DP IB Maths: AA HL

Your notes

2.2 Quadratic Functions & Graphs

Contents

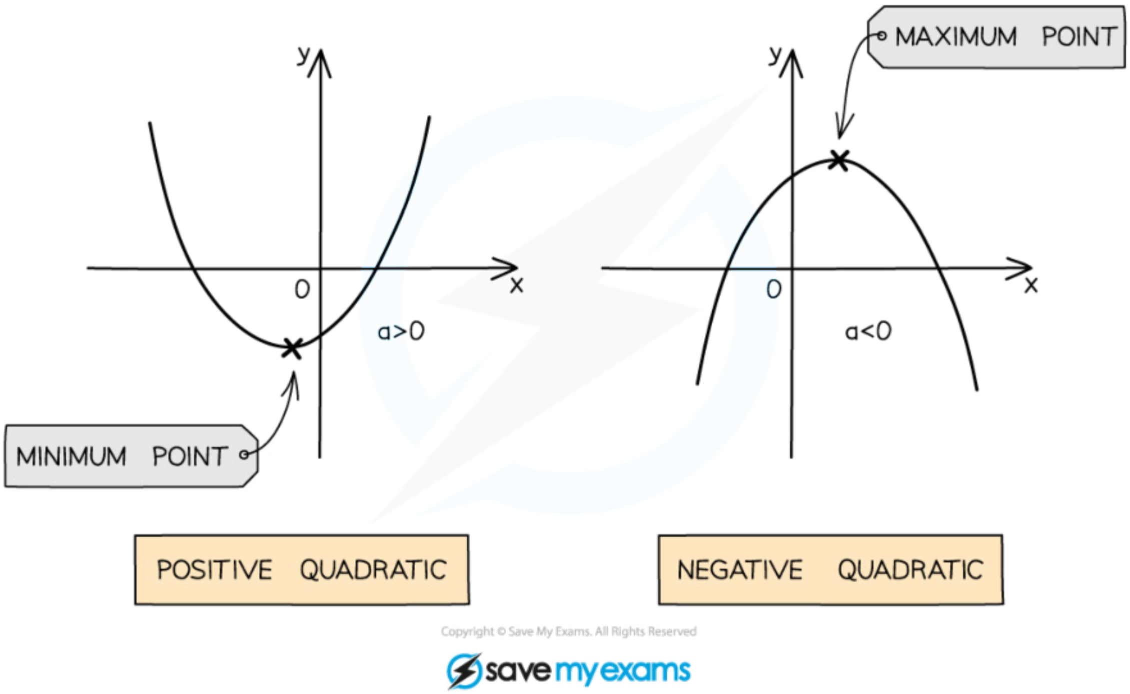
2.2.1 Quadratic Functions

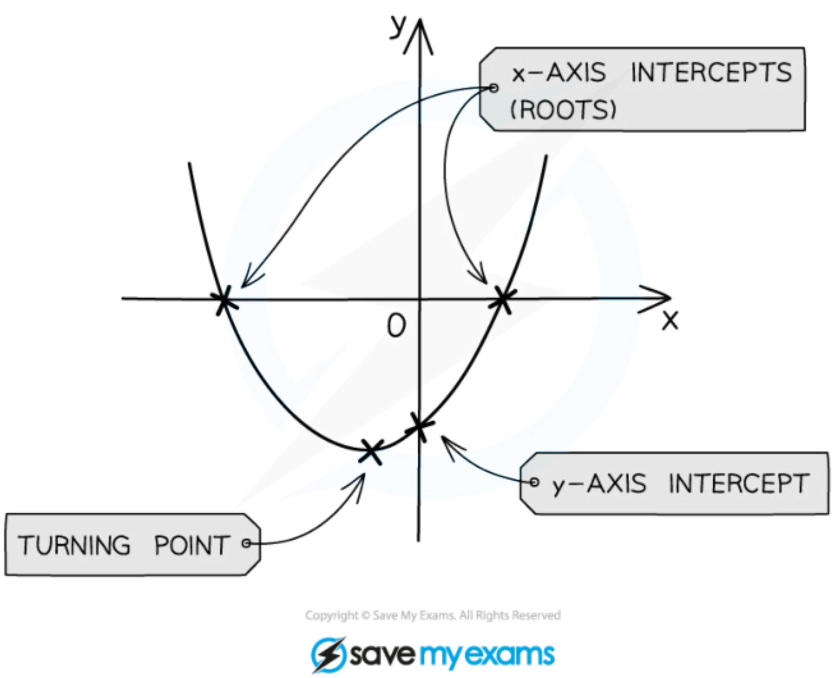
Your notes

Quadratic Functions & Graphs

What are the key features of quadratic graphs?

- A **quadratic** graph can be written in the form $y = ax^2 + bx + c$ where $a \neq 0$
- The value of *a* affects the shape of the curve
  - If *a* is **positive** the shape is **concave up** ∪
  - If *a* is **negative** the shape is **concave down** ∩
- The ***y*-intercept** is at the point (0, c)
- The **zeros or roots** are the solutions to $ax^2 + bx + c = 0$
  - These can be found by
    - Factorising
    - Quadratic formula
    - Using your GDC
  - These are also called the *x*-intercepts
  - There can be 0, 1 or 2 *x*-intercepts
    - This is determined by the value of the **discriminant**
- There is an **axis of symmetry** at $x = -\frac{b}{2a}$
  - This is given in your **formula booklet**
  - If there are two *x*-intercepts then the axis of symmetry goes through the midpoint of them
- The **vertex** lies on the axis of symmetry
  - It can be found by **completing the square**
  - The *x*-coordinate is $x = -\frac{b}{2a}$
  - The *y*-coordinate can be found using the GDC or by calculating *y* when $x = -\frac{b}{2a}$
  - If *a* is **positive** then the vertex is the **minimum point**
  - If *a* is **negative** then the vertex is the **maximum point**





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What are the equations of a quadratic function?

- $f(x) = ax^2 + bx + c$
  - This is the **general form**
  - It clearly shows the *y*-intercept (0, c)
  - You can find the axis of symmetry by $x = -\frac{b}{2a}$
    - This is given in the formula booklet
- $f(x) = a(x - p)(x - q)$
  - This is the **factorised form**
  - It clearly shows the roots (*p*, 0) & (*q*, 0)
  - You can find the axis of symmetry by $x = \frac{p+q}{2}$
- $f(x) = a(x - h)^2 + k$
  - This is the **vertex form**
  - It clearly shows the vertex (*h*, *k*)
  - The axis of symmetry is therefore $x = h$
  - It clearly shows how the function can be transformed from the graph $y = x^2$
    - Vertical stretch by scale factor *a*
    - Translation by vector $\begin{pmatrix} h \\ k \end{pmatrix}$

How do I find an equation of a quadratic?

- If you have the **roots** *x* = *p* and *x* = *q*...
  - Write in **factorised form** $y = a(x - p)(x - q)$
  - You will need a third point to find the value of *a*
- If you have the **vertex** (*h*, *k*) then...
  - Write in **vertex form** $y = a(x - h)^2 + k$
  - You will need a second point to find the value of *a*
- If you have **three random points** (x_1, y_1), (x_2, y_2) & (x_3, y_3) then...
  - Write in the **general form** $y = ax^2 + bx + c$
  - Substitute the three points into the equation
  - Form and solve a system of three linear equations to find the values of *a*, *b* & *c*

Examiner Tip

- Use your GDC to find the roots and the turning point of a quadratic function
  - You do not need to factorise or complete the square
  - It is good exam technique to sketch the graph from your GDC as part of your working

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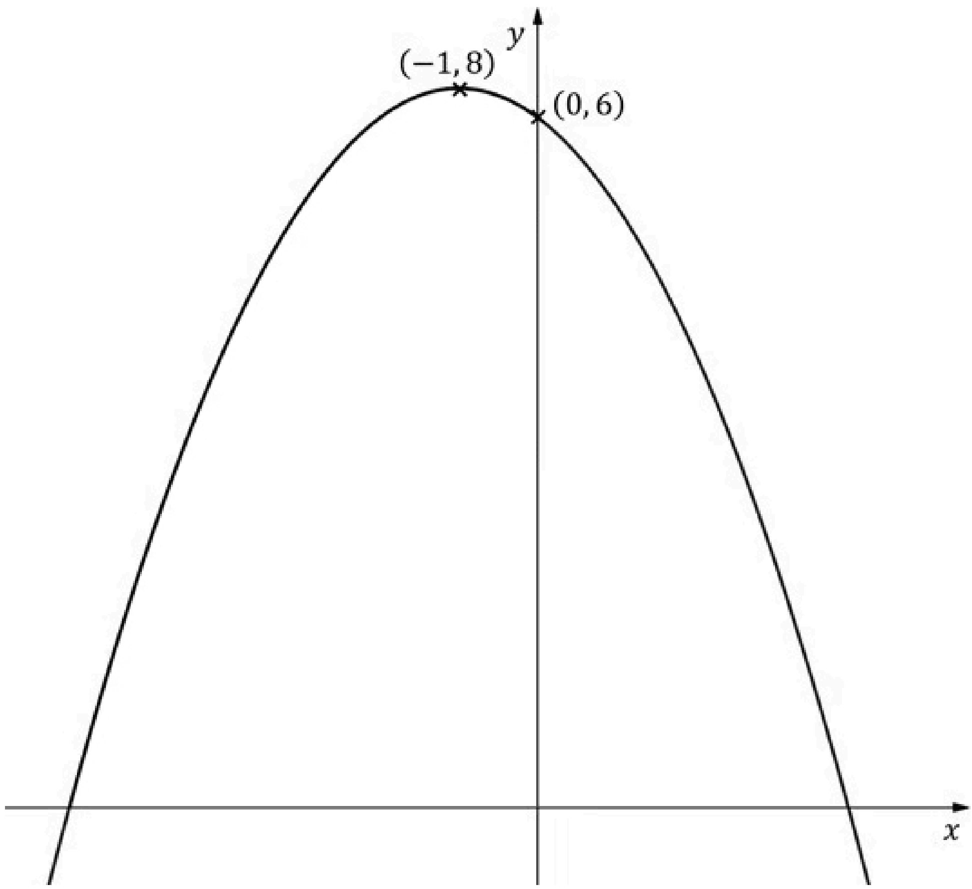
Worked example

The diagram below shows the graph of $y = f(x)$, where $f(x)$ is a quadratic function.

The intercept with the y-axis and the vertex have been labelled.



Write down an expression for $y = f(x)$.

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We have the vertex so use $y = a(x-h)^2 + k$

Vertex $(-1, 8)$: $y = a(x - (-1))^2 + 8$

$y = a(x+1)^2 + 8$

Substitute the second point

$x = 0, y = 6$: $6 = a(0+1)^2 + 8$

$6 = a + 8$

$a = -2$

$$y = -2(x+1)^2 + 8$$

2.2.2 Factorising & Completing the Square

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Factorising Quadratics

Why is factorising quadratics useful?

- Factorising gives **roots (zeroes or solutions)** of a quadratic
- It gives the ***x*-intercepts** when drawing the graph

How do I factorise a monic quadratic of the form $x^2 + bx + c$?

- A monic quadratic is a quadratic where the coefficient of the x^2 term is 1
- You might be able to spot the factors by **inspection**
  - Especially if *c* is a **prime number**
- Otherwise find two numbers *m* and *n* ..
  - A sum equal to *b*
    - $p + q = b$
  - A product equal to c
    - $pq = c$
- Rewrite *bx* as *mx* + *nx*
- Use this to factorise $x^2 + mx + nx + c$
- A shortcut is to write:
  - $(x + p)(x + q)$

How do I factorise a non-monic quadratic of the form $ax^2 + bx + c$?

- A non-monic quadratic is a quadratic where the coefficient of the x^2 term is not equal to 1
- If *a*, *b* & *c* have a common factor then first factorise that out to leave a quadratic with coefficients that have **no common factors**
- You might be able to spot the factors by **inspection**
  - Especially if *a* and/or *c* are **prime numbers**
- Otherwise find two numbers *m* and *n* ..
  - A sum equal to *b*
    - $m + n = b$
  - A product equal to *ac*
    - $mn = ac$
- Rewrite *bx* as *mx* + *nx*
- Use this to factorise $ax^2 + mx + nx + c$
- A shortcut is to write:
  - $\frac{(ax + m)(ax + n)}{a}$
  - Then factorise common factors from numerator to cancel with the *a* on the denominator

How do I use the difference of two squares to factorise a quadratic of the form $a^2x^2 - c^2$?

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- The **difference of two squares** can be used when...
  - There is **no *x* term**
  - The **constant term is a negative**
- Square root the two terms a^2x^2 and c^2
- The two factors are the **sum of square roots** and the **difference of the square roots**
- A shortcut is to write:
  - $(ax + c)(ax - c)$

Examiner Tip

- You can deduce the factors of a quadratic function by using your GDC to find the solutions of a quadratic equation
  - Using your GDC, the quadratic equation $6x^2 + x - 2 = 0$ has solutions $x = -\frac{2}{3}$ and $x = \frac{1}{2}$
  - Therefore the factors would be $(3x + 2)$ and $(2x - 1)$
  - i.e. $6x^2 + x - 2 = (3x + 2)(2x - 1)$

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Worked example

Factorise fully:

a) $x^2 - 7x + 12$.

Find two numbers m and n such that

$m + n = b = -7$ $mn = c = 12$

$-4 + -3 = -7$ $-4 \times -3 = 12$

Split $-7x$ up and factorise

$x^2 - 4x - 3x + 12$

$x(x-4) - 3(x-4)$

$(x-3)(x-4)$

Shortcut

$(x+m)(x+n)$

$(x-3)(x-4)$

b) $4x^2 + 4x - 15$.

Find two numbers m and n such that

$m + n = b = 4$ $mn = ac = 4 \times -15 = -60$

$10 + -6 = 4$ $10 \times -6 = -60$

Split $4x$ up and factorise

$4x^2 + 10x - 6x - 15$

$2x(2x+5) - 3(2x+5)$

$(2x-3)(2x+5)$

Shortcut

$\frac{(ax+m)(ax+n)}{a}$

$\frac{(4x+10)(4x-6)}{4}$

$\frac{2(2x+5)2(2x-3)}{4}$

$(2x-3)(2x+5)$

c) $18 - 50x^2$.

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Factorise the common factor

$2(9-25x^2)$

Use difference of two squares

$2(3-5x)(3+5x)$

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Completing the Square

Why is completing the square for quadratics useful?

- Completing the square gives the **maximum/minimum** of a quadratic function
  - This can be used to define the **range** of the function
- It gives the **vertex** when drawing the graph
- It can be used to **solve quadratic equations**
- It can be used to derive the **quadratic formula**

How do I complete the square for a monic quadratic of the form $x^2 + bx + c$?

- **Half the value of *b*** and write $\left(x + \frac{b}{2}\right)^2$
  - This is because $\left(x + \frac{b}{2}\right)^2 = x^2 + bx + \frac{b^2}{4}$
- **Subtract the unwanted** $\frac{b^2}{4}$ **term** and **add on the constant c**
  - $\left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} + c$

How do I complete the square for a non-monic quadratic of the form $ax^2 + bx + c$?

- **Factorise out the *a*** from the **terms involving *x***
  - $a\left(x^2 + \frac{b}{a}x\right) + x$
  - Leaving the c alone will **avoid working with lots of fractions**
- **Complete the square** on the quadratic term
  - **Half** $\frac{b}{a}$ and write $\left(x + \frac{b}{2a}\right)^2$
    - This is because $\left(x + \frac{b}{2a}\right)^2 = x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$
  - **Subtract the unwanted** $\frac{b^2}{4a^2}$ **term**
- **Multiply by *a*** and **add the constant c**
  - $a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right] + c$
  - $a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$

Your notes

Examiner Tip

- Some questions may not use the phrase "completing the square" so ensure you can recognise a quadratic expression or equation written in this form
  - $a(x-h)^2 + k\ (=0)$

Worked example

Complete the square:

a) $x^2 - 8x + 3$.

Half b and subtract its square

$(x-4)^2 - 4^2 + 3$

$(x-4)^2 - 13$

b) $3x^2 + 12x - 5$.

Factorise the 3 from the x terms

$3(x^2 + 4x) - 5$

Complete the square on $x^2 + 4x$

$3\left((x+2)^2 - 2^2\right) - 5$

Simplify

$3\left((x+2)^2 - 4\right) - 5$

$3(x+2)^2 - 12 - 5$

$3(x+2)^2 - 17$

2.2.3 Solving Quadratics

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Solving Quadratic Equations

How do I decide the best method to solve a quadratic equation?

- A quadratic equation is of the form $ax^2 + bx + c = 0$
- If it is a calculator paper then use your GDC to solve the quadratic
- If it is a non-calculator paper then...
  - you can always use the **quadratic formula**
  - you can **factorise** if it can be factorised with integers
  - you can always **complete the square**

How do I solve a quadratic equation by the quadratic formula?

- If necessary **rewrite** in the form $ax^2 + bx + c = 0$
- **Clearly identify** the values of *a*, *b* & c
- **Substitute** the values into the formula
  - $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
    - This is given in the **formula booklet**
- **Simplify** the solutions as much as possible

How do I solve a quadratic equation by factorising?

- **Factorise** to rewrite the quadratic equation in the form $a(x-p)(x-q) = 0$
- Set each factor to zero and **solve**
  - $x - p = 0 \Rightarrow x = p$
  - $x - q = 0 \Rightarrow x = q$

How do I solve a quadratic equation by completing the square?

- **Complete the square** to rewrite the quadratic equation in the form $a(x-h)^2 + k = 0$
- Get the squared term by itself
  - $(x-h)^2 = -\dfrac{k}{a}$
- If $\left(-\dfrac{k}{a}\right)$ is **negative** then there will be **no solutions**
- If $\left(-\dfrac{k}{a}\right)$ is **positive** then there will be **two values** for $x - h$

- $x - h = \pm\sqrt{-\frac{k}{a}}$
- **Solve** for *x*
  - $x = h \pm \sqrt{-\frac{k}{a}}$

Examiner Tip

- When using the quadratic formula with awkward values or fractions you may find it easier to deal with the " $b^2 - 4ac$ " (**discriminant**) first
  - This can help avoid numerical and negative errors, improving accuracy

Your notes

Your notes

Worked example

Solve the equations:

a) $4x^2 + 4x - 15 = 0$.

This can be factorised

$(2x+5)(2x-3) = 0$

$2x + 5 = 0$ or $2x - 3 = 0$

$x = -\frac{5}{2}$ or $x = \frac{3}{2}$

b) $3x^2 + 12x - 5 = 0$.

This can not be factorised but $3x^2$ and $12x$ have a common factor so complete the square

$3(x+2)^2 - 17 = 0$

$(x+2)^2 = \frac{17}{3}$ — Rearrange

$x + 2 = \pm\sqrt{\frac{17}{3}}$ — Remember $\pm$

$x = -2 \pm \sqrt{\frac{17}{3}}$

c) $7 - 3x - 5x^2 = 0$.

Your notes

This can not be factorised so use formula

Formula booklet

Solutions of a quadratic equation	$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad a \neq 0$

$a = -5 \quad b = -3 \quad c = 7$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(-5)(7)}}{2(-5)}$$

$$= \frac{3 \pm \sqrt{9 + 140}}{-10}$$

$$x = -\frac{3 \pm \sqrt{149}}{10}$$

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2.2.4 Quadratic Inequalities

Quadratic Inequalities

What affects the inequality sign when rearranging a quadratic inequality?

- The inequality sign is **unchanged** by...
  - **Adding/subtracting** a term to both sides
  - **Multiplying/dividing** both sides by a **positive term**
- The inequality sign **flips** (< changes to >) when...
  - **Multiplying/dividing** both sides by a **negative term**

How do I solve a quadratic inequality?

- **STEP 1**: **Rearrange** the inequality into quadratic form with a **positive squared term**
  - $ax^2 + bx + c > 0$
  - $ax^2 + bx + c \geq 0$
  - $ax^2 + bx + c < 0$
  - $ax^2 + bx + c \leq 0$
- **STEP 2**: Find the **roots** of the quadratic equation
  - Solve $ax^2 + bx + c = 0$ to get x_1 and x_2 where $x_1 < x_2$
- **STEP 3**: **Sketch** a graph of the quadratic and label the roots
  - As the squared term is positive it will be **concave up** so "U" shaped
- **STEP 4**: **Identify** the **region** that satisfies the inequality
  - If you want the graph to be **above the *x*-axis** then choose the region to be the **two intervals outside** of the two roots
  - If you want the graph to be **below the *x*-axis** then choose the region to be the **interval between** the two roots
  - For $ax^2 + bx + c > 0$
    - The solution is $\boldsymbol{x < x_1}$ **or** $\boldsymbol{x > x_2}$
  - For $ax^2 + bx + c \geq 0$
    - The solution is $\boldsymbol{x \leq x_1}$ **or** $\boldsymbol{x \geq x_2}$
  - For $ax^2 + bx + c < 0$
    - The solution is $\boldsymbol{x_1 < x < x_2}$
  - For $ax^2 + bx + c \leq 0$
    - The solution is $\boldsymbol{x_1 \leq x \leq x_2}$

How do I solve a quadratic inequality of the form $(x - h)^2 < n$ or $(x - h)^2 > n$?

- The safest way is by following the steps above
  - Expand and rearrange
- A **common mistake** is writing $x - h < \pm\sqrt{n}$ or $x - h > \pm\sqrt{n}$
  - This is **NOT correct**!
- The correct solution to $\boldsymbol{(x - h)^2 < n}$ is

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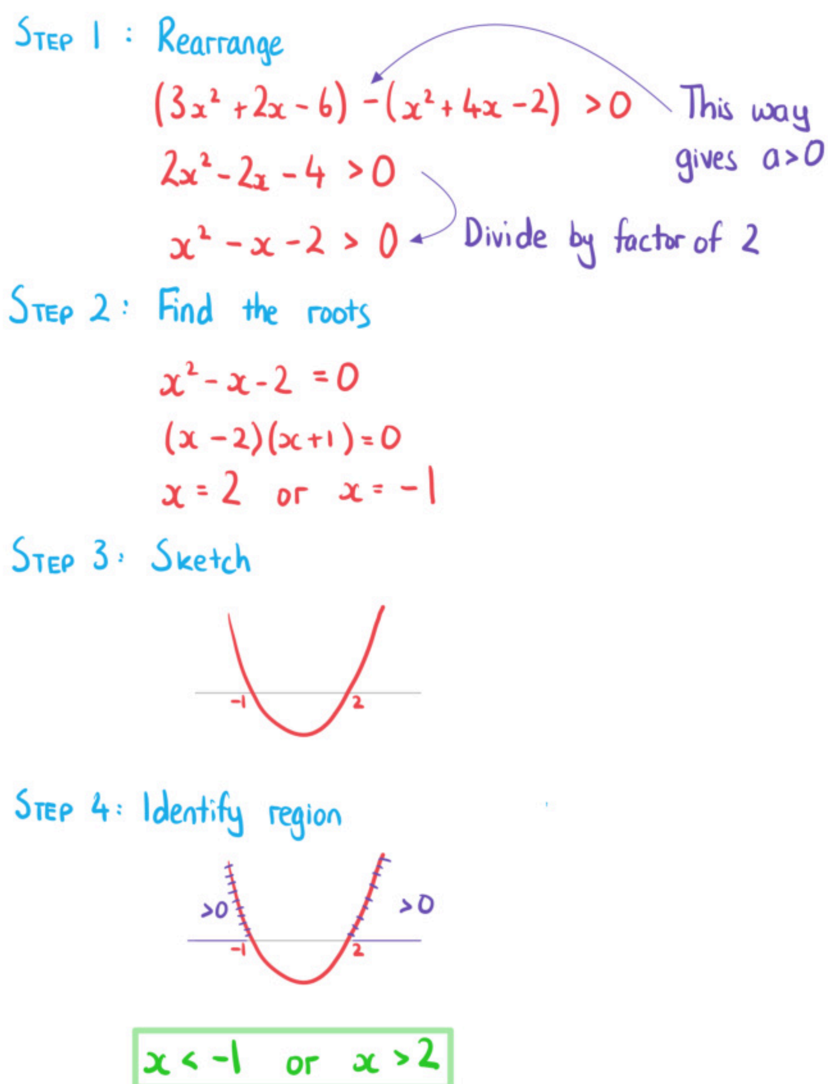
- $|x-h|<\sqrt{n}$ which can be written as $-\sqrt{n}<x-h<\sqrt{n}$
- The **final solution** is $h-\sqrt{n}<x<h+\sqrt{n}$
- The correct solution to **(x – h)^2 > *n*** is
  - $|x-h|>\sqrt{n}$ which can be written as $x-h<-\sqrt{n}$ or $x-h>\sqrt{n}$
  - The **final solution** is $x<h-\sqrt{n}$ **or** $x>h+\sqrt{n}$

Examiner Tip

- It is easiest to sketch the graph of a quadratic when it has a positive x^2 term, so rearrange first if necessary
- Use your GDC to help select the correct region(s) for the inequality
- Some makes/models of GDC may have the ability to solve inequalities directly
  - However unconventional notation may be used to display the answer (e.g. $6 > x > 3$ rather than $3 < x < 6$)
  - The safest method is to **always** sketch the graph

Worked example

Find the set of values which satisfy $3x^2 + 2x - 6 > x^2 + 4x - 2$.



Your notes

Your notes

2.2.5 Discriminants

Discriminants

What is the discriminant of a quadratic function?

- The discriminant of a quadratic is denoted by the Greek letter Δ (upper case delta)
- For the quadratic function the discriminant is given by
  - $\Delta = b^2 - 4ac$
    - This is given in the **formula booklet**
- The discriminant is the expression that is square rooted in the **quadratic formula**

How does the discriminant of a quadratic function affect its graph and roots?

- If Δ > 0 then $\sqrt{b^2 - 4ac}$ and $-\sqrt{b^2 - 4ac}$ are **two distinct values**
  - The equation $ax^2 + bx + c = 0$ has **two distinct real solutions**
  - The graph of $y = ax^2 + bx + c$ has **two distinct real roots**
    - This means the graph **crosses** the *x*-axis **twice**
- If Δ = 0 then $\sqrt{b^2 - 4ac}$ and $-\sqrt{b^2 - 4ac}$ are **both zero**
  - The equation $ax^2 + bx + c = 0$ has **one repeated real solution**
  - The graph of $y = ax^2 + bx + c$ has **one repeated real root**
    - This means the graph **touches** the *x*-axis at **exactly one point**
    - This means that the **x-axis** is a **tangent** to the graph
- If Δ < 0 then $\sqrt{b^2 - 4ac}$ and $-\sqrt{b^2 - 4ac}$ are **both undefined**
  - The equation $ax^2 + bx + c = 0$ has **no real solutions**
  - The graph of $y = ax^2 + bx + c$ has **no real roots**
    - This means the graph **never touches** the **x-axis**
    - This means that graph is **wholly above** (or **below**) the **x-axis**

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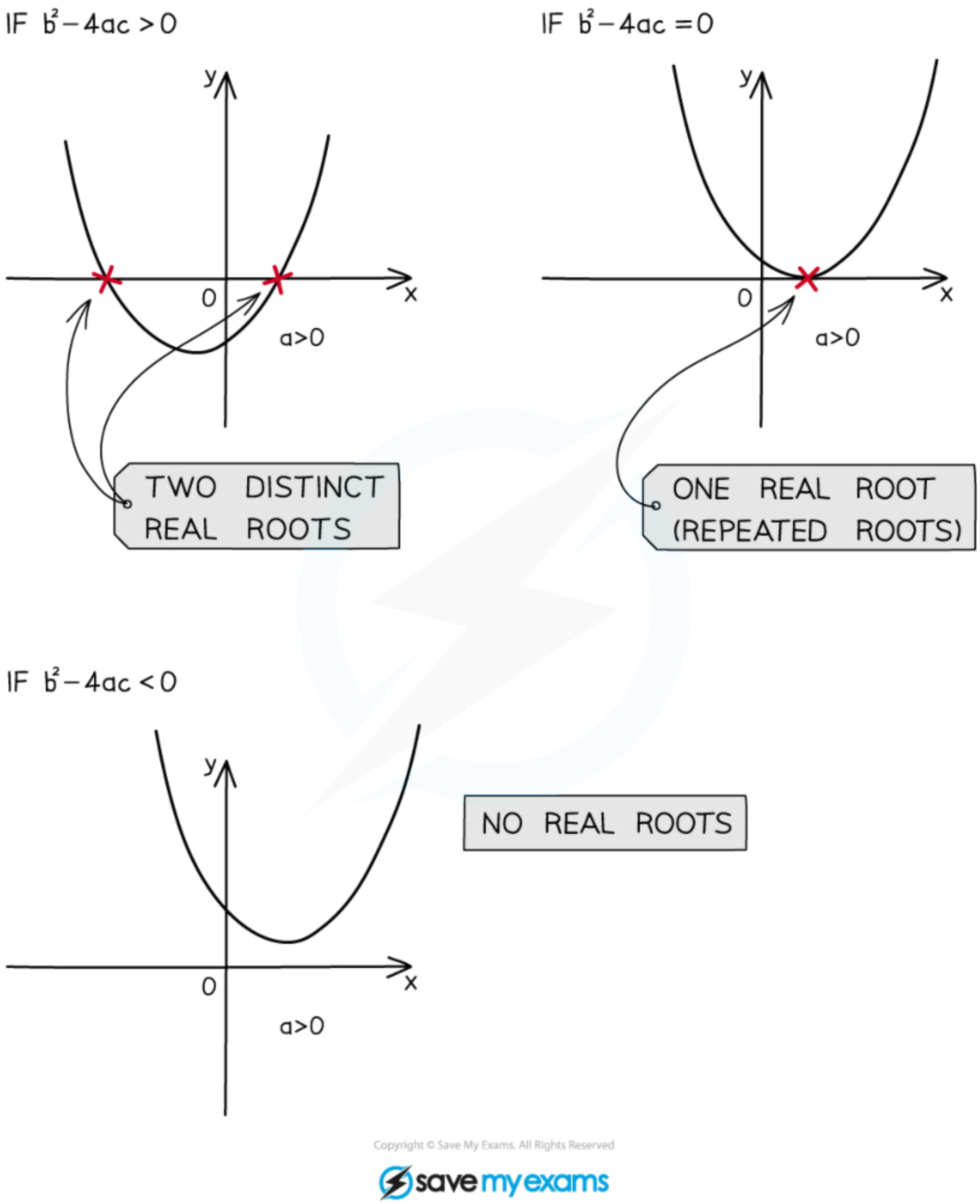

Forming equations and inequalities using the discriminant

- Often at least one of the coefficients of a quadratic is **unknown**
  - Questions usually use the letter k for the unknown constant
- You will be given a fact about the quadratic such as:
  - The **number of solutions** of the equation
  - The **number of roots** of the graph

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- To find the **value or range of values** of k
  - Find an **expression for the discriminant**
    - Use $\Delta = b^2 - 4ac$
  - Decide whether $\Delta > 0$, $\Delta = 0$ or $\Delta < 0$
    - If the question says there are **real roots** but does not specify how many then use $\Delta \geq 0$
  - **Solve** the resulting equation or inequality

Examiner Tip

- Questions will rarely use the word discriminant so it is important to recognise when its use is required
  - Look for
    - a number of roots or solutions being stated
    - whether and/or how often the graph of a quadratic function intercepts the x-axis
- Be careful setting up inequalities that concern "two real roots" ($\Delta \geq 0$) as opposed to "two real distinct roots" ($\Delta > 0$)

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Worked example

A function is given by $f(x) = 2kx^2 + kx - k + 2$, where k is a constant. The graph of $y = f(x)$ has two distinct real roots.

a) Show that $9k^2 - 16k > 0$.

Two distinct real roots $\Rightarrow$ $\Delta > 0$

Formula booklet

Discriminant	$\Delta = b^2 - 4ac$

$a = 2k \quad b = k \quad c = (-k + 2)$

$\Delta = k^2 - 4(2k)(-k+2)$

$= k^2 + 8k^2 - 16k$

$= 9k^2 - 16k$

$\Delta > 0 \Rightarrow 9k^2 - 16k > 0$

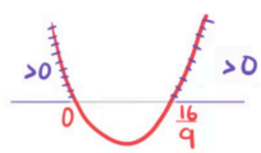
b) Hence find the set of possible values of k.

Solve the inequality

$9k^2 - 16k = 0$

$k(9k - 16) = 0$

$k = 0$ or $k = \frac{16}{9}$



$k < 0$ or $k > \frac{16}{9}$