DP IB Maths: AA HL

Your notes

1.6 Binomial Theorem

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1.6.1 Binomial Theorem

Binomial Theorem

What is the Binomial Theorem?

- The **binomial theorem (**sometimes known as the binomial expansion) gives a method for expanding a **two-term** expression in a bracket raised to a power
  - A **binomial expression** is in fact any two terms inside the bracket, however in IB the expression will usually be linear
- To expand a bracket with a two-term expression in:
  - First choose the most appropriate parts of the expression to assign to *a* and *b*
  - Then use the formula for the binomial theorem:

$$(a+b)^n = a^n + {}^nC_1\, a^{n-1} b + \ldots + {}^nC_r\, a^{n-r} b^r + \ldots + b^n$$

  - where ${}^nC_r = \dfrac{n!}{r!(n-r)!}$
    - See below for more information on ${}^nC_r$
    - You may also see ${}^nC_r$ written as $\binom{n}{r}$ or ${}_nC_r$

- You will usually be asked to find the first three or four terms of an expansion
- Look out for whether you should give your answer in **ascending** or **descending** powers of *x*
  - For **ascending** powers start with the constant term, a^n
  - For **descending** powers start with the term with *x* in
    - You may wish to swap *a* and *b* over so that you can follow the general formula given in the formula book
- If you are not writing the full expansion you can either
  - show that the sequence continues by putting an ellipsis (...) after your final term
  - or show that the terms you have found are an approximation of the full sequence by using the sign for approximately equals to (≈)

How do I find the coefficient of a single term?

- Most of the time you will be asked to find the coefficient of a term, rather than carry out the whole expansion
- Use the formula for the general term

$${}^nC_r\, a^{n-r} b^r$$

- The question will give you the power of *x* of the term you are looking for

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- Use this to choose which value of r you will need to use in the formula
- This will depend on where the x is in the bracket
- The laws of indices can help you decide which value of r to use:
  - For $(a + bx)^n$ to find the coefficient of x^r use $a^{n-r}(bx)^r$
  - For $(a + bx^2)^n$ to find the coefficient of x^r use $a^{n-\frac{r}{2}}(bx^2)^{\frac{r}{2}}$
  - For $\left(a + \frac{b}{x}\right)^n$ look at how the powers will cancel out to decide which value of r to use
  - So for $\left(3x + \frac{2}{x}\right)^8$ to find the coefficient of x^2 use the term with $r = 3$ and to find the constant term use the term with $r = 4$
  - There are a lot of variations of this so it is usually easier to see this by inspection of the exponents
- You may also be given the coefficient of a particular term and asked to find an unknown in the brackets
  - Use the laws of indices to choose the correct term and then use the binomial theorem formula to form and solve and equation

Examiner Tip

- Binomial expansion questions can get messy, use separate lines to keep your working clear and always put terms in brackets

Worked example

Find the first three terms, in ascending powers of x, in the expansion of $(3-2x)^5$.

$a = 3 \quad b = -2x \quad n = 5$

Substitute values into the formula for $(a+b)^n$

$$(a+b)^n = a^n + {}^nC_1 a^{n-1} b + \ldots + {}^nC_r a^{n-r} b^r + \ldots + b^n$$

Question asks for ascending powers of x, so start with the constant term, a^n.

$$(3-2x)^5 = 3^5 + {}^5C_1 (3)^{5-1}(-2x) + {}^5C_2 (3)^{5-2}(-2x)^2 + \ldots$$

Watch out for the negative

$$\approx 243 + 5 \times 81 \times -2x + 10 \times 27 \times 4x^2$$

$$\approx 243 - 810x + 1080x^2$$

$$(3-2x)^5 \approx 243 - 810x + 1080x^2$$

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The Binomial Coefficient nCr

What is ${}^{n}C_{r}$?

- If we want to find the number of ways to **choose** *r* items out of *n* different objects we can use the formula for ${}^{n}C_{r}$
  - The formula for *r* **combinations** of *n* items is ${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$
  - This formula is given in the formula booklet along with the formula for the binomial theorem
  - The function ${}^{n}C_{r}$ can be written $\binom{n}{r}$ or ${}_{n}C_{r}$ and is often read as '*n choose r*'
    - Make sure you can find and use the button on your GDC

How does ${}^{n}C_{r}$ relate to the binomial theorem?

- The formula ${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$ is also known as a **binomial** coefficient
- For a binomial expansion $(a + b)^{n}$ the coefficients of each term will be ${}^{n}C_{0}, {}^{n}C_{1}$ and so on up to ${}^{n}C_{n}$
  - The coefficient of the r^{th} term will be ${}^{n}C_{r}$
- ${}^{n}C_{n} = {}^{n}C_{0} = 1$
- The binomial coefficients are symmetrical, so ${}^{n}C_{r} = {}^{n}C_{n-r}$
  - This can be seen by considering the formula for ${}^{n}C_{r}$
  - ${}^{n}C_{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{r!(n-r)!} = nC_{r}$

Examiner Tip

- You will most likely need to use the formula for nCr at some point in your exam
  - Practice using it and don't always rely on your GDC
  - Make sure you can find it easily in the formula booklet

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Worked example

Without using a calculator, find the coefficient of the term in x^3 in the expansion of $(1 + x)^9$.

$n = 9, \quad a = 1, \quad b = x$

Substitute values into the formula for the binomial theorem:

$$(a+b)^n = a^n + \ldots + {}^nC_r\, a^{n-r} b^r + \ldots + b^n$$

where $\quad {}^nC_r = \dfrac{n!}{r!(n-r)!}$

$$(1+x)^9 = \sum_{r=0}^{9} {}^9C_r\,(1)^{9-r}\,(x)^r$$

← Coefficient of x^3 occurs when $r = 3$.

$r = 3$ gives $\quad {}^9C_3 \times (1)^{9-3}(x)^3$

Non-calculator, so work out ${}^nC_r$ separately:

$${}^9C_3 = \frac{9!}{3!(9-3)!} = \frac{9\times 8\times 7\times \cancel{6}\times \cancel{5}\times \cancel{4}\times \cancel{3}\times \cancel{2}}{(3\times 2)(\cancel{6}\times \cancel{5}\times \cancel{4}\times \cancel{3}\times \cancel{2})}$$

$$= \frac{9\times 8\times 7}{6} = 84$$

so the term when $r = 3$ is $\quad 84 \times (1)^6 \times x^3$

$$= 84x^3$$

Coefficient of x^3 = 84

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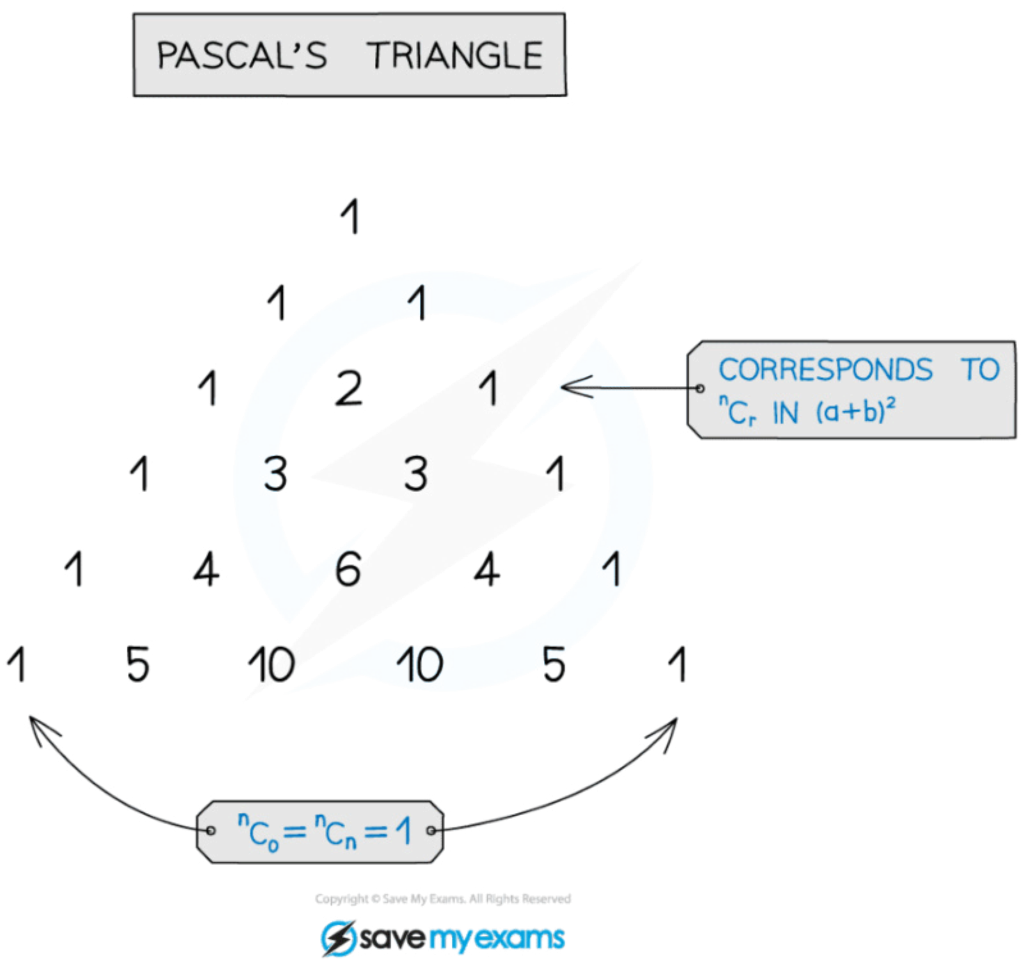
Pascal's Triangle

What is Pascal's Triangle?

- Pascal's triangle is a way of arranging the binomial coefficients and neatly shows how they are formed
  - Each term is formed by adding the two terms above it
  - The first row has just the number 1
  - Each row begins and ends with a number 1
  - From the third row the terms in between the 1s are the sum of the two terms above it



How does Pascal's Triangle relate to the binomial theorem?

- Pascal's triangle is an alternative way of finding the binomial coefficients, ^nC_r
  - It can be useful for finding for smaller values of n without a calculator

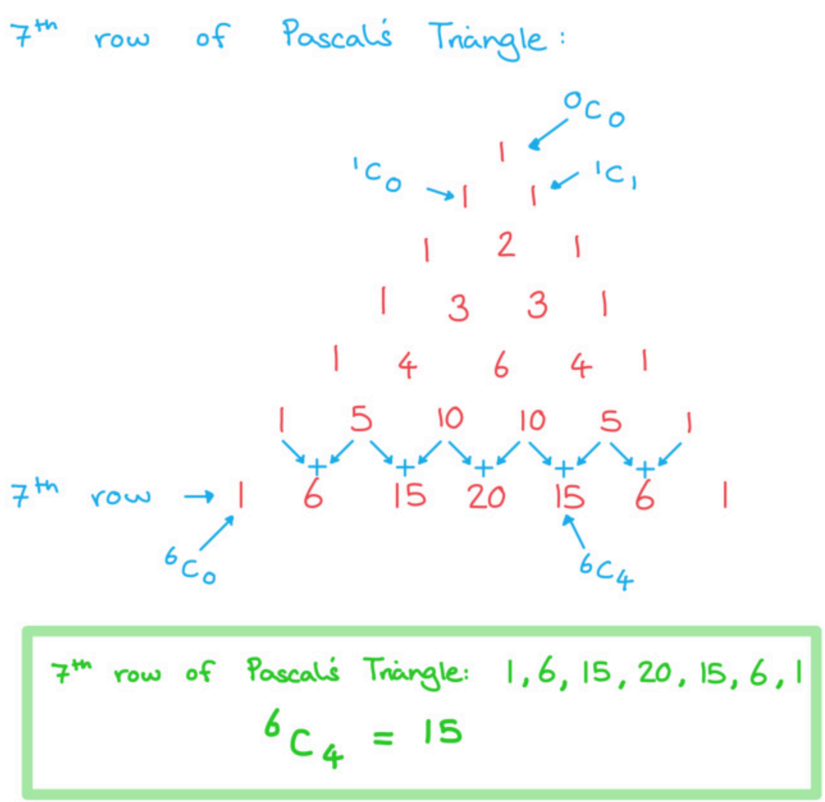
- However for larger values of n it is slow and prone to arithmetic errors
- Taking the first row as zero, $\left({}^{0}C_{0} = 1\right)$, each row corresponds to the n^{th} row and the term within that row corresponds to the r^{th} term

Examiner Tip

- In the non-calculator exam Pascal's triangle can be helpful if you need to get the coefficients of an expansion quickly, provided the value of *n* is not too big

Worked example

Write out the 7^{th} row of Pascal's triangle and use it to find the value of ${}^{6}C_{4}$.



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1.6.2 Extension of The Binomial Theorem

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Binomial Theorem: Fractional & Negative Indices

How do I use the binomial theorem for fractional and negative indices?

- The formula given in the formula booklet for the binomial theorem applies to positive integers only
  - $(a+b)^n = a^n + {}^nC_1 a^{n-1}b + \ldots + {}^nC_r a^{n-r}b^r + \ldots + b^n$
  - where ${}^nC_r = \dfrac{n!}{r!(n-r)!}$
- For **negative** or **fractional powers** the expression in the brackets must first be changed such that the value for *a* is 1
  - $(a+b)^n = a^n\left(1+\dfrac{b}{a}\right)^n$
  - $(a+b)^n = a^n\left(1+n\left(\dfrac{b}{a}\right)+\dfrac{n(n-1)}{2!}\left(\dfrac{b}{a}\right)^2+\ldots\right),\ n\in\mathbb{Q}$
  - This **is given in the formula booklet**
- If *a* = 1 and *b* = *x* the binomial theorem is simplified to
  - $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 + \ldots,\ \ n\in\mathbb{Q},\ \ |x|<1$
  - This is **not** in the formula booklet, you must remember it or be able to derive it from the formula given
- You need to be able to recognise a negative or fractional power
  - The expression may be on the denominator of a fraction
    - $\dfrac{1}{(a+b)^n} = (a+b)^{-n}$
  - Or written as a surd
    - $\sqrt[n]{(a+b)^m} = (a+b)^{\frac{m}{n}}$
- For $n \notin \mathbb{N}$ the expansion is infinitely long
  - You will usually be asked to find the first three terms
- The expansion is only valid for $|x| < 1$
  - This means $-1 < x < 1$
  - This is known as the **interval of convergence**
  - For an expansion $(a + bx)^n$ the interval of convergence would be $-\dfrac{a}{b} < x < \dfrac{a}{b}$

How do we use the binomial theorem to estimate a value?

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- The binomial expansion can be used to form an approximation for a value raised to a power
- Since $|x| < 1$ higher powers of *x* will be very small
  - Usually only the first three or four terms are needed to form an approximation
  - The more terms used the closer the approximation is to the true value
- The following steps may help you use the binomial expansion to approximate a value
  - STEP 1: Compare the value you are approximating to the expression being expanded
    - e.g. $(1 - x)^{\frac{1}{2}} = 0.96^{\frac{1}{2}}$
  - STEP 2: Find the value of *x* by solving the appropriate equation
    - e.g. $1 - x = 0.96$
      $x = 0.04$
  - STEP 3: Substitute this value of *x* into the expansion to find the approximation
    - e.g. $1 - \frac{1}{2}(0.04) - \frac{1}{8}(0.04)^2 = 0.9798$
- Check that the value of *x* is within the **interval of convergence** for the expression
  - If *x* is outside the interval of convergence then the approximation may not be valid

Examiner Tip

- Students often struggle with the extension of the binomial theorem questions in the exam, however the formula is given in the formula booklet
  - Make sure you can locate the formula easily and practice substituting values in
  - Mistakes are often made with negative numbers or by forgetting to use brackets properly
    - Writing one term per line can help with both of these

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Worked example

Consider the binomial expansion of $\dfrac{1}{\sqrt{9-3x}}$.

a) Write down the first three terms.

Rewrite $\dfrac{1}{\sqrt{9-3x}}$ in the form $k\left(1+\frac{x}{a}\right)^n$

$$\frac{1}{\sqrt{9-3x}} = (9-3x)^{-\frac{1}{2}} = 9^{-\frac{1}{2}}\left(1-\frac{3x}{9}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{3}\left(1-\frac{x}{3}\right)^{-\frac{1}{2}}$$

Substitute values into the formula for $(1+x)^n$

$$\frac{1}{3}\left(1-\frac{x}{3}\right)^{-\frac{1}{2}} = \frac{1}{3}\left[1+\left(-\frac{1}{2}\right)\left(-\frac{x}{3}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{x}{3}\right)^2+\dots\right]$$

$$= \frac{1}{3}\left[1+\frac{x}{6}+\frac{x^2}{24}+\dots\right]$$

$$= \frac{1}{3}+\frac{x}{18}+\frac{x^2}{72}+\dots$$

$$\frac{1}{\sqrt{9-3x}} \approx \frac{1}{3}+\frac{x}{18}+\frac{x^2}{72}$$

b) State the interval of convergence for the complete expansion.

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$n \geq 0$ and $n \notin \mathbb{N}$, so the series converges when $|x| < 1$

$\frac{1}{3}\left(1 - \frac{x}{3}\right)^{-\frac{1}{2}}$ ← x-term

$\left|-\frac{x}{3}\right| < 1$

$|x| < 3 \Rightarrow -3 < x < 3$

Converges for $-3 < x < 3$

c) Use the terms found in part (a) to estimate $\frac{1}{\sqrt{10}}$. Give your answer as a fraction.

Find the value of x for which $\frac{1}{\sqrt{9-3x}} = \frac{1}{\sqrt{10}}$

$9 - 3x = 10$

$x = -\frac{1}{3}$ ← $-3 < x < 3$ so can use the expansion

Substitute $x = -\frac{1}{3}$ into the expansion for $\frac{1}{\sqrt{9-3x}}$

$$\frac{1}{\sqrt{9-3\left(-\frac{1}{3}\right)}} \approx \frac{1}{3} + \frac{\left(-\frac{1}{3}\right)}{18} + \frac{\left(-\frac{1}{3}\right)^2}{72}$$

$$\frac{1}{\sqrt{10}} \approx \frac{205}{648}$$