DP IB Maths: AI HL

Your notes

5.6 Differential Equations

Contents

5.6.1 Modelling with Differential Equations

Your notes

Modelling with Differential Equations

Why are differential equations used to model real-world situations?

- A **differential equation** is an equation that contains one or more derivatives
- Derivatives deal with rates of change, and with the way that variables change with respect to one another
- Therefore differential equations are a natural way to model real-world situations involving change
  - Most frequently in real-world situations we are interested in how things change over time, so the derivatives used will usually be with respect to time t

How do I set up a differential equation to model a situation?

- An exam question may require you to create a differential equation from information provided
- The question will provide a context from which the differential equation is to be created
- Most often this will involve the rate of change of a variable being proportional to some function of the variable
    - For example, the rate of change of a population of bacteria, P, at a particular time may be proportional to the size of the population at that time
- The expression 'rate of' ('rate of change of…', 'rate of growth of…', etc.) in a modelling question is a strong hint that a differential equation is needed, involving derivatives with respect to time t
    - So with the bacteria example above, the equation will involve the derivative $\frac{\mathrm{d}P}{\mathrm{d}t}$
- Recall the basic equation of proportionality
  - If y is proportional to x, then $y = kx$ for some **constant of proportionality** k
    - So for the bacteria example above the differential equation needed would be $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$
  - The precise value of k will generally not be known at the start, but will need to be found as part of the process of solving the differential equation
  - It can often be useful to assume that $k > 0$ when setting up your equation
    - In this case, $-k$ will be used in the differential equation in situations where the rate of change is expected to be negative
    - So in the bacteria example, if it were known that the population of bacteria was decreasing, then the equation could instead be written $\frac{\mathrm{d}P}{\mathrm{d}t} = -kP$

Your notes

Worked example

a) In a particular pond, the rate of change of the area covered by algae, A, at any time t is directly proportional to the square root of the area covered by algae at that time. Write down a differential equation to model this situation.

$$\frac{dA}{dt} = k\sqrt{A}$$ (where k is a constant of proportionality)

b) Newton's Law of Cooling states that the rate of change of the temperature of an object, T, at any time t is proportional to the difference between the temperature of the object and the ambient temperature of its surroundings, T_a, at that time. Assuming that the object starts off warmer than its surroundings, write down the differential equation implied by Newton's Law of Cooling.

The object is assumed to be warmer than its surroundings, so $T - T_a > 0$

$$\frac{dT}{dt} = -k(T - T_a)$$

(where $k > 0$ is a constant of proportionality)

We expect the temperature to be decreasing, so $-k$ in the equation combined with $k > 0$ assures that $\frac{dT}{dt}$ is negative.

5.6.2 Separation of Variables

Your notes

Separation of Variables

What is separation of variables?

- **Separation of variables** can be used to solve certain types of first order differential equations
- Look out for equations of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)h(y)$
  - i.e. $\frac{\mathrm{d}y}{\mathrm{d}x}$ is a function of x multiplied by a function of y
  - be careful – the 'function of x' $g(x)$ may just be a constant!
    - For example in $\frac{\mathrm{d}y}{\mathrm{d}x} = 6y$, $g(x) = 6$ and $h(y) = y$
- If the equation is in that form you can use separation of variables to try to solve it
- If the equation is not in that form you will need to use another solution method

How do I solve a differential equation using separation of variables?

- STEP 1: Rearrange the equation into the form $\left(\frac{1}{h(y)}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)$
- STEP 2: Take the integral of both sides to change the equation into the form

$$\int \frac{1}{h(y)}\ \mathrm{d}y = \int g(x)\ \mathrm{d}x$$

  - You can think of this step as 'multiplying the $\mathrm{d}x$ across and integrating both sides'
    - Mathematically that's not *quite* what is actually happening, but it will get you the right answer here!
- STEP 3: Work out the integrals on both sides of the equation to find the **general solution** to the differential equation
  - Don't forget to include a constant of integration
    - Although there are two integrals, you only need to include one constant of integration
- STEP 4: Use any boundary or initial conditions in the question to work out the value of the integration constant
- STEP 5: If necessary, rearrange the solution into the form required by the question

Examiner Tip

- Be careful with letters – the equation on an exam may not use x and y as the variables
- Unless the question asks for it, you don't have to change your solution into $y = f(x)$ form – sometimes it might be more convenient to leave your solution in another form

Your notes

Your notes

Worked example

For each of the following differential equations, either (i) solve the equation by using separation of variables giving your answer in the form $y = f(x)$, or (ii) state why the equation may not be solved using separation of variables.

a) $\dfrac{dy}{dx} = \dfrac{e^x + 4x}{3y^2}$.

STEP 1: $3y^2 \dfrac{dy}{dx} = e^x + 4x$ $g(x) = e^x + 4x$ $h(y) = \dfrac{1}{3y^2}$

STEP 2: $\int 3y^2 \, dy = \int (e^x + 4x) \, dx$

STEP 3: $y^3 = e^x + 2x^2 + c$ ← Don't forget constant of integration

STEP 4: No boundary conditions given, so skip step

STEP 5: $y = \sqrt[3]{e^x + 2x^2 + c}$ $y = f(x)$

b) $\dfrac{dy}{dx} = 4xy - 2\ln x$.

$4xy - 2\ln x$ is not of the form $g(x)h(y)$, so it may not be solved using separation of variables.

c) $\dfrac{dy}{dx} = 3y$, given that $y = 2$ when $x = 0$.

STEP 1: $\frac{1}{y}\frac{dy}{dx} = 3 \qquad g(x) = 3 \quad h(y) = y$

STEP 2: $\int \frac{1}{y}\, dy = \int 3\, dx$

STEP 3: $\ln|y| = 3x + c$ ← Don't forget constant of integration

STEP 4: $\ln|2| = 3(0) + c \implies c = \ln 2$ ← $y = 2$ when $x = 0$

STEP 5: For the boundary condition $y = 2$, $y > 0$. Therefore we can drop the modulus sign from $|y|$.

$$y = e^{3x + \ln 2} = (e^{3x})(e^{\ln 2})$$

$$\implies \boxed{y = 2e^{3x}} \qquad y = f(x)$$

Your notes

5.6.3 Slope Fields

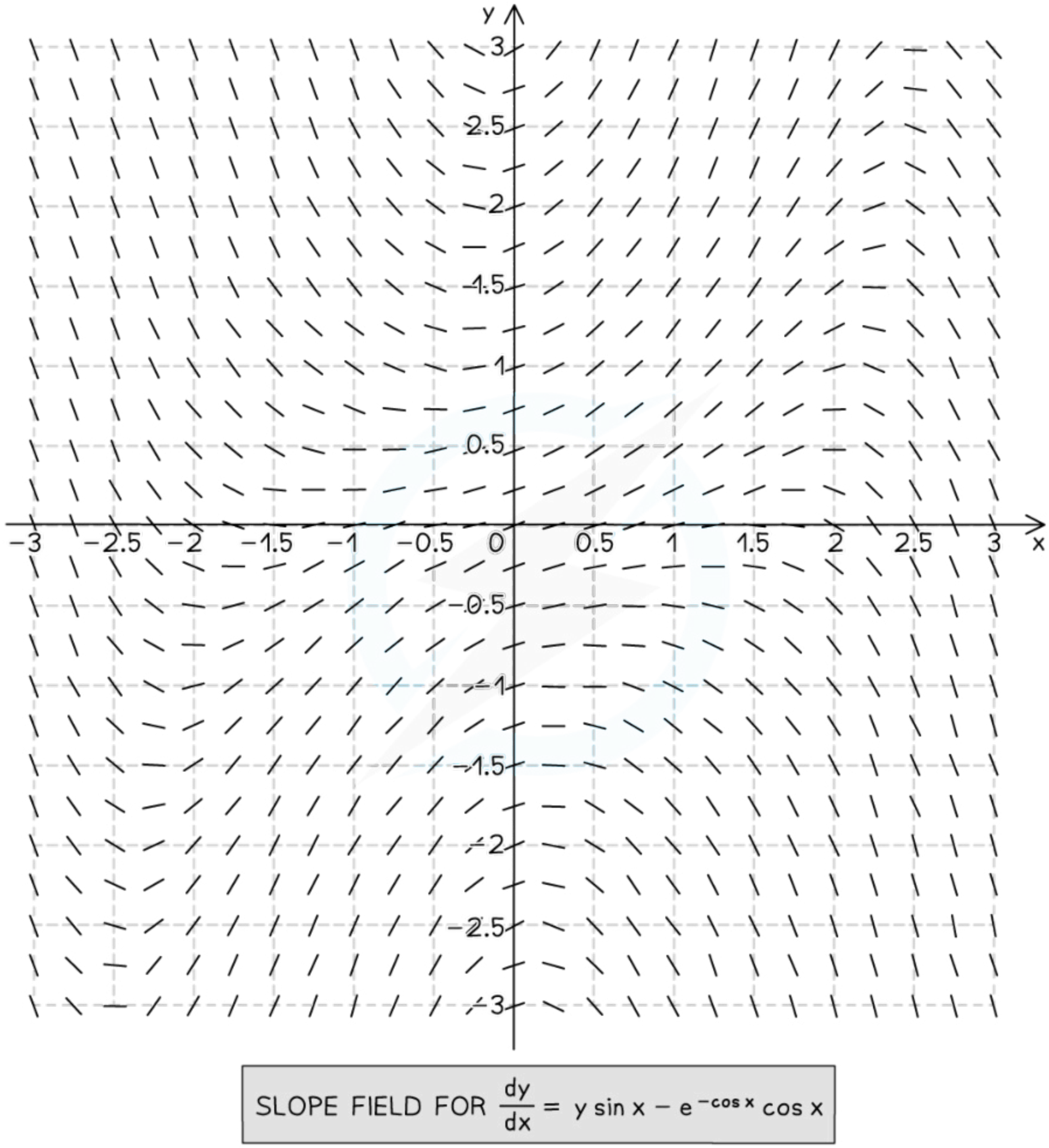
Your notes

Slope Fields

What are slope fields?

- We are considering here a differential equation involving two variables of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = g(x, y)$
  - I.e., the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ is equal to some function of x and y
- In some cases it may be possible to solve the differential equation analytically, while in other cases this is not possible
- Whether or not the equation can be analytically solved, however, it is always possible to calculate the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ at any point (x, y) by putting the x and y values into $g(x, y)$
  - This means that we can calculate the gradient of the solution curve at any point that the solution might go through
- A **slope field** for a differential equation is a diagram with short tangent lines drawn at a number of points
  - The gradient of the tangent line drawn at any given point will be equal to the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at that point
  - Normally the tangent lines will be drawn for points that form a regularly-spaced grid of x and y values

Your notes



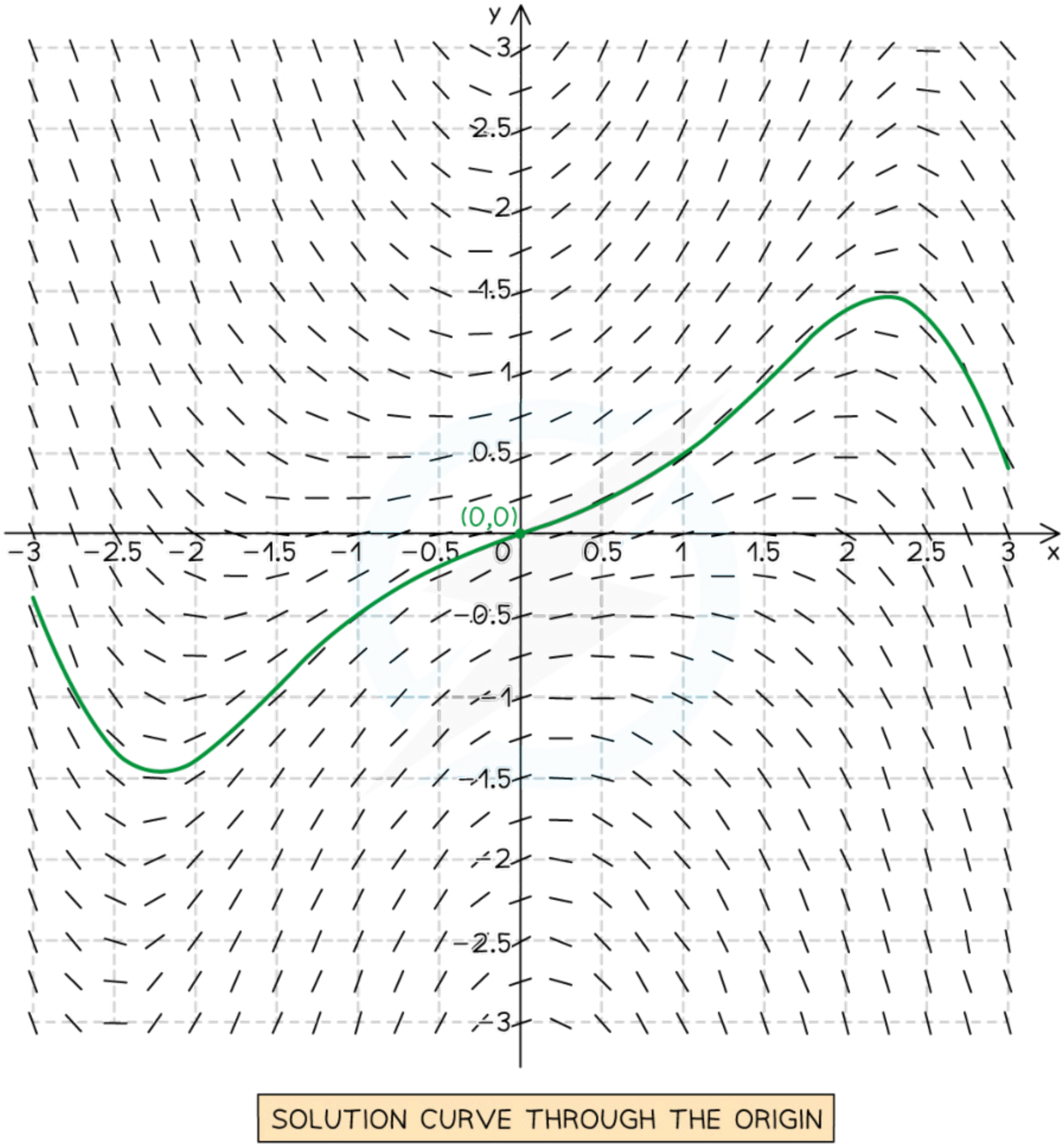
How can I use slope fields to study the solutions of a differential equation?

- Looking at the tangent lines in a slope field diagram will give you a general sense for what the solution curves to the differential equation will look like
  - Remember that the solution to a given differential equation is actually a family of solutions
  - We need appropriate boundary conditions or initial conditions to determine which of that family of solutions is the precise solution in a particular situation
- You can think of the tangent lines in a slope diagram as ‘flow lines’

Your notes

- From a given point the solution curve through that point will 'flow' away from the point in the direction of the tangent line
- For a given point, you can use a slope field to sketch the general shape of the solution curve that goes through that point
  - The given point here serves as a boundary condition, letting you know which of all the possible solution curves is the one you want to sketch
  - The sketch should go through the given point, and follow the general 'flow' of the tangent lines through the rest of the slope field diagram
  - In general, the sketched solution curve should **not** attempt to connect together a number of different tangent lines in the diagram
    - There is no guarantee that the solution curve will go through any exact point in the 'grid' of points at which tangent lines have been drawn
    - The only tangent line that your solution curve should definitely go through is one at the given 'boundary condition' point
  - The sketched solution curve may go along some of the tangent lines, but it should not should not cut across any of them

Your notes



- Look out for places where the tangent lines are horizontal
  - At such points $\frac{dy}{dx} = 0$
  - Therefore such points may indicate local minimum or maximum points for a solution curve
    - Be careful – not every point where $\frac{dy}{dx} = 0$ is a local minimum or maximum

Your notes

- But every local minimum or maximum will be at a point where $\frac{\mathrm{d}y}{\mathrm{d}x}=0$
- Don't forget that you can also solve the equation $\frac{\mathrm{d}y}{\mathrm{d}x}=g(x,\ y)=0$ directly to identify points where the gradient is zero
    - For example if $\frac{\mathrm{d}y}{\mathrm{d}x}=\sin(x-y)$, then the gradient will be zero anywhere where $x-y=0,\pm\pi,\ \pm2\pi,\ \pm3\pi,\ \ldots$
    - This is another way to identify possible local minimum and maximum points for the solution curves
    - If such a point falls between the 'grid points' at which the tangent lines have been drawn, this may be the only way to identify such a point exactly

Your notes

Worked example

Consider the differential equation

$$\frac{dy}{dx} = -0.4(y-2)^{\frac{1}{3}}(x-1)e^{-\frac{(x-1)^2}{25}}.$$

a) Using the equation, determine the set of points for which the solutions to the differential equation will have horizontal tangents.

The solution will have horizontal tangents wherever $\frac{dy}{dx} = 0$.

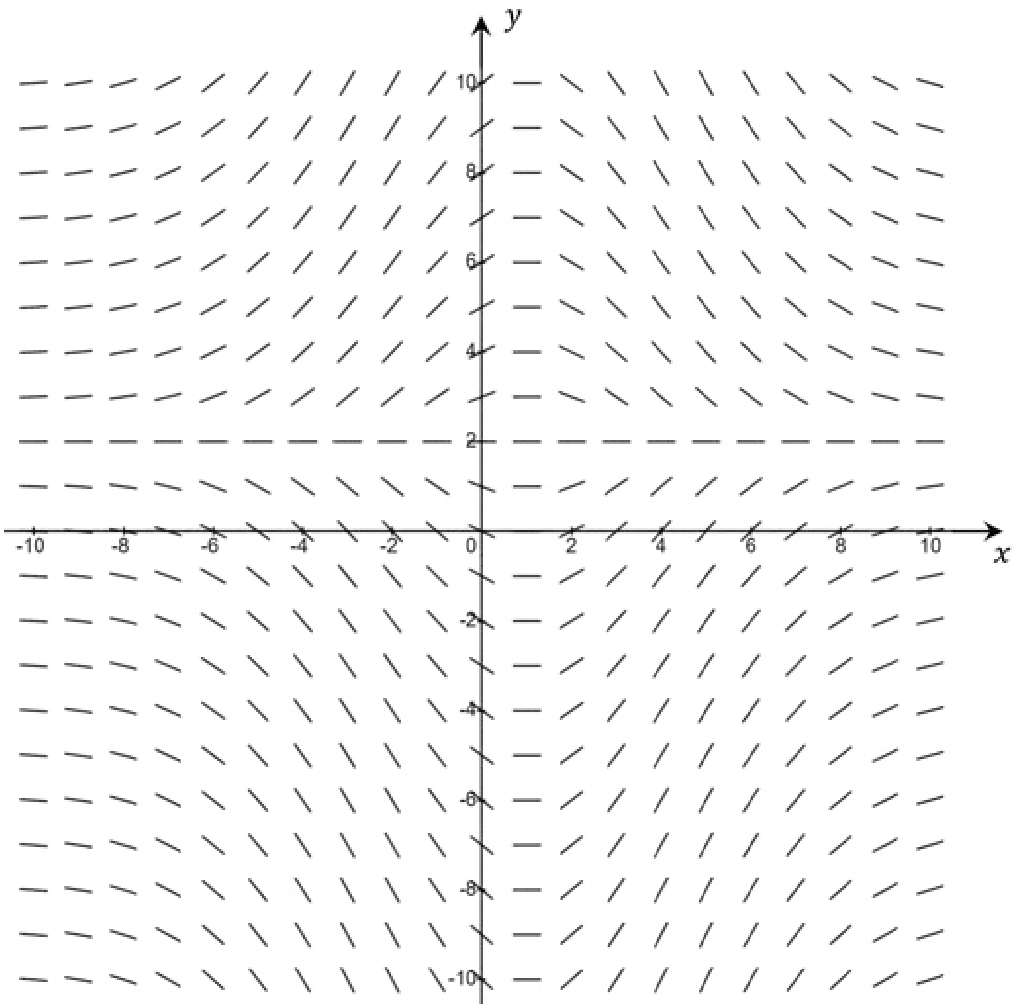
The exponential function is never equal to zero.

$\frac{dy}{dx} = 0$ when $y - 2 = 0$ or $x - 1 = 0$.

The solutions will have horizontal tangents at any point where $y = 2$ or $x = 1$.

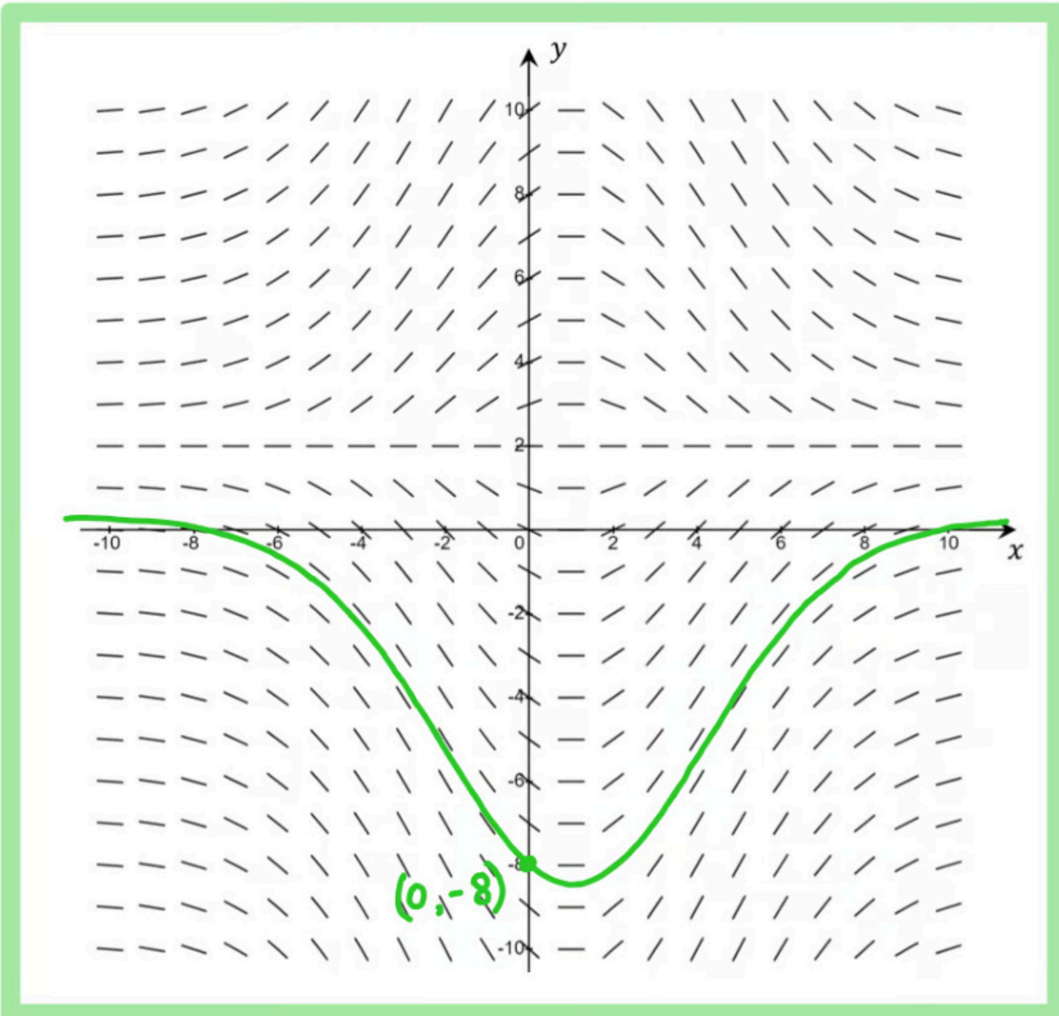
The diagram below shows the slope field for the differential equation, for $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$.

Your notes



b)Sketch the solution curve for the solution to the differential equation that passes through the point $(0, -8)$.

Your notes



5.6.4 Approximate Solutions to Differential Equations

Your notes

Euler's Method: First Order

What is Euler's method?

- **Euler's method** is a numerical method for finding approximate solutions to differential equations
- It treats the derivatives in the equation as being constant over short 'steps'
- The accuracy of the Euler's Method approximation can be improved by making the step sizes smaller

How do I use Euler's method with a first order differential equation?

- STEP 1: Make sure your differential equation is in $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$ form
- STEP 2: Write down the recursion equations using the formulae $y_{n+1} = y_n + h \times f(x_n, y_n)$ and $x_{n+1} = x_n + h$ from the exam formula booklet
  - h in those equations is the **step size**
  - the exam question will usually tell you the correct value of h to use
- STEP 3: Use the recursion feature on your GDC to calculate the Euler's method approximation over the correct number of steps
  - the values for x_0 and y_0 will come from the boundary conditions given in the question

Examiner Tip

- Be careful with letters – in the equations in the exam, and in your GDC's recursion calculator, the variables may not be *x* and *y*
- If an exam question asks you how to improve an Euler's method approximation, the answer will almost always have to do with decreasing the step size!

Your notes

Worked example

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + y = x + 1$ with the boundary condition $y(0) = 0.5$.

a) Apply Euler's method with a step size of $h = 0.2$ to approximate the solution to the differential equation at $x = 1$.

Your notes

Euler's method	$y_{n+1} = y_n + h \times f(x_n, y_n)$; $x_{n+1} = x_n + h$	h is a constant (step length)

} from formula booklet

STEP 1: $\frac{dy}{dx} = \underbrace{x - y + 1}_{f(x,y)}$

STEP 2: $y_{n+1} = y_n + \underbrace{0.2}_{h \text{ (from question)}} \times \underbrace{(x_n - y_n + 1)}_{f(x_n, y_n)}$ $\quad x_{n+1} = x_n + 0.2$

STEP 3: We need to get x from 0 to 1, so we will need $\frac{1-0}{0.2} = 5$ steps.

$y(0) = 0.5$

n	x_n	y_n
0	0	0.5
1	0.2	0.6
2	0.4	0.72
3	0.6	0.856
4	0.8	1.0048
5	1	1.16384

} from GDC

$y(1) = 1.16$ (3 s.f.)

b) Explain how the accuracy of the approximation in part (a) could be improved.

Your notes

Make the step size smaller.

Your notes

Euler's Method: Coupled Systems

How do I use Euler's method with coupled first order differential equations?

- STEP 1: Make sure your coupled differential equations are in $\frac{\mathrm{d}x}{\mathrm{d}t} = f_1(x,y,t)$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = f_2(x,y,t)$ form
- STEP 2: Write down the recursion equations using the formulae $x_{n+1} = x_n + h \times f_1(x_n, y_n, t_n)$, $y_{n+1} = y_n + h \times f_1(x_n, y_n, t_n)$ and $t_{n+1} = t_n + h$ from the exam formula booklet
  - *h* in those equations is the **step size**
  - the exam question will usually tell you the correct value of *h* to use
- STEP 3: Use the recursion feature on your GDC to calculate the Euler's method approximation over the correct number of steps
  - the values for x_0, y_0 and t_0 will come from the boundary conditions given in the question
  - frequently you will be given an initial condition
    - look out for terms like 'initially' or 'at the start'
    - in this case t_0

Examiner Tip

- Be careful with letters – in the equations in the exam, and in your GDC's recursion calculator, the variables may not be *x*, *y* and *t*.
- If an exam question asks you how to improve an Euler's method approximation, the answer will almost always have to do with decreasing the step size!

Your notes

Worked example

Consider the following system of differential equations:

$$\frac{dx}{dt} = 2x - 3y + 1$$

$$\frac{dy}{dt} = x + y + \frac{1}{t+1}$$

Initially $x = 10$ and $y = 2$.

Use Euler's method with a step size of 0.1 to find approximations for the values of x and y when $t = 0.5$.

Euler's method for coupled systems	$x_{n+1} = x_n + h \times f_1(x_n, y_n, t_n)$ $y_{n+1} = y_n + h \times f_2(x_n, y_n, t_n)$ $t_{n+1} = t_n + h$	h is a constant (step length)

from formula booklet

STEP 1: Equations are already in the proper form

STEP 2: $x_{n+1} = x_n + 0.1 \times \underbrace{(2x_n - 3y_n + 1)}_{f_1(x_n,\, y_n,\, t_n)}$

$y_{n+1} = y_n + \underbrace{0.1}_{h \text{ (from question)}} \times \underbrace{\left(x_n + y_n + \frac{1}{t_n + 1}\right)}_{f_2(x_n,\, y_n,\, t_n)}$ $t_{n+1} = t_n + 0.1$

STEP 3: To get t from 0 to 0.5 we need $\frac{0.5 - 0}{0.1} = 5$ steps.

Initially $x = 10$ and $y = 2$

n	t_n	x_n	y_n
0	0	10	2
1	0.1	11.5	3.3
2	0.2	12.91	4.8709
3	0.3	14.13	6.7323
4	0.4	15.037	8.8955
5	0.5	15.475	11.36

from GDC

$x(0.5) = 15.5$ (3 s.f.) $y(0.5) = 11.4$ (3 s.f.)

Your notes