
- The ratios for angles of  $0^\circ$ ,  $90^\circ$ ,  $180^\circ$ ,  $270^\circ$  and  $360^\circ$  are either 0, 1, -1 or undefined
  - You should know these ratios or know how to derive them without a calculator

#### Examiner Tip

- Being able to sketch out the unit circle and remembering CAST can help you to find all solutions to a problem in an exam question



Your notes

### Worked example

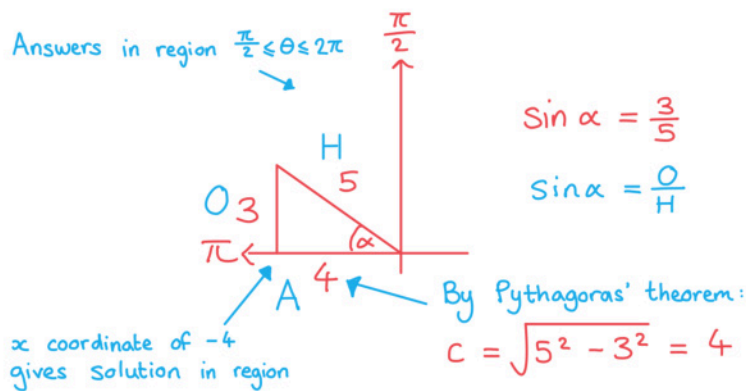
The value of  $\sin \alpha = \frac{3}{5}$  for  $\frac{\pi}{2} \leq \alpha \leq \pi$ . Find:

i)  $\cos \alpha$

Method 1: Use right-triangle:

$$\frac{\pi}{2} \leq \alpha \leq \pi$$

Answers in region  $\frac{\pi}{2} \leq \theta \leq 2\pi$



$$\sin \alpha = \frac{3}{5}$$

$$\sin \alpha = \frac{O}{H}$$

$$\cos \alpha = \frac{A}{H} = -\frac{4}{5}$$

$$\boxed{\cos \alpha = -\frac{4}{5}}$$

Method 2: Use Pythagorean identity:

$$\cos^2 \alpha = 1 - \sin^2 \alpha = 1 - \left(\frac{3}{5}\right)^2$$

$$\cos \alpha = \pm \sqrt{\frac{16}{25}} = \pm \frac{4}{5}$$

Check which solution is in range.

ii)  $\tan \alpha$

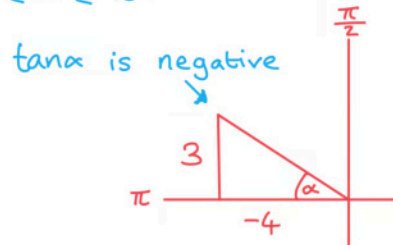




Your notes

$$\text{Use } \tan \alpha = \frac{\sin \alpha}{\cos \alpha} = \frac{\frac{3}{5}}{-\frac{4}{5}} = -\frac{3}{4}$$

Check if  $\tan \alpha = -\frac{3}{4}$  is in the correct range for  $\frac{\pi}{2} \leq \alpha \leq \pi$ :



$$\tan \alpha = -\frac{3}{4}$$

iii)  $\sin 2\alpha$

Double angle identity:  $\sin 2\theta = 2\sin \theta \cos \theta$

$$\begin{aligned} \sin 2\alpha &= 2\sin \alpha \cos \alpha \\ &= 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right) \\ &= -\frac{24}{25} \end{aligned}$$

$$\begin{aligned} \sin \alpha &= \frac{3}{5} \\ \cos \alpha &= -\frac{4}{5} \end{aligned}$$

$$\sin 2\alpha = -\frac{24}{25}$$

iv)  $\cos 2\alpha$



Your notes

Double angle identity:  $\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$ 

$$\cos 2\alpha = \left(-\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{7}{25}$$

$$\cos 2\alpha = \frac{7}{25}$$

v)  $\tan 2\alpha$ Using identity  $\tan \theta = \frac{\sin \theta}{\cos \theta}$ 

$$\tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha} = \frac{-\frac{24}{25}}{\frac{7}{25}} = -\frac{24}{7}$$

$$\tan 2\alpha = -\frac{24}{7}$$



Your notes

## 3.6.4 Linear Trigonometric Equations

### Trigonometric Equations: $\sin x = k$

#### How are trigonometric equations solved?

- Trigonometric equations can have an infinite number of solutions
  - For an equation in  $\sin$  or  $\cos$  you can add  $360^\circ$  or  $2\pi$  to each solution to find more solutions
  - For an equation in  $\tan$  you can add  $180^\circ$  or  $\pi$  to each solution
- When solving a trigonometric equation you will be given a range of values within which you should find all the values
- Solving the equation normally and using the inverse function on your calculator or your knowledge of **exact values** will give you the **primary value**
- The **secondary values** can be found with the help of:
  - The **unit circle**
  - The **graphs of trigonometric functions**

#### How are trigonometric equations of the form $\sin x = k$ solved?

- It is a good idea to sketch the graph of the trigonometric function first
  - Use the given range of values as the domain for your graph
  - The intersections of the graph of the function and the line  $y = k$  will show you
    - The location of the solutions
    - The number of solutions
  - You will be able to use the symmetry properties of the graph to find all secondary values within the given range of values
- The method for finding secondary values are:
  - For the equation  $\sin x = k$  the primary value is  $x_1 = \sin^{-1} k$ 
    - A secondary value is  $x_2 = 180^\circ - \sin^{-1} k$
    - Then all values within the range can be found using  $x_1 \pm 360n$  and  $x_2 \pm 360n$  where  $n \in \mathbb{N}$
  - For the equation  $\cos x = k$  the primary value is  $x_1 = \cos^{-1} k$ 
    - A secondary value is  $x_2 = -\cos^{-1} k$
    - Then all values within the range can be found using  $x_1 \pm 360n$  and  $x_2 \pm 360n$  where  $n \in \mathbb{N}$
  - For the equation  $\tan x = k$  the primary value is  $x = \tan^{-1} k$ 
    - All secondary values within the range can be found using  $x \pm 180n$  where  $n \in \mathbb{N}$



Your notes

### Examiner Tip

- If you are using your GDC it will only give you the principal value and you need to find all other solutions for the given interval
- Sketch out the CAST diagram and the trig graphs on your exam paper to refer back to as many times as you need to

### Worked example

Solve the equation  $2\cos x = -1$ , finding all solutions in the range  $-\pi \leq x \leq \pi$ .

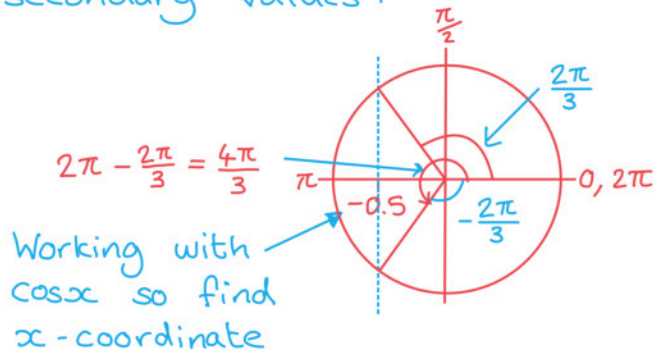
Isolate  $\cos x$ :  $\cos x = \frac{-1}{2}$

use GDC or  
knowledge of  
exact values

$$x = \cos^{-1}\left(-\frac{1}{2}\right)$$

$$= \frac{2\pi}{3} \leftarrow \text{Primary value}$$

Find secondary values:



$$\frac{2\pi}{3} \pm 2\pi n \quad \text{and} \quad \frac{4\pi}{3} \pm 2\pi n$$

Find all answers in range  $-\pi \leq x \leq \pi$

$$\boxed{-\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}}$$

## Trigonometric Equations: $\sin(ax + b) = k$

### How can I solve equations with transformations of trig functions?

- Trigonometric equations in the form  $\sin(ax + b)$  can be solved in more than one way
- The easiest method is to consider the transformation of the angle as a substitution
  - For example let  $u = ax + b$
- Transform the given interval for the solutions in the same way as the angle
  - For example if the given interval is  $0^\circ \leq x \leq 360^\circ$  the new interval will be
  - $(a(0^\circ) + b) \leq u \leq (a(360^\circ) + b)$
- Solve the function to find the primary value for  $u$
- Use either the unit circle or sketch the graph to find all the other solutions in the range for  $u$
- Undo the substitution to convert all of the solutions back into the corresponding solutions for  $x$
- Another method would be to sketch the transformation of the function
  - If you use this method then you will not need to use a substitution for the range of values

#### Examiner Tip

- If you transform the interval, remember to convert the found angles back to the final values at the end!
- If you are using your GDC it will only give you the principal value and you need to find all other solutions for the given interval
- Sketch out the CAST diagram and the trig graphs on your exam paper to refer back to as many times as you need to



Your notes



Your notes

### Worked example

Solve the equation  $2\cos(2x - 30^\circ) = -1$ , finding all solutions in the range  $-360^\circ \leq x \leq 360^\circ$ .

$$2\cos(2x - 30^\circ) = -1 \quad -360^\circ \leq x \leq 360^\circ$$

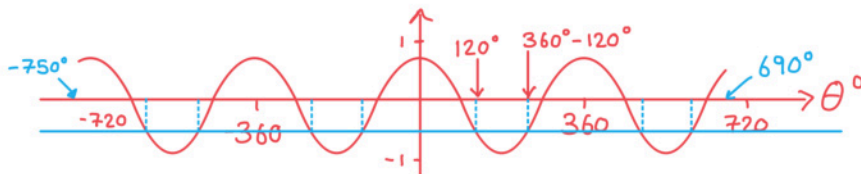
Start by changing the range:  $-750^\circ \leq 2x - 30 \leq 690^\circ$

Substitute  $\theta = 2x - 30$ :

$$2\cos\theta = -1 \quad -750^\circ \leq \theta \leq 690^\circ$$

$$\cos\theta = -\frac{1}{2}$$

$$\theta = \cos^{-1}\left(-\frac{1}{2}\right) = 120^\circ \leftarrow \text{Primary value}$$



From the sketch you can see there are 8 solutions:

$$\theta = 120^\circ \pm 360^\circ \text{ and } \theta = 240^\circ \pm 360^\circ$$

$$\theta = -600^\circ, -480^\circ, -240^\circ, -120^\circ, 120^\circ, 240^\circ, 480^\circ, 600^\circ$$

Solve for  $x$ :  $x = \frac{\theta + 30}{2}$

$$x = -285^\circ, -225^\circ, -105^\circ, -45^\circ, 75^\circ, 135^\circ, 255^\circ, 315^\circ$$



Your notes

## 3.6.5 Quadratic Trigonometric Equations

### Quadratic Trigonometric Equations

#### How are quadratic trigonometric equations solved?

- A quadratic trigonometric equation is one that includes either  $\sin^2 \theta$ ,  $\cos^2 \theta$  or  $\tan^2 \theta$
- Often the **identity**  $\sin^2 \theta + \cos^2 \theta = 1$  can be used to rearrange the equation into a form that is possible to solve
  - If the equation involves both sine and cosine then the **Pythagorean identity** should be used to write the equation in terms of just one of these functions
- Solve the **quadratic equation** using your GDC, the quadratic equation or factorisation
  - This can be made easier by changing the function to a single letter
    - Such as changing  $2\cos^2 \theta - 3\cos \theta - 1 = 0$  to  $2c^2 - 3c - 1 = 0$
- A quadratic can give up to two solutions
  - You must consider both solutions to see whether a real value exists
  - Remember that solutions for  $\sin \theta = k$  and  $\cos \theta = k$  only exist for  $-1 \leq k \leq 1$
  - Solutions for  $\tan \theta = k$  exist for all values of  $k$
- Find all solutions within the given interval
  - There will often be more than two solutions for one quadratic equation
  - The best way to check the number of solutions is to sketch the graph of the function

#### Examiner Tip

- Sketch the trig graphs on your exam paper to refer back to as many times as you need to!
- Be careful to make sure you have found **all** of the solutions in the given interval, being super-careful if you get a negative solution but have a positive interval



Your notes

### Worked example

Solve the equation  $11\sin x - 7 = 5\cos^2 x$ , finding all solutions in the range  $0 \leq x \leq 2\pi$ .

Use the identity  $\cos^2 x = 1 - \sin^2 x$  to write equation in terms of  $\sin x$ :

$$\begin{aligned} 11\sin x - 7 &= 5(1 - \sin^2 x) \text{ in formula booklet.} \\ &= 5 - 5\sin^2 x \end{aligned}$$

Move all terms to one side:

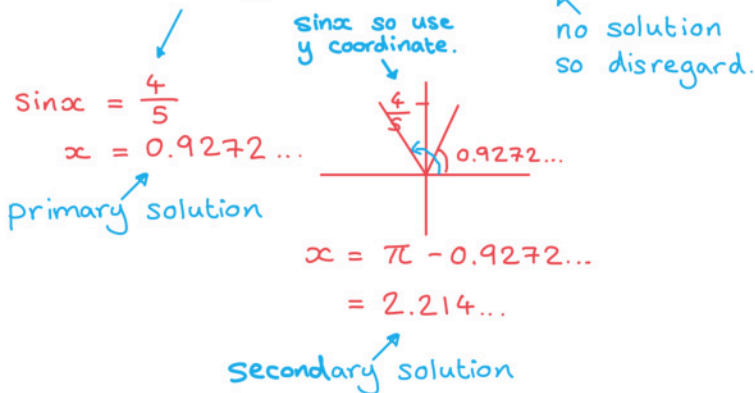
$$11\sin x - 7 - (5 - 5\sin^2 x) = 0$$

Spot the hidden quadratic:

$$11\sin x - 7 - 5 + 5\sin^2 x = 0$$

$$5\sin^2 x + 11\sin x - 12 = 0$$

$$\sin x = \frac{4}{5} \text{ or } \sin x = -3$$



$$x = 0.927, 2.21 \text{ (3 s.f.)}$$