DP IB Maths: AA HL

Your notes

5.8 Advanced Differentiation

Contents

5.8.1 First Principles Differentiation

Your notes

First Principles Differentiation

What is differentiation from first principles?

- Differentiation from **first principles** uses the **definition** of the **derivative** of a function ***f(x)***
- The definition is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- $\lim_{h \to 0}$ means the '*limit as **h** tends to zero*'
- When $h = 0$, $\frac{f(x+h) - f(x)}{h} = \frac{f(x) - f(x)}{0} = \frac{0}{0}$ which is **undefined**
  - Instead we consider what happens as ***h*** gets closer and closer to zero
- Differentiation **from first principles** means using that definition to show what the **derivative** of a function is
- The first principles definition (formula) is in the **formula booklet**

How do I differentiate from first principles?

STEP 1: Identify the function ***f(x)*** and substitute this into the first principles formula

e.g. Show, from first principles, that the derivative of $3x^2$ is $6x$

$$f(x) = 3x^2 \text{ so } f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{3(x+h)^2 - 3x^2}{h}$$

STEP 2: Expand ***f(x+h)*** in the numerator

$$f'(x) = \lim_{h \to 0} \frac{3(x^2 + 2hx + h^2) - 3x^2}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{3x^2 + 6hx + 3h^2 - 3x^2}{h}$$

STEP 3: Simplify the numerator, factorise and cancel h with the denominator

$$f'(x) = \lim_{h \to 0} \frac{h(6x + 3h)}{h}$$

STEP 4: Evaluate the remaining expression as ***h*** *tends to zero*

$$f'(x) = \lim_{h \to 0}(6x + 3h) = 6x \quad \text{As } h \to 0,\ (6x + 3h) \to (6x + 0) \to 6x$$

$\therefore$ The derivative of $3x^2$ is $6x$

Examiner Tip

- Most of the time you will not use first principles to find the derivative of a function (there are much quicker ways!)
  However, you can be asked to demonstrate differentiation from first principles
- To get full marks make sure you are are writing lim h -> 0 right up until the concluding sentence!

Your notes

Your notes

Worked example

Prove, from first principles, that the derivative of $5x^3$ is $15x^2$.

STEP 1: For $f(x) = 5x^3$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This is given in the formula booklet

$$= \lim_{h \to 0} \frac{5(x+h)^3 - 5x^3}{h}$$

STEP 2:

$$= \lim_{h \to 0} \frac{5(x^3 + 3x^2h + 3xh^2 + h^3) - 5x^3}{h}$$

Expand $(x+h)^3$ using binomial theorem

$$= \lim_{h \to 0} \frac{5x^3 + 15x^2h + 15xh^2 + 5h^3 - 5x^3}{h}$$

STEP 3:

$$= \lim_{h \to 0} \frac{15x^2h + 15xh^2 + 5h^3}{h}$$

$$= \lim_{h \to 0} (15x^2 + 15xh + 5h^2)$$

STEP 4: As $h \to 0$

$$(15x^2 + 15xh + 5h^2) \to (15x^2 + 15x(0) + 5(0)^2) = 15x^2$$

$\therefore$ The derivative of $5x^3$ is $15x^2$

5.8.2 Applications of Chain Rule

Your notes

Related Rates of Change

What is meant by rates of change?

- A rate of change is a measure of how a quantity is changing with respect to another quantity
- Mathematically rates of change are derivatives
  - $\frac{\mathrm{d}V}{\mathrm{d}r}$ could be the rate at which the volume of a sphere changes relative to how its radius is changing
- Context is important when interpreting positive and negative rates of change
  - A positive rate of change would indicate an increase
    - e.g. the change in volume of water as a bathtub fills
  - A negative rate of change would indicate a decrease
    - e.g. the change in volume of water in a leaking bucket

What is meant by related rates of change?

- Related rates of change are connected by a linking variable or parameter
  - this is often time, represented by t
  - seconds is the standard unit for time but this will depend on context
- e.g. Water running into a large hemi-spherical bowl
  - both the height and volume of water in the bowl are changing with time
    - time is the linking parameter between the rate of change of height and the rate of change of volume

How do I solve problems involving related rates of change?

- Use of chain rule and product rule are common in such problems
- Be clear about which variables are representing which quantities

STEP 1
Write down any variables and derivatives involved in the problem

e.g. x, y, t, $\frac{\mathrm{d}y}{\mathrm{d}x}$, $\frac{\mathrm{d}x}{\mathrm{d}t}$, $\frac{\mathrm{d}y}{\mathrm{d}t}$

STEP 2
Use an appropriate differentiation rule to set up an equation linking 'rates of change'

e.g. Chain rule: $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t}$

STEP 3
Substitute in known values

Your notes

e.g. If, when $t=3$, $\frac{\mathrm{d}x}{\mathrm{d}t}=2$ and $\frac{\mathrm{d}y}{\mathrm{d}t}=8$, then $8=\frac{\mathrm{d}y}{\mathrm{d}x}\times 2$

STEP 4
Solve the problem and interpret the answer in context if required
e.g. $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{8}{2}=4$ 'when $t=3$, y changes at a rate of 4, with respect to x'

Examiner Tip

- If you struggle to determine which rate to use then you can look at the units to help
  - e.g. A rate of 5 **cm^3** *per* **second** implies **volume** per **time** so the rate would be $\frac{\mathrm{d}V}{\mathrm{d}t}$

Your notes

Worked example

In a manufacturing process a metal component is heated such that it's cross-sectional area expands but always retains the shape of a right-angled triangle. At time t seconds the triangle has base b cm and height h cm.

At the time when the component's cross-sectional area is changing at 4 cm s^{-1}, the base of the triangle is 3 cm and its height is 6 cm. Also at this time, the rate of change of the height is twice the rate of change of the base.

Find the rate of change of the base at this point of time.

Your notes

STEP 1: List variables and derivatives

$A, b, h, t, \frac{dA}{dt}, \frac{db}{dt}, \frac{dh}{dt}$

$A = \frac{1}{2}bh$

STEP 2: Use a differentiation rule to link 'rates of change'

$A = \frac{1}{2}bh$ is a product - so use product rule

$$\frac{dA}{dt} = \frac{1}{2}\left[b\frac{dh}{dt} + h\frac{db}{dt}\right]$$

STEP 3: Substitute known values

$$4 = \frac{1}{2}\left[3\left(2\frac{db}{dt}\right) + 6\frac{db}{dt}\right]$$

↑ $\frac{dh}{dt} = 2\frac{db}{dt}$ in question

STEP 4: Solve and interpret

$$8 = 12\frac{db}{dt}$$

$\therefore \frac{db}{dt} = \frac{2}{3}$ cm s^{-1}

The rate of change of the base is $\frac{2}{3}$ cm per second.

Your notes

Differentiating Inverse Functions

What is meant by an inverse function?

- Some functions are easier to process with x (rather than y) as the subject
  - i.e. in the form $x = f(y)$
- This is particularly true when dealing with inverse functions
  - e.g. If $y = f(x)$ the inverse would be written as $y = f^{-1}(x)$
    - finding $f^{-1}(x)$ can be awkward
    - so write $x = f(y)$ instead

How do I differentiate inverse functions?

- Since $x = f(y)$ it is easier to differentiate "x with respect to y" rather than "y with respect to x"
  - i.e. find $\frac{dx}{dy}$ rather than $\frac{dy}{dx}$
  - Note that $\frac{dx}{dy}$ will be in terms of y but can be substituted

STEP 1

For the function $y = f(x)$, the inverse will be $y = f^{-1}(x)$

Rewrite this as $x = f(y)$

STEP 2

From $x = f(y)$ find $\frac{dx}{dy}$

STEP 3

Find $\frac{dy}{dx}$ using $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ - this will usually be in terms of y

- If an algebraic solution in terms of x is required substitute $f(x)$ for y in $\frac{dy}{dx}$
- If a numerical derivative (e.g. a gradient) is required then use the y-coordinate
  - If the y-coordinate is not given, you should be able to work it out from the orginal function and x-coordinate

Your notes

Examiner Tip

- With x's and y's everywhere this can soon get confusing!
  - Be clear of the key information and steps - and set your wokring out accordingly
    - The orginal function, $y = f(x)$
    - Its inverse, $y = f^{-1}(x)$
    - Rewriting the inverse, $x = f(y)$
    - Finding $\frac{dx}{dy}$ first, then finding its reciprocal for $\frac{dy}{dx}$
- Your GDC can help when numerical derivatives (gradients) are required

Your notes

Worked example

a) Find the gradient of the curve at the point where $y = 3$ on the graph of $y = f^{-1}(x)$ where $f(x) = \sqrt{(5x+1)^3}$.

STEP 1: Rewrite inverse as $x = f(y)$

$f(x) = \sqrt{(5x+1)^3}$

$\therefore$ For $y = f^{-1}(x)$, $x = f(y)$

$x = \sqrt{(5y+1)^3}$

STEP 2: Find $\frac{dx}{dy}$

$x = (5y+1)^{3/2}$ — Write as powers

$\frac{dx}{dy} = \frac{3}{2}(5y+1)^{1/2} \times 5$ — Using chain rule

$\frac{dx}{dy} = \frac{15}{2}\sqrt{5y+1}$

STEP 3: Find $\frac{dy}{dx}$

A gradient is required - substitute $y=3$

$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{15}{2}\sqrt{5y+1}} = \frac{2}{15\sqrt{5y+1}}$

At $y=3$, $\frac{dy}{dx} = \frac{2}{15\sqrt{5(3)+1}}$

$\therefore$ Gradient, at $y=3$, on the graph of $y = f^{-1}(x)$ is $\frac{1}{30}$.

Your notes

b) Given that $y = e^x$ show that the derivative of $y = \ln x$ is $\frac{1}{x}$.

The key to this question is realising that e^x and $\ln x$ are inverses

$y = e^x$ so $y = \ln x$ will be its inverse

STEP 1: $y = e^x$ "$y = f(x)$"

$\therefore$ The inverse will be $x = e^y$ "$y = f^{-1}(x)$, $x = f(y)$"

STEP 2: $\frac{dx}{dy} = e^y$

STEP 3: $\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{e^{\ln x}} = \frac{1}{x}$ $e^{\ln x} = x$ since e^x and $\ln x$ are inverses

$\therefore$ If $y = \ln x$, $\frac{dy}{dx} = \frac{1}{x}$

Your notes

5.8.3 Implicit Differentiation

Implicit Differentiation

What is implicit differentiation?

- An equation connecting x and y is not always easy to write **explicitly** in the form $y = f(x)$ or $x = f(y)$
  - In such cases the equation is written **implicitly**
    - as a function of x and y
    - in the form $f(x, y) = 0$
- Such equations can be **differentiated implicitly** using the **chain rule**

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(y)] = f'(y)\frac{\mathrm{d}y}{\mathrm{d}x}$$

- A shortcut way of thinking about this is that 'y is a function of a x'
  - when differentiating a function of y chain rule says "differentiate with respect to y, then multiply by the derivative of y" (which is $\frac{\mathrm{d}y}{\mathrm{d}x}$)

Your notes

Applications of Implicit Differentiation

What type of problems could involve implicit differentiation?

- Broadly speaking there are three types of problem that could involve implicit differentiation
  - algebraic problems involving graphs, derivatives, tangents, normals, etc
    - where it is not practical to write y explicitly in terms of x
    - usually in such cases, $\frac{dy}{dx}$ will be in terms of x and y
  - **optimisation** problems that involve **time derivatives**
    - more than one variable may be involved too

      e.g. Volume of a cylinder, $V = \pi r^2 h$

      e.g. The side length and (so) area of a square increase over time
  - any problem that involves differentiating with respect to an extraneous variable
    - e.g. $y = f(x)$ but the derivative $\frac{dy}{d\theta}$ is required (rather than $\frac{dy}{dx}$)

How do I apply implicit differentiation to algebraic problems?

- Algebraic problems revolve around values of the derivative (gradient) $\left(\frac{dy}{dx}\right)$
  - if not required to find this value it will either be given or implied
- Particular problems focus on special case tangent values
  - horizontal tangents
    - also referred to as tangents parallel to the x-axis
    - this is when $\frac{dy}{dx} = 0$
  - vertical tangents
    - also referred to as tangents parallel to the y-axis
    - this is when $\frac{dx}{dy} = 0$
    - In such cases it may appear that $\frac{1}{\frac{dy}{dx}} = 0$ but this has no solutions; this occurs when for nearby values of x, $\frac{dy}{dx} \rightarrow \pm\infty$

      (i.e. very steep gradients, near vertical)
- Other problems may involve finding equations of (other) tangents and/or normals

Your notes

- For problems that involve finding the coordinates of points on a curve with a specified gradient the method below can be used

  STEP 1
  Differentiate the equation of the curve implicitly

  STEP 2
  Substitute the given or implied value of $\frac{dy}{dx}$ to create an equation linking x and y

  STEP 3
  There are now two equations

  - the original equation
  - the linking equation

  Solve them simultaneously to find the x and y coordinates as required

Examiner Tip

- After some rearranging, $\frac{dy}{dx}$ will be in terms of both x and y
  - There is usually no need (unless asked to by the question) to write $\frac{dy}{dx}$ in terms of x (or y) only
- If evalutaing derivatives, you'll need both x and y coordinates, so one may have to be found from the other using the original function

Your notes

Worked example

The curve C has equation $x^2 + 2y^2 = 16$.

a) Find the exact coordinates of the points where the normal to curve C has gradient 2.

Your notes

STEP 1: Differentiate implicitly

$2x + 4y\frac{dy}{dx} = 0$

STEP 2: Substitute value of $\frac{dy}{dx}$ in

Gradient of normal is 2

$\therefore$ gradient of tangent $\left(\frac{dy}{dx}\right)$ is $-\frac{1}{2}$

(Normal and tangent are perpendicular, so product of their gradients is −1)

$\therefore 2x + 4y\left(-\frac{1}{2}\right) = 0$

$2x - 2y = 0$

$x = y$ Equation linking x and y

STEP 3: Solve simultaneously, obtain coordinates

$x^2 + 2y^2 = 16$

$y = x$

$\therefore x^2 + 2x^2 = 16$

$3x^2 = 16 \quad x = \pm\frac{4}{\sqrt{3}} = \pm\frac{4}{3}\sqrt{3} \quad \therefore y = \pm\frac{4}{3}\sqrt{3}$

$\therefore$ Coordinates are $\left(\frac{4}{3}\sqrt{3}, \frac{4}{3}\sqrt{3}\right)$ and $\left(-\frac{4}{3}\sqrt{3}, -\frac{4}{3}\sqrt{3}\right)$

b) Find the equations of the tangents to the curve that are

(i) parallel to the x-axis

Your notes

(ii) parallel to the y-axis.

These are special cases,

in part (i), $\frac{dy}{dx} = 0$ (parallel to x-axis)

in part (ii), $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = 0$ (parallel to y-axis)

$x^2 + 2y^2 = 16$

$2x + 4y\frac{dy}{dx} = 0$

(i) $\frac{dy}{dx} = 0$ $\therefore$ $2x = 0$

$x = 0$ $\therefore$ $2y^2 = 16$

$y = \pm\sqrt{8} = \pm 2\sqrt{2}$

(ii) $\frac{dy}{dx} = \frac{-2x}{4y} = \frac{-x}{2y}$

$\therefore \frac{dx}{dy} = \frac{2y}{-x}$

$\frac{dx}{dy} = 0$ $\therefore$ $2y = 0$

$y = 0$ $\therefore$ $x^2 = 16$

$x = \pm 4$

$\therefore$ Tangents (i) parallel to x-axis are $y = 2\sqrt{2}$ and $y = -2\sqrt{2}$ and (ii) parallel to y-axis are $x = 4$ and $x = -4$

How do I apply implicit differentiation to optimisation problems?

- For a single variable use chain rule to differentiate implicitly
  - e.g. A square with side length changing over time, $A = x^2$

Your notes

$$\frac{dA}{dt} = 2x\frac{dx}{dt}$$

- For more than one variable use product rule (and chain rule) to differentiate implicitly
  - e.g. A square-based pyramid with base length and height changing over time, $V = \frac{1}{3}x^2h$

$$\frac{dV}{dt} = \frac{1}{3}\left[x^2\frac{dh}{dt} + 2x\frac{dx}{dt}h\right] = \frac{1}{3}x\left(x\frac{dh}{dt} + 2h\frac{dx}{dt}\right)$$

- After differentiating implicitly the rest of the question should be similar to any other optimisation problem
  - be aware of phrasing
    - "the rate of change of the height of the pyramid" (over time) is $\frac{dh}{dt}$
  - when finding the location of minimum and maximum problems
    - there is not necessarily a turning point
    - the minimum or maximum could be at the start or end of a given or appropriate interval

Examiner Tip

- If you are struggling to tell which derivative is needed for a question, writing all possibilities down may help you
  - You don't need to work them out at this stage but if you conisder them it may nudge you to the next stage of the solution
  - e.g. For $V = \pi r^2 h$, possible derivatives are $\frac{dV}{dr}$, $\frac{dV}{dh}$ and $\frac{dV}{dt}$

Your notes

Worked example

The radius, r cm, and height, h cm, of a cylinder are increasing with time. The volume, V cm^3, of the cylinder at time t seconds is given by $V = \pi r^2 h$.

a) Find an expression for $\frac{\mathrm{d}V}{\mathrm{d}t}$.

Using implicit differentiation with product rule

$$\frac{dV}{dt} = \pi\left[2r\frac{dr}{dt}h + \frac{dh}{dt}r^2\right]$$

$$\therefore \frac{dV}{dt} = \pi r\left(2h\frac{dr}{dt} + r\frac{dh}{dt}\right)$$

b) At time T seconds, the radius of the cylinder is 4 cm, expanding at a rate of 2 cm s^{-1}. At the same time, the height of the cylinder is 10 cm, expanding at a rate of 3 cm s^{-1}.

Find the rate at which the volume is expanding at time T seconds.

At time T, $r = 4$, $\frac{dr}{dt} = 2$

$h = 10$, $\frac{dh}{dt} = 3$

$$\therefore \frac{dV}{dt} = \pi(4)(2\times10\times2 + 4\times3)$$

$\therefore$ At time T seconds, the volume is expanding at a rate of 208π cm^3 s^{-1}

5.8.4 Differentiating Further Functions

Your notes

This Revision Note focuses on the results and derivations of results involving the less common trigonometric, exponential and logarithmic functions. As with any function, questions may go on to ask about gradients, tangents, normals and stationary points.

Your notes

Differentiating Reciprocal Trigonometric Functions

What are the reciprocal trigonometric functions?

- **Secant**, **cosecant** and **cotangent** and abbreviated and defined as

$$\sec x = \frac{1}{\cos x} \qquad \text{cosec } x = \frac{1}{\sin x} \qquad \cot x = \frac{1}{\tan x}$$

- Remember that for calculus, angles need to be measured in **radians**
  - θ may be used instead of x
- $\text{cosec } x$ is sometimes further abbreviated to $\csc x$

What are the derivatives of the reciprocal trigonometric functions?

- $f(x) = \sec x$
  - $f'(x) = \sec x \tan x$
- $f(x) = \text{cosec } x$
  - $f'(x) = -\text{cosec } x \cot x$
- $f(x) = \cot x$
  - $f'(x) = -\text{cosec}^2 x$
- These are given in the **formula booklet**

How do I show or prove the derivatives of the reciprocal trigonometric functions?

- For $y = \sec x$
  - Rewrite, $y = \frac{1}{\cos x}$
  - Use quotient rule, $\frac{dy}{dx} = \frac{\cos x(0) - (1)(-\sin x)}{\cos^2 x}$
  - Rearrange, $\frac{dy}{dx} = \frac{\sin x}{\cos^2 x}$
  - Separate, $\frac{dy}{dx} = \frac{1}{\cos x} \times \frac{\sin x}{\cos x}$
  - Rewrite, $\frac{dy}{dx} = \sec x \tan x$
- Similarly, for $y = \text{cosec } x$
  - $y = \frac{1}{\sin x}$

Your notes

- $\frac{dy}{dx} = \frac{\sin x(0) - (1)\cos x}{\sin^2 x}$
- $\frac{dy}{dx} = \frac{-\cos x}{\sin^2 x}$
- $\frac{dy}{dx} = -\frac{1}{\sin x} \times \frac{\cos x}{\sin x}$
- $\frac{dy}{dx} = -\operatorname{cosec} x \cot x$

What do the derivatives of reciprocal trig look like with a linear functions of x?

- For linear functions of the form ax+b
  - $f(x) = \sec(ax + b)$
    - $f'(x) = a \sec(ax + b) \tan(ax + b)$
  - $f(x) = \operatorname{cosec}(ax + b)$
    - $f'(x) = -a \operatorname{cosec}(ax + b) \cot(ax + b)$
  - $f(x) = \cot(ax + b)$
    - $f'(x) = -a \operatorname{cosec}^2(ax + b)$
  - These are not given in the formula booklet
    - they can be derived from chain rule
    - they are not essential to remember

Examiner Tip

- Even if you think you have remembered these derivatives, always use the formula booklet to double check
  - those squares and negatives are easy to get muddled up!
- Where two trig functions are involved in the derivative be careful with the angle multiple; $x, 2x, 3x$, etc
  - An example of a common mistake is differentiating $y = cosec\ 3x$
    - $\frac{dy}{dx} = -3\ cosec\ x\ cot\ 3x$ instead of $\frac{dy}{dx} = -3\ cosec\ 3x\ cot\ 3x$

Worked example

Your notes

Curve C has equation $y = 2\cot\left(3x - \frac{\pi}{8}\right)$.

a) Show that the derivative of $\cot x$ is $-\operatorname{cosec}^2 x$.

$y = \cot x = \frac{\cos x}{\sin x}$

Quotient rule

$u = \cos x \quad v = \sin x$

$u' = -\sin x \quad v' = \cos x$

$\therefore \frac{dy}{dx} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$

$\frac{dy}{dx} = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$

$\sin^2 x + \cos^2 x = 1$

$\therefore \frac{dy}{dx} = -\frac{1}{\sin^2 x}$

$\frac{dy}{dx} = -\operatorname{cosec}^2 x$

b) Find $\frac{dy}{dx}$ for curve C.

Your notes

Chain rule/Linear function of x

$$\frac{dy}{dx} = -2\operatorname{cosec}^2\left(3x - \frac{\pi}{8}\right) \times 3$$

$$\therefore \frac{dy}{dx} = -6\operatorname{cosec}^2\left(3x - \frac{\pi}{8}\right)$$

c) Find the gradient of curve C at the point where $x = \dfrac{7\pi}{24}$.

When $x = \frac{7\pi}{24}$

$$\frac{dy}{dx} = -6\operatorname{cosec}^2\left(\frac{21\pi}{24} - \frac{\pi}{8}\right)$$

$$\frac{dy}{dx} = \frac{-6}{\sin^2\left(\frac{3\pi}{4}\right)} = \frac{-6}{\left(\frac{\sqrt{2}}{2}\right)^2}$$

$$\therefore \left.\frac{dy}{dx}\right|_{x=\frac{7\pi}{24}} = -12$$

Your GDC may be able to do this directly

Your notes

Differentiating Inverse Trigonometric Functions

What are the inverse trigonometric functions?

- **arcsin**, **arccos** and **arctan** are functions defined as the inverse functions of sine, cosine and tangent respectively
  - $\arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ which is equivalent to $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$
  - $\arctan(-1) = \frac{3\pi}{4}$ which is equivalent to $\tan\left(\frac{3\pi}{4}\right) = -1$

What are the derivatives of the inverse trigonometric functions?

- $f(x) = \arcsin x$
  - $f'(x) = \frac{1}{\sqrt{1-x^2}}$
- $f(x) = \arccos x$
  - $f'(x) = -\frac{1}{\sqrt{1-x^2}}$
- $f(x) = \arctan x$
  - $f'(x) = \frac{1}{1+x^2}$
- Unlike other derivatives these look completely unrelated at first
  - their derivation involves use of the identity $\cos^2 x + \sin^2 x \equiv 1$
  - hence the squares and square roots!
- All three are given in the **formula booklet**
- Note with the derivative of $\arctan x$ that $(1 + x^2)$ is the same as $(x^2 + 1)$

How do I show or prove the derivatives of the inverse trigonometric functions?

- For $y = \arcsin x$
  - Rewrite, $\sin y = x$
  - Differentiate implicitly, $\cos y \frac{dy}{dx} = 1$
  - Rearrange, $\frac{dy}{dx} = \frac{1}{\cos y}$
  - Using the identity $\cos^2 y \equiv 1 - \sin^2 y$ rewrite, $\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}}$

Your notes

- Since, $\sin y = x$, $\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$
- Similarly, for $y = \arccos x$
  - $\cos y = x$
  - $-\sin y \frac{dy}{dx} = 1$
  - $\frac{dy}{dx} = -\frac{1}{\sin y}$
  - $\frac{dy}{dx} = -\frac{1}{\sqrt{1 - \cos^2 y}}$
  - $\frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}$
- Notice how the derivative of $y = \arcsin x$ is positive but is negative for $y = \arccos x$
  - This subtle but crucial difference can be seen in their graphs
    - $y = \arcsin x$ has a positive gradient for all values of x in its domain
    - $y = \arccos x$ has a negative gradient for all values of x in its domain

What do the derivative of inverse trig look like with a linear function of x?

- For linear functions of the form $ax + b$
- $f(x) = \arcsin(ax + b)$
  - $f'(x) = \frac{a}{\sqrt{1 - (ax + b)^2}}$
- $f(x) = \arccos(ax + b)$
  - $f'(x) = \frac{a}{\sqrt{1 - (ax + b)^2}}$
- $f(x) = \arctan(ax + b)$
  - $f'(x) = \frac{a}{1 + (ax + b)^2}$
- These are **not** in the formula booklet
  - they can be derived from chain rule
  - they are not essential to remember
  - they are not commonly used

Examiner Tip

- For $f(x) = \arctan x$ the terms on the denominator can be reversed (as they are being added rather than subtracted)
  - $f'(x) = \dfrac{1}{1 + x^2} = \dfrac{1}{x^2 + 1}$
  - Don't be fooled by this, it sounds obvious but on awkward "show that" questions it can be off-putting!

Your notes

Your notes

Worked example

a) Show that the derivative of $\arctan x$ is $\dfrac{1}{1+x^2}$

$y = \arctan x$

$\tan y = x$

Differentiate implicitly

$\sec^2 y \dfrac{dy}{dx} = 1$

$\dfrac{dy}{dx} = \dfrac{1}{\sec^2 y}$

Using the identity $\tan^2 y + 1 \equiv \sec^2 y$

$\dfrac{dy}{dx} = \dfrac{1}{\tan^2 y + 1}$

Since $\tan y = x$

$\dfrac{dy}{dx} = \dfrac{1}{x^2 + 1}$

$$\therefore \frac{dy}{dx} = \frac{1}{1+x^2}$$

b) Find the derivative of $\arctan(5x^3 - 2x)$.

Your notes

$5x^3 - 2x$ is not a linear function of x, use chain rule

$y = \arctan(5x^3 - 2x)$

$$\frac{dy}{dx} = \frac{1}{1 + (5x^3 - 2x)^2} \times (15x^2 - 2)$$

$$\therefore \frac{dy}{dx} = \frac{15x^2 - 2}{1 + (5x^3 - 2x)^2}$$

Your notes

Differentiating Exponential & Logarithmic Functions

What are exponential and logarithmic functions?

- Exponential functions have term(s) where the variable (x) is the power (exponent)
  - In general, these would be of the form $y = a^x$
    - The special case of this is when $a = e$, i.e. $y = e^x$
- Logarithmic functions have term(s) where the logarithms of the variable (x) are involved
  - In general, these would be of the form $y = \log_a x$
    - The special case of this is when $a = e$, i.e. $y = \log_e x = \ln x$

What are the derivatives of exponential functions?

- The first two results, of the special cases above, have been met before
  - $f(x) = e^x, \ f'(x) = e^x$
  - $f(x) = \ln x, \ f'(x) = \frac{1}{x}$
  - These are given in the formula booklet
- For the general forms of exponentials and logarithms
  - $f(x) = a^x$
    - $f'(x) = a^x(\ln a)$
  - $f(x) = \log_a x$
    - $f'(x) = \frac{1}{x\ln a}$
  - These are also given in the **formula booklet**

How do I show or prove the derivatives of exponential and logarithmic functions?

- For $y = a^x$
  - Take natural logarithms of both sides, $\ln y = x\ln a$
  - Use the laws of logarithms, $\ln y = x\ln a$
  - Differentiate, implicitly, $\frac{1}{y}\frac{dy}{dx} = \ln a$
  - Rearrange, $\frac{dy}{dx} = y\ln a$
  - Substitute for y, $\frac{dy}{dx} = a^x\ln a$
- For $y = \log_a x$

Your notes

- Rewrite, $x = a^y$
- Differentiate x with respect to y, using the above result, $\frac{dx}{dy} = a^y \ln a$
- Using $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$, $\frac{dy}{dx} = \frac{1}{a^y \ln a}$
- Substitute for y, $\frac{dy}{dx} = \frac{1}{a^{\log_a x} \ln a}$
- Simplify, $\frac{dy}{dx} = \frac{1}{x \ln a}$

What do the derivatives of exponentials and logarithms look like with a linear functions of x?

- For linear functions of the form $px + q$
    - $f(x) = a^{px+q}$
        - $f'(x) = pa^{px+q}(\ln a)$
    - $f(x) = \log_a(px + q)$
        - $f'(x) = \frac{p}{(px+q)\ln a}$
    - These are **not** in the formula booklet
        - they can be derived from chain rule
        - they are not essential to remember

Examiner Tip

- For questions that require the derivative in a particular format, you may need to use the laws of logarithms
    - With ln appearing in denominators be careful with the division law
        - $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$
        - but $\frac{\ln a}{\ln b}$ cannot be simplified (unless there is some numerical connection between a and b)

Your notes

Worked example

a) Find the derivative of a^{3x-2}.

Chain rule or '$px + q$ shortcut' is required

$$\frac{\mathrm{d}}{\mathrm{d}x}[a^{3x-2}] = a^{3x-2} \ln a \times 3$$

∴ The derivative of a^{3x-2} is $3a^{3x-2} \ln a$

b) Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ given that $y = \log_5(2x^3)$

Chain rule is needed

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2x^3 \ln 5} \times 6x^2$$

Simplify by cancelling

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{x \ln 5}$$