



# DP IB Maths: AI HL

  
Your notes

## 2.2 Further Functions & Graphs

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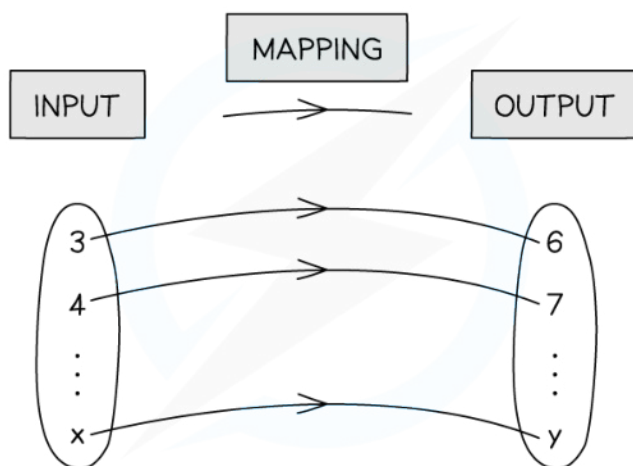
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## 2.2.1 Functions

### Language of Functions

#### What is a mapping?

- A **mapping transforms** one set of values (**inputs**) into another set of values (**outputs**)
- Mappings can be:
  - **One-to-one**
    - Each input gets mapped to **exactly one unique** output
    - No two inputs are mapped to the same output
    - For example: A mapping that cubes the input
  - **Many-to-one**
    - Each input gets mapped to **exactly one** output
    - Multiple inputs can be mapped to the same output
    - For example: A mapping that squares the input
  - **One-to-many**
    - An input can be mapped to **more than one** output
    - No two inputs are mapped to the same output
    - For example: A mapping that gives the numbers which when squared equal the input
  - **Many-to-many**
    - An input can be mapped to **more than one** output
    - Multiple inputs can be mapped to the same output
    - For example: A mapping that gives the factors of the input



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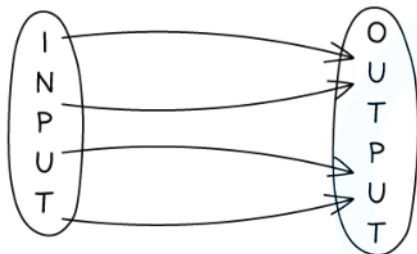


#### What is a function?

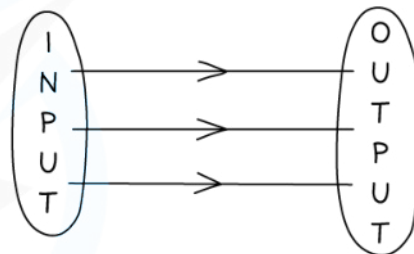


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- A **function** is a mapping between two sets of numbers where **each input** gets mapped to **exactly one output**
  - The output does not need to be unique
- **One-to-one** and **many-to-one** mappings are functions
- A mapping is a function if its graph passes the **vertical line test**
  - Any **vertical line** will intersect with the graph **at most once**



MANY-TO-ONE MAPPINGS ARE FUNCTIONS



ONE-TO-ONE MAPPINGS ARE FUNCTIONS

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### What notation is used for functions?

- Functions are denoted using letters (such as  $f$ ,  $v$ ,  $g$ , etc)
  - A function is followed by a variable in a bracket
  - This shows the input for the function
  - The letter  $f$  is used most commonly for functions and will be used for the remainder of this revision note
- $f(x)$  represents an expression for the value of the function  $f$  when evaluated for the variable  $x$
- Function notation gets rid of the need for words which makes it **universal**
  - $f = 5$  when  $x = 2$  can simply be written as  $f(2) = 5$

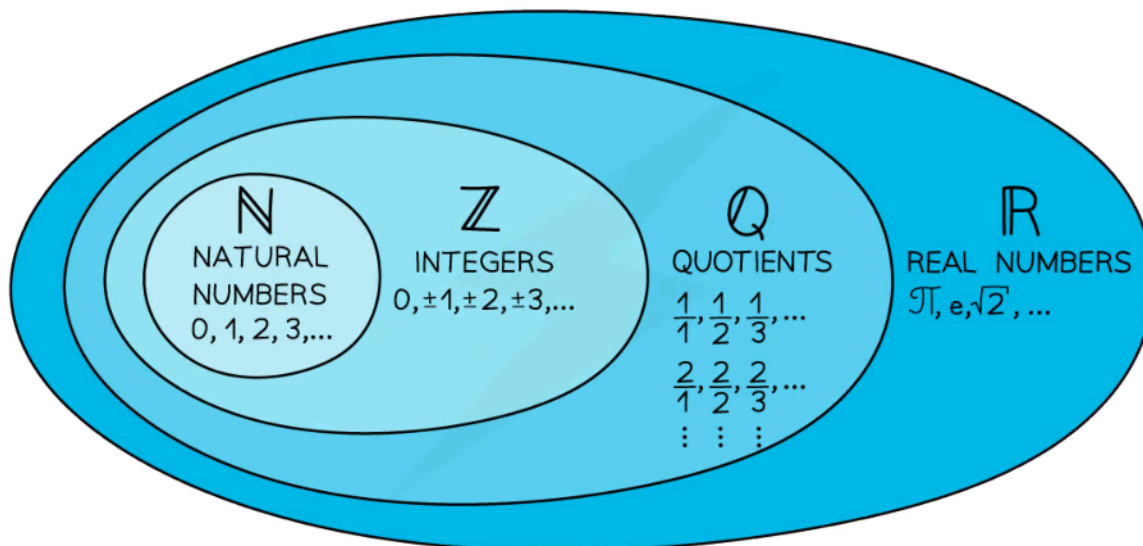
### What are the domain and range of a function?

- The **domain** of a function is the set of values that are used as **inputs**
- A domain should be stated with a function
  - If a domain is not stated then it is assumed the domain is all the real values which would work as inputs for the function
  - Domains are expressed in terms of the input
    - $x \leq 2$
- The **range** of a function is the set of values that are given as **outputs**
  - The range depends on the domain
  - Ranges are expressed in terms of the output



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- $f(x) \geq 0$
- To graph a function we use the **inputs as the x-coordinates** and the **outputs as the y-coordinates**
  - $f(2) = 5$  corresponds to the coordinates (2, 5)
- Graphing the function can help you visualise the range
- Common sets of numbers have special symbols:
  - $\mathbb{R}$  represents all the real numbers that can be placed on a number line
    - $x \in \mathbb{R}$  means  $x$  is a real number
  - $\mathbb{Q}$  represents all the rational numbers  $\frac{a}{b}$  where  $a$  and  $b$  are integers and  $b \neq 0$
  - $\mathbb{Z}$  represents all the integers (positive, negative and zero)
    - $\mathbb{Z}^+$  represents positive integers
  - $\mathbb{N}$  represents the natural numbers (0,1,2,3...)



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### Examiner Tip

- Questions may refer to "the largest possible domain"
  - This would usually be  $x \in \mathbb{R}$  unless natural numbers, integers or quotients has already been stated
  - There are usually some exceptions
    - e.g.  $x \geq 0$  for functions involving a square root (so the function can be 1-to-1 and have an inverse)
    - e.g.  $x \neq 2$  for a reciprocal function with denominator  $x-2$



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 **Worked example**For the function  $f(x) = x^3 + 1$ ,  $2 \leq x \leq 10$ :

- a) write down the value of
- $f(7)$
- .

Substitute  $x = 7$ 

$$f(7) = 7^3 + 1$$

$$f(7) = 344$$

- b) find the range of
- $f(x)$
- .

Find the values of  $x^3 + 1$  when  $2 \leq x \leq 10$ 

$$2 \leq x \leq 10$$

$$8 \leq x^3 \leq 1000$$

$$9 \leq x^3 + 1 \leq 1001$$

$$9 \leq f(x) \leq 1001$$



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## Piecewise Functions

### What are piecewise functions?

- **Piecewise functions** are defined by different functions depending on which interval the input is in
  - E.g.  $f(x) = \begin{cases} x+1 & x \leq 5 \\ 2x-4 & 5 < x < 10 \end{cases}$
- The region for the individual functions **cannot overlap**
- To evaluate a piecewise function for a particular value  $x = k$ 
  - Find which interval includes  $k$
  - Substitute  $x = k$  into the corresponding function

### Worked example

For the piecewise function

$$f(x) = \begin{cases} 2x - 5 & -10 \leq x \leq 10 \\ 3x + 1 & x > 10 \end{cases}$$

- a) find the values of  $f(0)$ ,  $f(10)$ ,  $f(20)$ .

Identify the correct function to use

$$x=0 \text{ is in } -10 \leq x \leq 10 \Rightarrow f(0) = 2(0) - 5 = -5$$

$$x=10 \text{ is in } -10 \leq x \leq 10 \Rightarrow f(10) = 2(10) - 5 = 15$$

$$x=20 \text{ is in } x > 10 \Rightarrow f(20) = 3(20) + 1 = 61$$

$$f(0) = -5 \quad f(10) = 15 \quad f(20) = 61$$

- b) state the domain.

Domain is the set of inputs

$$-10 \leq x \leq 10 \text{ and } x > 10$$

$$x \geq -10$$



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## 2.2.2 Graphing Functions

### Graphing Functions

#### How do I graph the function $y = f(x)$ ?

- A point  $(a, b)$  lies on the graph  $y = f(x)$  if  $f(a) = b$
- The **horizontal axis** is used for the **domain**
- The **vertical axis** is used for the **range**
- You will be able to graph some functions by hand
- For some functions you will need to use your GDC
- You might be asked to graph the **sum** or **difference** of two functions
  - Use your GDC to graph  $y = f(x) + g(x)$  or  $y = f(x) - g(x)$
  - Just type the functions into the graphing mode

#### What is the difference between “draw” and “sketch”?

- If asked to sketch you should:
  - Show the general shape
  - Label any key points such as the intersections with the axes
  - Label the axes
- If asked to draw you should:
  - Use a pencil and ruler
  - Draw to scale
  - Plot any points **accurately**
  - Join points with a straight line or smooth curve
  - Label any key points such as the intersections with the axes
  - Label the axes

#### How can my GDC help me sketch/draw a graph?

- You use your GDC to plot the graph
  - Check the scales on the graph to make sure you see the full shape
- Use your GDC to find any key points
- Use your GDC to check specific points to help you plot the graph

## Key Features of Graphs

### What are the key features of graphs?

- You should be familiar with the following key features and know how to use your GDC to find them
- Local minimums/maximiums
  - These are points where the graph has a minimum/maximum for a small region
  - They are also called **turning points**
    - This is where the graph changes its direction between upwards and downwards directions
  - A graph can have multiple local minimums/maximiums
  - A local minimum/maximum is not necessarily the minimum/maximum of the whole graph
    - This would be called the **global** minimum/maximum
  - For quadratic graphs the minimum/maximum is called the **vertex**
- Intercepts
  - $y$  - intercepts are where the graph crosses the  $y$ -axis
    - At these points  $x = 0$
  - $x$  - intercepts are where the graph crosses the  $x$ -axis
    - At these points  $y = 0$
    - These points are also called the **zeros of the function** or **roots of the equation**
- Symmetry
  - Some graphs have lines of symmetry
    - A quadratic will have a vertical line of symmetry
- Asymptotes
  - These are lines which the graph will get closer to but not cross
  - These can be horizontal or vertical
    - Exponential graphs have horizontal asymptotes
    - Graphs of variables which vary inversely can have vertical and horizontal asymptotes

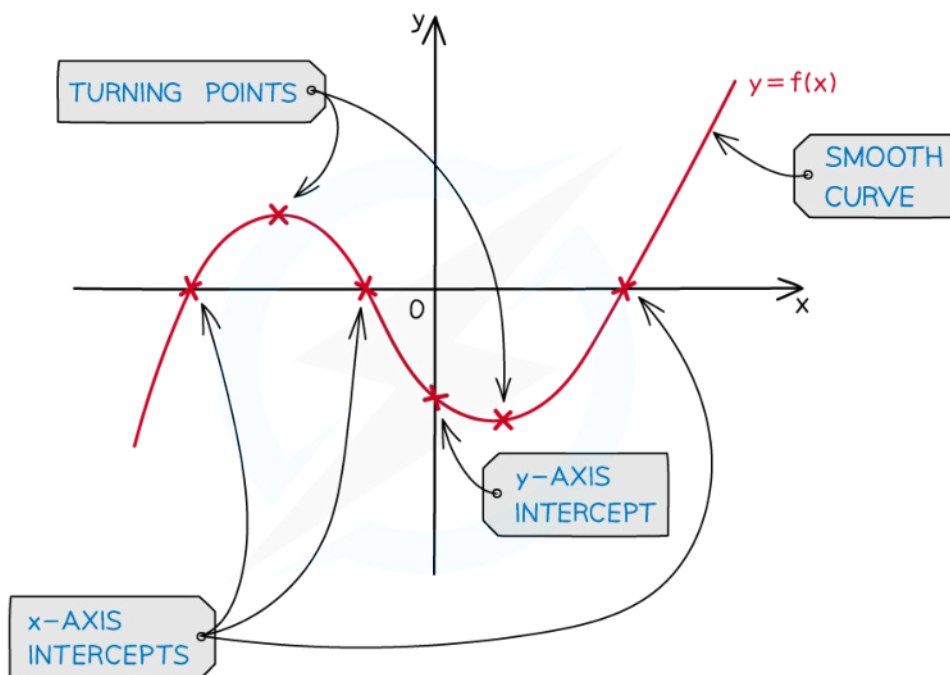


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### Examiner Tip

- Most GDC makes/models will not plot/show asymptotes just from inputting a function
  - Add the asymptotes as additional graphs for your GDC to plot
  - You can then check the equations of your asymptotes visually
  - You may have to zoom in or change the viewing window options to confirm an asymptote
- Even if using your GDC to plot graphs and solve problems sketching them as part of your working is good exam technique
  - Label the key features of the graph and anything else relevant to the question on your sketch



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### Worked example

Two functions are defined by

$$f(x) = x^2 - 4x - 5 \text{ and } g(x) = 2 + \frac{1}{x+1}.$$

a) Draw the graph  $y = f(x)$ .

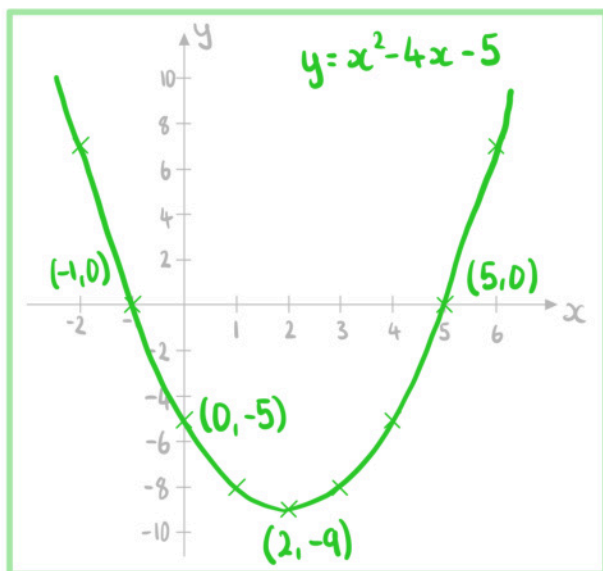
Draw means accurately

Use GDC to find vertex, roots and y-intercepts

Vertex =  $(2, -9)$

Roots =  $(-1, 0)$  and  $(5, 0)$

y-intercept =  $(0, -5)$



b) Sketch the graph  $y = g(x)$ .



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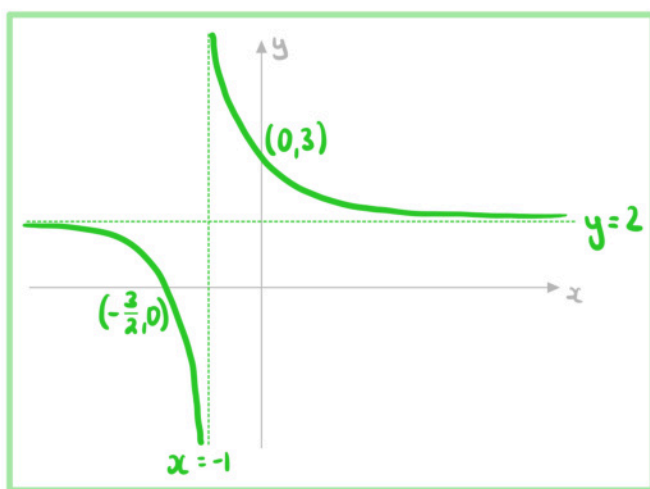
Sketch means rough but showing key points

Use GDC to find  $x$  and  $y$ -intercepts and asymptotes

$$x\text{-intercept} = \left(-\frac{3}{2}, 0\right)$$

$$y\text{-intercept} = (0, 3)$$

Asymptotes :  $x = -1$  and  $y = 2$



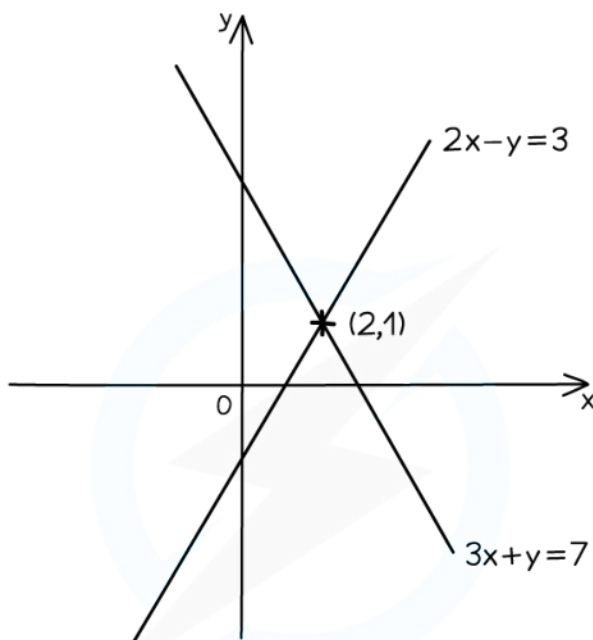


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## Intersecting Graphs

### How do I find where two graphs intersect?

- Plot both graphs on your GDC
- Use the intersect function to find the intersections
- Check if there is more than one point of intersection



- LINES INTERSECT AT (2,1)
- SOLVING  $2x - y = 3$  AND  $3x + y = 7$  SIMULTANEOUSLY IS  $x = 2, y = 1$

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### How can I use graphs to solve equations?

- One method to solve equations is to use graphs
- To solve  $f(x) = a$ 
  - Plot the two graphs  $y = f(x)$  and  $y = a$  on your GDC
  - Find the points of intersections
  - The **x-coordinates** are the **solutions** of the equation
- To solve  $f(x) = g(x)$

- Plot the two graphs  $y = f(x)$  and  $y = g(x)$  on your GDC
- Find the points of intersections
- The **x-coordinates** are the **solutions** of the equation
- Using graphs makes it easier to see **how many solutions** an equation will have

### Examiner Tip

- You can use graphs to solve equations
  - Questions will not necessarily ask for a drawing/sketch or make reference to graphs
  - Use your GDC to plot the equations and find the intersections between the graphs



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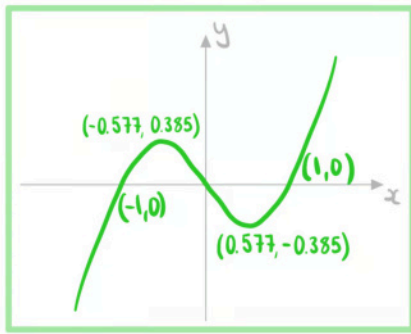
**Worked example**

Two functions are defined by

$$f(x) = x^3 - x \text{ and } g(x) = \frac{4}{x}$$

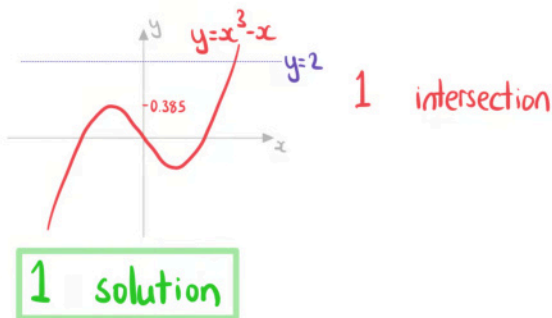
a) Sketch the graph  $y = f(x)$ .

Use GDC to find max, min, intercepts



b) Write down the number of real solutions to the equation  $x^3 - x = 2$ .

Identify the number of intersections between  $y = x^3 - x$  and  $y = 2$

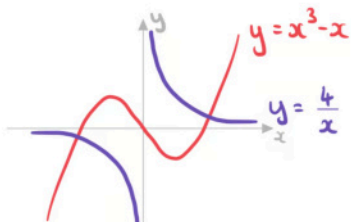


c) Find the coordinates of the points where  $y = f(x)$  and  $y = g(x)$  intersect.



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Use GDC to sketch both graphs



$(-1.60, -2.50)$  and  $(1.60, 2.50)$

d)

Write down the solutions to the equation  $x^3 - x = \frac{4}{x}$ .

Solutions to  $x^3 - x = \frac{4}{x}$  are the  $x$  coordinates of the points of intersection.

$x = -1.60$  and  $x = 1.60$



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## 2.2.3 Properties of Graphs

### Quadratic Functions & Graphs

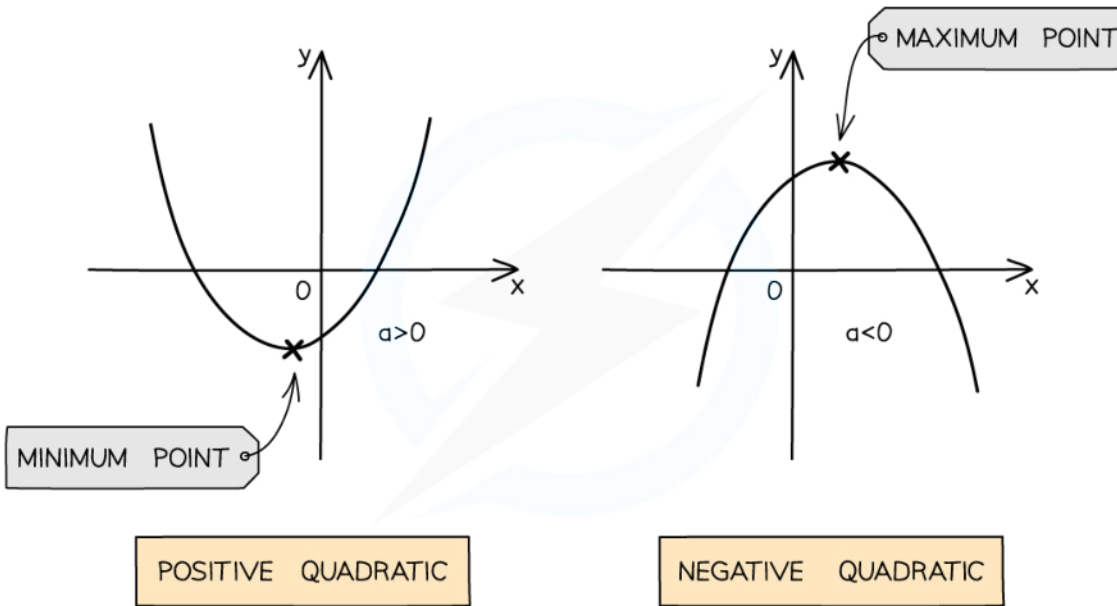
#### What are the key features of quadratic graphs?

- A **quadratic** graph is of the form  $y = ax^2 + bx + c$  where  $a \neq 0$ .
- The value of  $a$  affects the shape of the curve
  - If  $a$  is positive the shape is  $\cup$
  - If  $a$  is negative the shape is  $\cap$
- The **y-intercept** is at the point  $(0, c)$
- The **zeros or roots** are the solutions to  $ax^2 + bx + c = 0$ 
  - These can be found using your GDC or the quadratic formula
  - These are also called the x-intercepts
  - There can be 0, 1 or 2 x-intercepts
- There is an **axis of symmetry** at  $x = -\frac{b}{2a}$ 
  - This is given in your **formula booklet**
  - If there are two x-intercepts then the axis of symmetry goes through the midpoint of them
- The **vertex** lies on the axis of symmetry
  - The x-coordinate is  $-\frac{b}{2a}$
  - The y-coordinate can be found using the GDC or by calculating  $y$  when  $x = -\frac{b}{2a}$
  - If  $a$  is **positive** then the vertex is the **minimum** point
  - If  $a$  is **negative** then the vertex is the **maximum** point

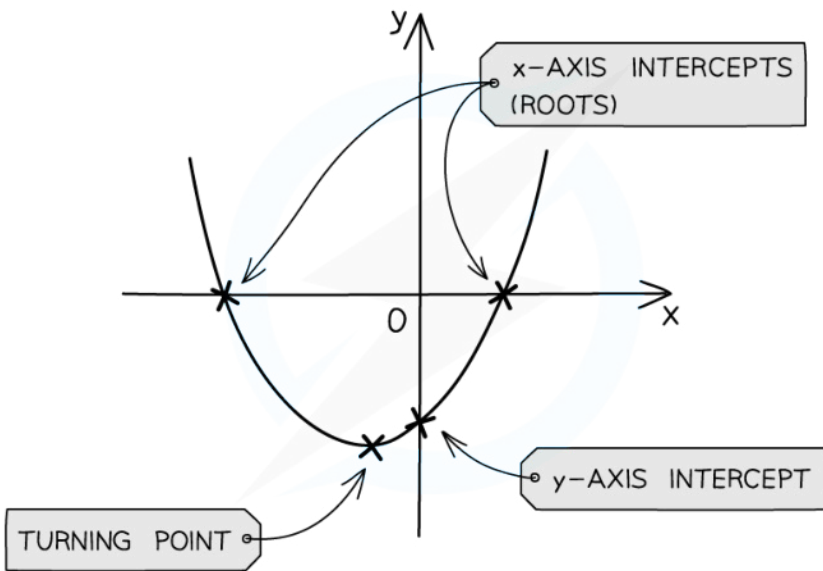




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### Examiner Tip

- Use your GDC to find the roots and the turning point of a quadratic function
  - You do not need to factorise or complete the square
  - It is good exam technique to sketch the graph from your GDC as part of your working

### Worked example

- a) Write down the equation of the axis of symmetry for the graph  $y = 4x^2 - 4x - 3$ .

Formula booklet

Axis of symmetry of the graph of a quadratic function

$$f(x) = ax^2 + bx + c \Rightarrow \text{axis of symmetry is } x = -\frac{b}{2a}$$

$$a = 4 \quad b = -4 \quad c = -3$$

$$x = -\frac{-4}{2(4)}$$

$$x = \frac{1}{2}$$

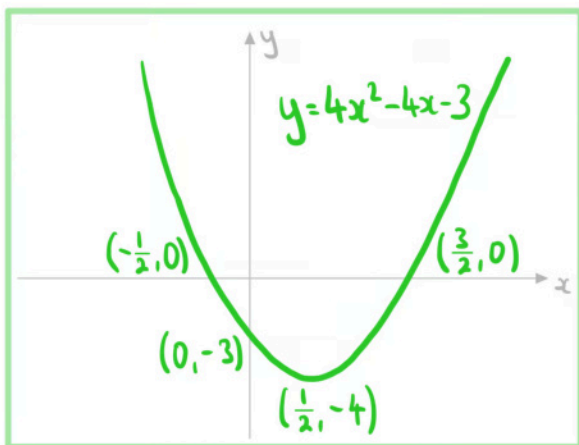
- b) Sketch the graph  $y = 4x^2 - 4x - 3$ .

Use GDC to find vertex, roots and y-intercepts

$$\text{Vertex} = \left(\frac{1}{2}, -4\right)$$

$$\text{Roots} = \left(-\frac{1}{2}, 0\right) \text{ and } \left(\frac{3}{2}, 0\right)$$

$$\text{y-intercept} = (0, -3)$$



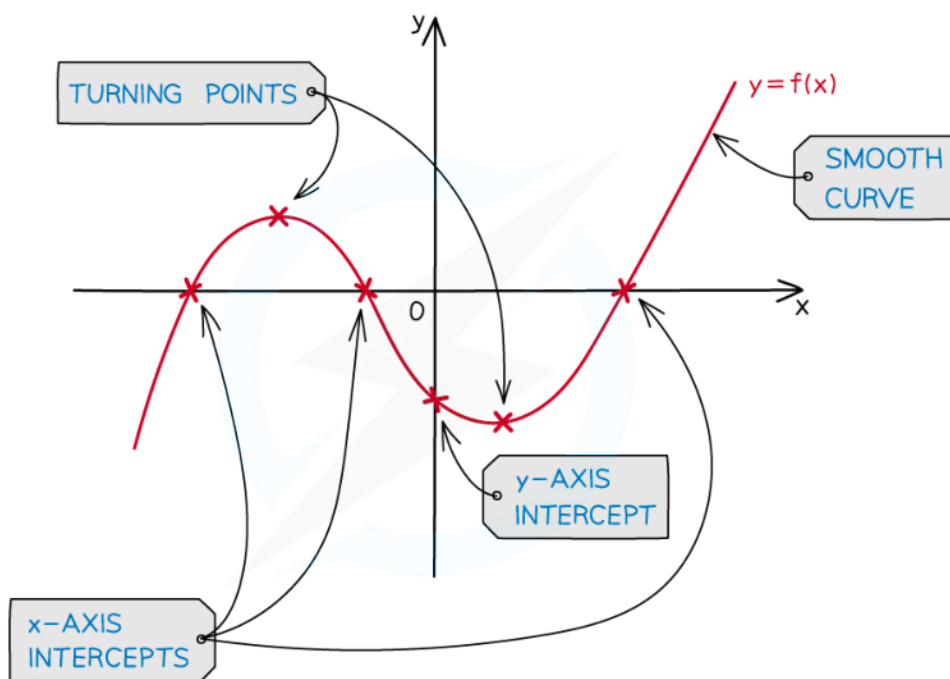


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## Cubic Functions & Graphs

### What are the key features of cubic graphs?

- A **cubic** graph is of the form  $y = ax^3 + bx^2 + cx + d$  where  $a \neq 0$ .
- The value of  $a$  affects the shape of the curve
  - If  $a$  is **positive** the graph goes from **bottom left to top right**
  - If  $a$  is **negative** the graph goes from **top left to bottom right**
- The **y-intercept** is at the point  $(0, d)$
- The **zeros or roots** are the solutions to  $ax^3 + bx^2 + cx + d = 0$ 
  - These can be found using your GDC
  - These are also called the x-intercepts
  - There can be 1, 2 or 3 x-intercepts
    - There is always at least 1
- There are either **0 or 2 local minimums/maximums**
  - If there are 0 then the curve is **monotonic** (always increasing or always decreasing)
  - If there are 2 then one is a local minimum and one is a local maximum



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### Examiner Tip

- Use your GDC to find the roots, the local maximum and local minimum of a cubic function
- When drawing/sketching the graph of a cubic function be sure to label all the key features
  - $x$  and  $y$  axes intercepts
  - the local maximum point
  - the local minimum point

### Worked example

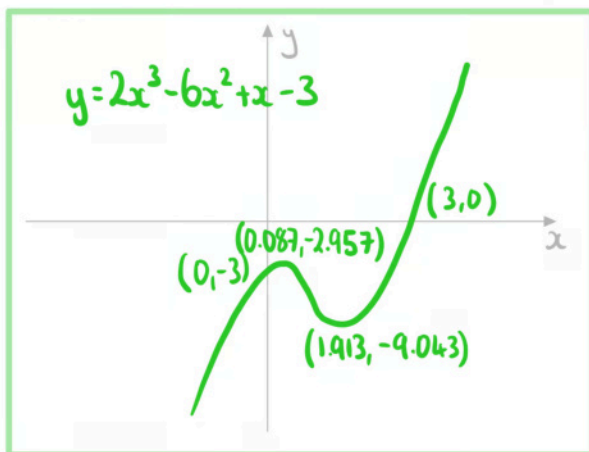
Sketch the graph  $y = 2x^3 - 6x^2 + x - 3$ .

Use GDC to find min, max, roots and y-intercept

$$\text{Min} = (1.913, -9.043) \quad \text{Max} = (0.087, -2.957)$$

$$\text{Root} = (3, 0)$$

$$\text{y-intercept} = (0, -3)$$





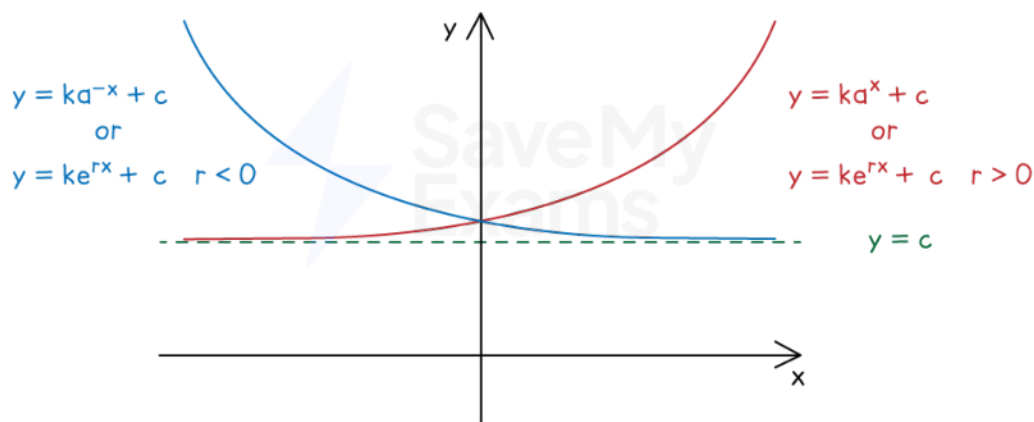
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## Exponential Functions & Graphs

### What are the key features of exponential graphs?

- An **exponential** graph is of the form
  - $y = ka^x + c$  or  $y = ka^{-x} + c$  where  $a > 0$
  - $y = ke^{rx} + c$ 
    - Where  $e$  is the mathematical constant 2.718...
- The **y-intercept** is at the point  $(0, k + c)$
- There is a **horizontal asymptote** at  $y = c$
- The value of  $k$  determines whether the graph is **above or below the asymptote**
  - If  **$k$  is positive** the graph is **above the asymptote**
    - So the range is  $y > c$
  - If  **$k$  is negative** the graph is **below the asymptote**
    - So the range is  $y < c$
- The coefficient of  $x$  and the constant  $k$  determine whether the graph is **increasing or decreasing**
  - If the coefficient of  $x$  and  $k$  have the **same sign** then **graph is increasing**
  - If the coefficient of  $x$  and  $k$  have **different signs** then the **graph is decreasing**
- There is at **most 1 root**
  - It can be found using your GDC

IN BOTH CURVES,  $k > 0$  AND  $a > 1$



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 **Examiner Tip**

- You may have to change the viewing window settings on your GDC to make asymptotes clear
  - A small scale can make it look as though the curve and an asymptote intersect
- Be careful about how two exponential graphs drawn on the same axes look
  - Particularly which one is "on top" either side of the  $y$ -axis



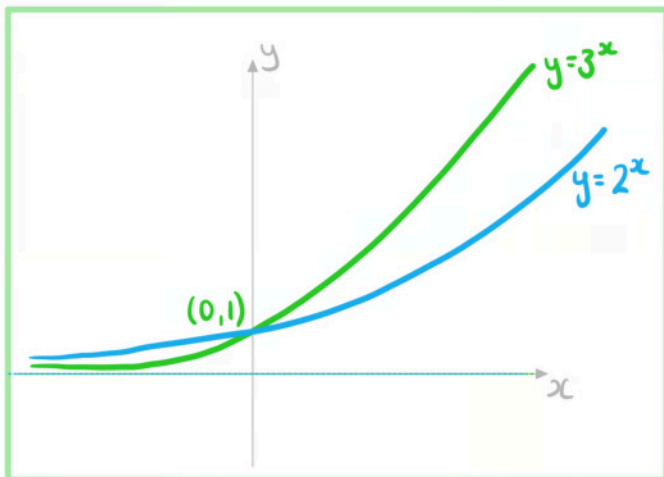
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 **Worked example**

- a) On the same set of axes sketch the graphs  $y = 2^x$  and  $y = 3^x$ . Clearly label each graph.

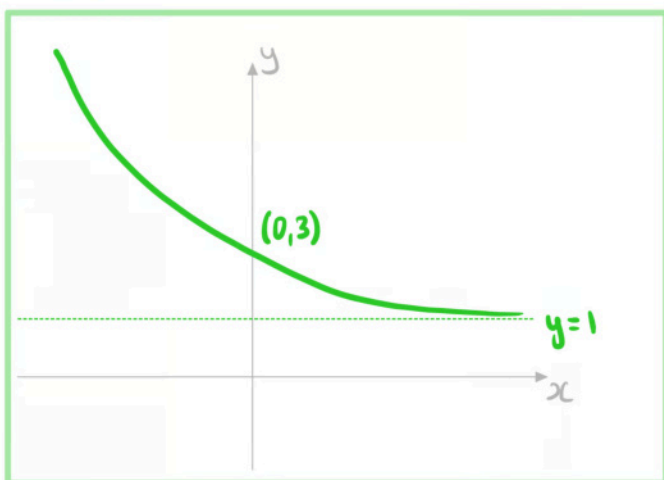


- b) Sketch the graph  $y = 2e^{-3x} + 1$ .

Use GDC to find intercept and asymptote

y-intercept = (0, 3)

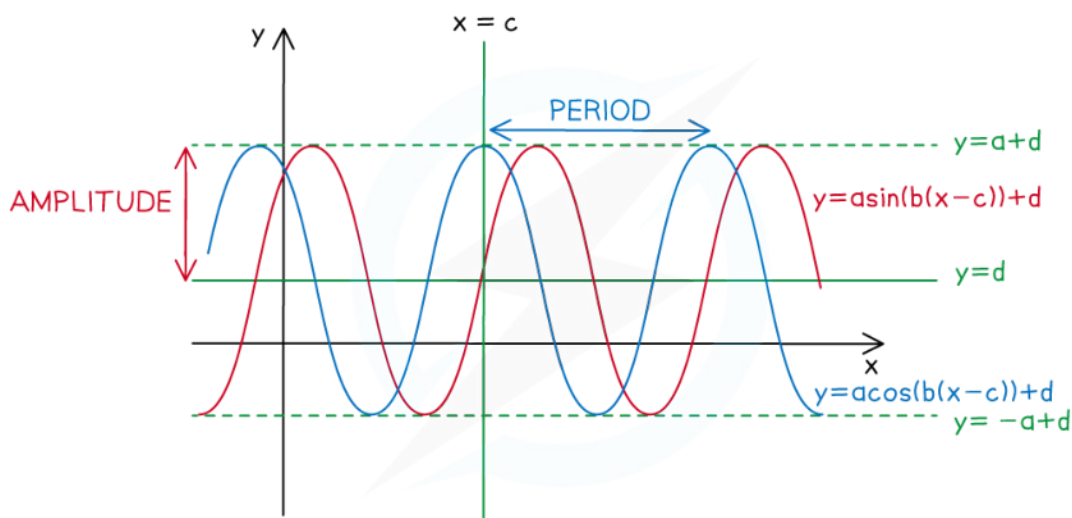
Asymptote:  $y = 1$



## Sinusoidal Functions & Graphs

### What are the key features of sinusoidal graphs?

- A **sinusoidal** graph is of the form
  - $y = a\sin(b(x - c)) + d$
  - $y = a\cos(b(x - c)) + d$
- The **y-intercept** is at the point where  $x = 0$ 
  - $(0, -a\sin(bc) + d)$  for  $y = a\sin(b(x - c)) + d$
  - $(0, a\cos(bc) + d)$  for  $y = a\cos(b(x - c)) + d$
- The **period** of the graph is the length of the interval of a full cycle
  - This is  $\frac{360^\circ}{b}$  (in degrees) or  $\frac{2\pi}{b}$
- The **maximum value** is  $y = a + d$
- The **minimum value** is  $y = -a + d$
- The **principal axis** is the horizontal line halfway between the maximum and minimum values
  - This is  $y = d$
- The **amplitude** is the vertical distance from the principal axis to the maximum value
  - This is  $a$
- The **phase shift** is the horizontal distance from its usual position
  - This is  $c$



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 **Examiner Tip**

- Make sure your angle setting is in the correct mode (degrees or radians) at the start of a question involving sinusoidal functions
- Pay careful attention to the angles between which you are required to use or draw/sketch a sinusoidal graph
  - e.g.  $0^\circ \leq x \leq 360^\circ$



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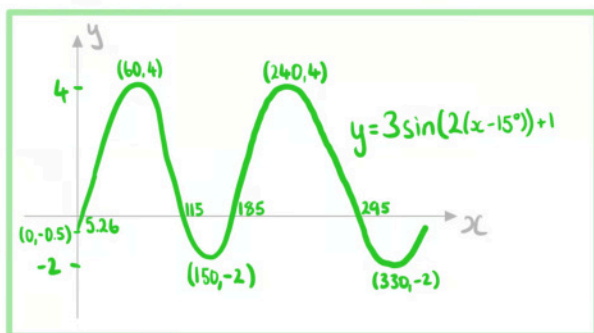


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### Worked example

- a) Sketch the graph  $y = 3\sin(2(x^\circ - 15^\circ)) + 1$  for the values  $0 \leq x \leq 360$ .

Use GDC to find max and min



- b) State the equation of the principal axis of the curve.

Principal axis is in middle of maximum and minimum points

$$\frac{4 + (-2)}{2} = 1$$

$$y = 1$$

- c) State the period and amplitude.

Period is how often it repeats

$$\frac{360}{2} = 180$$

$$\text{Period} = 180^\circ$$

Amplitude is distance from principal axis to maximum or minimum

$$4 - 1 = 1 - (-2) = 3$$

$$\text{Amplitude} = 3$$