DP IB Maths: AI HL

Your notes

2.4 Functions Toolkit

Contents

2.4.1 Composite & Inverse Functions

Your notes

Composite Functions

What is a composite function?

- A **composite function** is where a function is applied to another function
- A composite function can be denoted
  - $(f \circ g)(x)$
  - $fg(x)$
  - $f(g(x))$
- The order matters
  - $(f \circ g)(x)$ means:
    - First apply *g* to *x* to get $g(x)$
    - Then apply *f* to the previous output to get $f(g(x))$
    - Always start with the function **closest to the variable**
  - $(f \circ g)(x)$ is not usually equal to $(g \circ f)(x)$

How do I find the domain and range of a composite function?

- The domain of $f \circ g$ is the set of values of x...
  - which are a **subset** of the **domain of *g***
  - which maps *g* to a value that is in the **domain of *f***
- The range of $f \circ g$ is the set of values of x...
  - which are a **subset** of the **range of *f***
  - found by **applying *f*** to the **range of *g***
- To find the **domain** and **range** of $f \circ g$
  - First find the **range of *g***
  - **Restrict** these values to the values that are **within the domain** of ***f***
    - The **domain** is the set of values that **produce the restricted range** of *g*
    - The **range** is the set of values that are **produced using the restricted range** of *g* as the domain for *f*
- For example: let $f(x) = 2x + 1,\ -5 \le x \le 5$ and $g(x) = \sqrt{x},\ 1 \le x \le 49$
  - The **range of g** is $1 \le g(x) \le 7$
    - **Restricting** this to fit the **domain of *f*** results in $1 \le g(x) \le 5$
  - The **domain** of $f \circ g$ is therefore $1 \le x \le 25$
    - These are the values of *x* which map to $1 \le g(x) \le 5$
  - The **range** of $f \circ g$ is therefore $3 \le (f \circ g)(x) \le 11$
    - These are the values which *f* maps $1 \le g(x) \le 5$ to

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Examiner Tip

- Make sure you know what your GDC is capable of with regard to functions
  - You may be able to store individual functions and find composite functions and their values for particular inputs
  - You may be able to graph composite functions directly and so deduce their domain and range from the graph
- The link between the domains and ranges of a function and its inverse can act as a check for your solution
- $ff(x)$ is not the same as $[f(x)]^2$

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Worked example

Given $f(x)=\sqrt{x+4}$ and $g(x)=3+2x$:

a) Write down the value of $(g \circ f)(12)$.

First apply function closest to input

$(g \circ f)(12) = g(f(12))$

$f(12) = \sqrt{12+4} = \sqrt{16} = 4$

$g(4) = 3 + 2(4) = 11$

$(g \circ f)(12) = 11$

b) Write down an expression for $(f \circ g)(x)$.

First apply function closest to input

$(f \circ g)(x) = f(g(x))$

$= f(3+2x)$

$= \sqrt{3+2x+4}$

$(f \circ g)(x) = \sqrt{7+2x}$

c) Write down an expression for $(g \circ g)(x)$.

$(g \circ g)(x) = g(g(x))$

$= g(3+2x)$

$= 3 + 2(3+2x)$

$= 3 + 6 + 4x$

$(g \circ g)(x) = 9 + 4x$

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Inverse Functions

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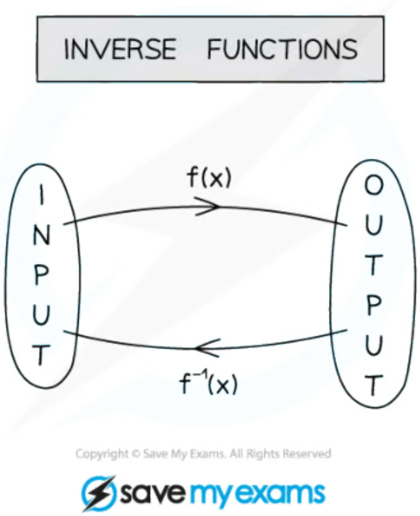
What is an inverse function?

- **Only one-to-one** functions have inverses
- A function has an inverse if its graph passes the **horizontal line test**
  - Any **horizontal line** will intersect with the graph **at most once**
- The **identity function** id maps each value to itself
  - $\mathrm{id}(x) = x$
- If $f \circ g$ and $g \circ f$ have the **same effect as the identity function** then f and g are **inverses**
- Given a function $f(x)$ we denote the **inverse function** as $f^{-1}(x)$
- An inverse function **reverses the effect** of a function
  - $f(2) = 5$ means $f^{-1}(5) = 2$
- Inverse functions are used to solve equations
  - The solution of $f(x) = 5$ is $x = f^{-1}(5)$
- A composite function made of f and f^{-1} has the **same effect as the identity function**
  - $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$



What are the connections between a function and its inverse function?

- The **domain of a function** becomes the **range of its inverse**
- The **range of a function** becomes the **domain of its inverse**
- The graph of $y = f^{-1}(x)$ is a **reflection** of the graph $y = f(x)$ in the line $y = x$
  - Therefore solutions to $f(x) = x$ or $f^{-1}(x) = x$ will also be solutions to $f(x) = f^{-1}(x)$
    - There could be other solutions to $f(x) = f^{-1}(x)$ that don't lie on the line $y = x$

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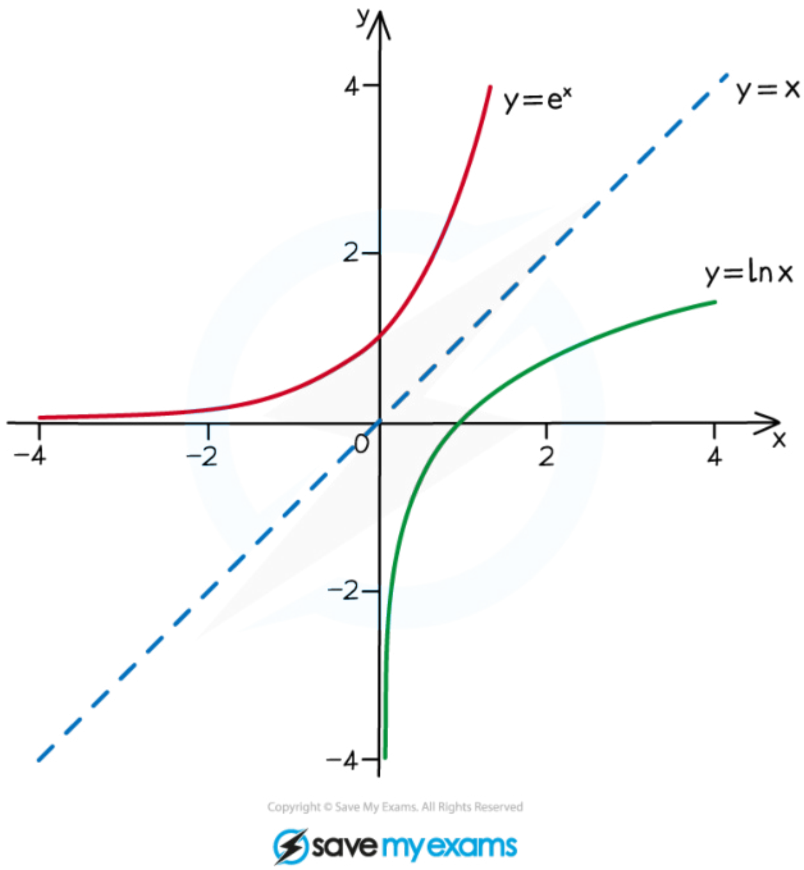

How do I find the inverse of a function?

- STEP 1: **Swap** the *x* and *y* in $y = f(x)$
  - If $y = f^{-1}(x)$ then $x = f(y)$
- STEP 2: **Rearrange** $x = f(y)$ to make y the subject
- Note this can be done in any order
  - Rearrange $y = f(x)$ to make x the subject
  - Swap x and y

Can many-to-one functions ever have inverses?

- You can **restrict the domain** of a many-to-one function so that it has an inverse
- Choose a subset of the domain where the function is one-to-one
  - The inverse will be determined by the restricted domain
  - Note that a many-to-one function can **only** have an inverse if its domain is restricted first
- For **quadratics** – use the **vertex** as the upper or lower bound for the **restricted domain**
  - For $f(x) = x^2$ restrict the domain so 0 is either the maximum or minimum value
    - For example: $x \geq 0$ or $x \leq 0$

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- For $f(x) = a(x - h)^2 + k$ restrict the domain so *h* is either the maximum or minimum value
  - For example: $x \geq h$ or $x \leq h$
- For **trigonometric functions** – use part of a cycle as the **restricted domain**
  - For $f(x) = \sin x$ restrict the domain to half a cycle between a maximum and a minimum
    - For example: $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$
  - For $f(x) = \cos x$ restrict the domain to half a cycle between maximum and a minimum
    - For example: $0 \leq x \leq \pi$
  - For $f(x) = \tan x$ restrict the domain to one cycle between two asymptotes
    - For example: $-\frac{\pi}{2} < x < \frac{\pi}{2}$

How do I find the inverse function after restricting the domain?

- The range of the inverse is the same as the restricted domain of the original function
- The inverse function is determined by the restricted domain
  - Restricting the domain differently will change the inverse function
- Use the range of the inverse to help find the inverse function
  - Restricting the domain of $f(x) = x^2$ to $x \geq 0$ means the range of the inverse is $f^{-1}(x) \geq 0$
    - Therefore $f^{-1}(x) = \sqrt{x}$
  - Restricting the domain of $f(x) = x^2$ to $x \leq 0$ means the range of the inverse is $f^{-1}(x) \leq 0$
    - Therefore $f^{-1}(x) = -\sqrt{x}$

Examiner Tip

- Remember that an inverse function is a reflection of the original function in the line $y = x$
  - Use your GDC to plot the function and its inverse on the same graph to visually check this
- $f^{-1}(x)$ is not the same as $\frac{1}{f(x)}$

Worked example

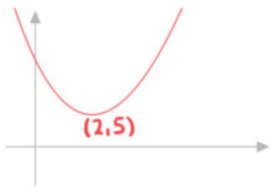
The function $f(x) = (x-2)^2 + 5, \;\; x \leq m$ has an inverse.

a) Write down the largest possible value of m.

Sketch $y = f(x)$

The graph is one-to-one for $x \leq 2$

$m = 2$



b) Find the inverse of $f(x)$.

Let $y = f^{-1}(x)$ and rearrange $x = f(y)$

$x = (y-2)^2 + 5$

$x - 5 = (y-2)^2$

$\pm\sqrt{x-5} = y - 2$

$2 \pm \sqrt{x-5} = y$

Range of f^{-1} is the domain of f

$f^{-1}(x) \leq 2 \quad \therefore \; y = 2 - \sqrt{x-5}$

$f^{-1}(x) = 2 - \sqrt{x-5}$

c) Find the domain of $f^{-1}(x)$.

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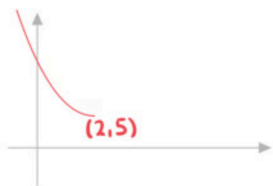
Domain of f^{-1} is the range of f

Sketch $y = f(x)$ to see range

For $x \le 2$, $f(x) \ge 5$



Domain of f^{-1} : $x \ge 5$

d) Find the value of k such that $f(k) = 9$.

Use inverse $f(a) = b \Leftrightarrow a = f^{-1}(b)$

$k = f^{-1}(9) = 2 - \sqrt{9-5}$

$k = 0$