

# DP IB Maths: AI HL



Your notes

## 1.3 Sequences & Series

### Contents

- \* 1.3.1 Language of Sequences & Series
- \* 1.3.2 Arithmetic Sequences & Series
- \* 1.3.3 Geometric Sequences & Series
- \* 1.3.4 Applications of Sequences & Series



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## 1.3.1 Language of Sequences & Series

### Language of Sequences & Series

#### What is a sequence?

- A **sequence** is an ordered set of numbers with a well-defined rule for getting from one number to the next
  - For example 1, 3, 5, 7, 9, ... is a sequence with the rule 'start at one and add two to get each subsequent number'
- The numbers in a sequence are often called **terms**
- The terms of a sequence are often referred to by letters with a subscript
  - In IB this will be the letter  $u$
  - So in the sequence above,  $u_1 = 1$ ,  $u_2 = 3$ ,  $u_3 = 5$  and so on
- Each term in a sequence can be found by **substituting** the term number into the **formula for the  $n^{\text{th}}$  term**

#### What is a series?

- You get a **series** by summing up the terms in a sequence
  - E.g. For the sequence 1, 3, 5, 7, ... the associated series is  $1 + 3 + 5 + 7 + \dots$
- We use the notation  $S_n$  to refer to the sum of the first  $n$  terms in the series
  - $S_n = u_1 + u_2 + u_3 + \dots + u_n$
  - So for the series above  $S_5 = 1 + 3 + 5 + 7 + 9 = 25$



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### Worked example

Determine the first five terms and the value of  $S_5$  in the sequence with terms defined by  $u_n = 5 - 2n$ .

$$u_n = 5 - 2n$$

find the term you want by replacing  $n$  with its value.

term number

first term

$$\begin{aligned} \rightarrow u_1 &= 5 - 2(1) = 3 \\ u_2 &= 5 - 2(2) = 1 \\ u_3 &= 5 - 2(3) = -1 \\ u_4 &= 5 - 2(4) = -3 \\ u_5 &= 5 - 2(5) = -5 \end{aligned}$$

recognise the pattern.

-2

-2 ← rule is subtract 2

'start with 3 and subtract 2 from each number'.

$$S_5 = 3 + 1 + (-1) + (-3) + (-5) = -5$$

the sum of the first 5 terms

$$\boxed{3, 1, -1, -3, -5}$$

$$S_5 = -5$$



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## Sigma Notation

### What is sigma notation?

- Sigma notation is used to show the sum of a certain number of terms in a sequence
- The symbol  $\Sigma$  is the capital Greek letter sigma
- $\Sigma$  stands for 'sum'
  - The expression to the right of the  $\Sigma$  tells you what is being summed, and the limits above and below tell you which terms you are summing

THESE LIMITS TELL YOU THAT YOU ARE SUMMING  $(2r-1)$  USING THE VALUES  $r=1, r=2, \dots$  UP TO  $r=5$

$$\sum_{r=1}^5 (2r-1) = 1 + 3 + 5 + 7 + 9$$

SUBSTITUTE  $r=1, r=2, r=3, r=4, r=5$  INTO  $(2r-1)$  TO FIND THE FIVE TERMS THAT ARE BEING SUMMED

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- Be careful, the limits don't have to start with 1
  - For example  $\sum_{k=0}^4 (2k+1)$  or  $\sum_{k=7}^{14} (2k-13)$
  - $r$  and  $k$  are commonly used variables within sigma notation

### Examiner Tip

- Your GDC will be able to use sigma notation, familiarise yourself with it and practice using it to check your work



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### Worked example

A sequence can be defined by  $u_n = 2 \times 3^{n-1}$  for  $n \in \mathbb{Z}^+$ .

- a) Write an expression for  $u_1 + u_2 + u_3 + \dots + u_6$  using sigma notation.

$$u_n = 2 \times 3^{n-1}, n \in \mathbb{Z}^+ \leftarrow n \text{ is the set of all positive integers}$$

Using sigma notation

$$u_1 + u_2 + \dots + u_6 = \sum_{k=1}^6 u_k$$

$$\sum_{k=1}^6 (2 \times 3^{k-1})$$

- b) Write an expression for  $u_7 + u_8 + u_9 + \dots + u_{12}$  using sigma notation.

$$u_n = 2 \times 3^{n-1}, n \in \mathbb{Z}^+ \leftarrow n \text{ is the set of all positive integers}$$

Using sigma notation

$$u_7 + u_8 + \dots + u_{12} = \sum_{k=7}^{12} u_k$$

$$\sum_{k=7}^{12} (2 \times 3^{k-1})$$



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## 1.3.2 Arithmetic Sequences & Series

### Arithmetic Sequences

#### What is an arithmetic sequence?

- In an **arithmetic sequence**, the difference between consecutive terms in the sequence is constant
- This **constant difference** is known as the **common difference**,  $d$ , of the sequence
  - For example, 1, 4, 7, 10, ... is an arithmetic sequence with the rule 'start at one and add three to each number'
    - The **first term**,  $u_1$ , is 1
    - The **common difference**,  $d$ , is 3
  - An arithmetic sequence can be **increasing** (positive common difference) or **decreasing** (negative common difference)
  - Each term of an arithmetic sequence is referred to by the letter  $u$  with a subscript determining its place in the sequence

#### How do I find a term in an arithmetic sequence?

- The  $n^{\text{th}}$  term formula for an arithmetic sequence is given as
$$u_n = u_1 + (n - 1)d$$
  - Where  $u_1$  is the first term, and  $d$  is the common difference
  - This is given in the formula booklet, you do not need to know how to derive it
- Enter the information you have into the formula and use your GDC to find the value of the term
- Sometimes you will be given a term and asked to find the first term or the common difference
  - Substitute the information into the formula and solve the equation
    - You could use your GDC for this
- Sometimes you will be given two terms and asked to find both the first term and the common difference
  - Substitute the information into the formula and set up a **system of linear equations**
  - Solve the simultaneous equations
    - You could use your GDC for this

#### Examiner Tip

- Simultaneous equations are often needed within arithmetic sequence questions, make sure you are confident solving them with and without the GDC



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### Worked example

The fourth term of an arithmetic sequence is 10 and the ninth term is 25, find the first term and the common difference of the sequence.

$$u_4 = 10, \quad u_9 = 25$$

Formula for  $n^{\text{th}}$  term of an arithmetic series:

$$u_n = u_1 + (n-1)d$$

Sub in  $u_4 = 10$  and  $u_9 = 25$

$$u_4 = u_1 + (4-1)d = u_1 + 3d = 10$$

$$u_9 = u_1 + (9-1)d = u_1 + 8d = 25$$

Solve using aOC:

let  $u_1 = x$  and  $d = y$

$$x + 3y = 10$$

$$x + 8y = 25$$

$$x = 1, \quad y = 3$$

$$\boxed{\begin{array}{l} u_1 = 1 \\ d = 3 \end{array}}$$



## Arithmetic Series

### How do I find the sum of an arithmetic series?

- An **arithmetic series** is the sum of the terms in an **arithmetic sequence**
  - For the arithmetic sequence 1, 4, 7, 10, ... the arithmetic series is  $1 + 4 + 7 + 10 + \dots$
- Use the following formulae to find the sum of the first  $n$  terms of the arithmetic series:

$$S_n = \frac{n}{2}(2u_1 + (n-1)d) \quad ; \quad S_n = \frac{n}{2}(u_1 + u_n)$$

- $u_1$  is the first term
- $d$  is the common difference
- $u_n$  is the last term
- Both formulae are given in the formula booklet, you do not need to know how to derive them
- You can use whichever formula is more convenient for a given question
  - If you know the first term and common difference use the first version
  - If you know the first and last term then the second version is easier to use
- A question will often give you the sum of a certain number of terms and ask you to find the value of the first term or the common difference
  - Substitute the information into the formula and solve the equation
    - You could use your GDC for this

#### Examiner Tip

- The formulae you need for arithmetic series are in the formula book, you do not need to remember them
  - Practice finding the formulae so that you can quickly locate them in the exam



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### Worked example

The sum of the first 10 terms of an arithmetic sequence is 630.

- a) Find the common difference,  $d$ , of the sequence if the first term is 18.

$$S_{10} = 630$$

Formula for the sum of  
an arithmetic series:

$$S_n = \frac{n}{2} (2u_1 + (n-1)d)$$

Sub in  $S_{10} = 630$ ,  $u_1 = 18$

$$S_{10} = \frac{10}{2} (2(18) + (10-1)d) = 630$$

$$5(36 + 9d) = 630$$

Solve:  $36 + 9d = 126$

$$9d = 90$$

$$d = 10$$

$$d = 10$$

- b) Find the first term of the sequence if the common difference,  $d$ , is 11.

$$\text{Sub in } S_{10} = 630, \quad d = 11$$

$$S_{10} = \frac{10}{2} (2u_1 + (10-1)(11)) = 630$$

$$5(2u_1 + 99) = 630$$

$$\text{Solve: } \quad 2u_1 + 99 = 126$$

$$2u_1 = 27$$

$$u_1 = 13.5$$



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## 1.3.3 Geometric Sequences & Series

### Geometric Sequences

#### What is a geometric sequence?

- In a **geometric sequence**, there is a **common ratio,  $r$** , between consecutive terms in the sequence
  - For example, 2, 6, 18, 54, 162, ... is a sequence with the rule 'start at two and multiply each number by three'
    - The **first term,  $u_1$** , is 2
    - The **common ratio,  $r$** , is 3
- A geometric sequence can be **increasing** ( $r > 1$ ) or **decreasing** ( $0 < r < 1$ )
- If the common ratio is a **negative number** the terms will alternate between positive and negative values
  - For example, 1, -4, 16, -64, 256, ... is a sequence with the rule 'start at one and multiply each number by negative four'
    - The **first term,  $u_1$** , is 1
    - The **common ratio,  $r$** , is -4
- Each term of a geometric sequence is referred to by the letter  $u$  with a subscript determining its place in the sequence

#### How do I find a term in a geometric sequence?

- The  $n^{\text{th}}$  term formula for a geometric sequence is given as

$$u_n = u_1 r^{n-1}$$

- Where  $u_1$  is the first term, and  $r$  is the common ratio
- This formula allows you to find **any term** in the geometric sequence
- It is given in the formula booklet, you do not need to know how to derive it
- Enter the information you have into the formula and use your GDC to find the value of the term
- Sometimes you will be given a term and asked to find the first term or the common ratio
  - Substitute the information into the formula and solve the equation
    - You could use your GDC for this
- Sometimes you will be given two or more consecutive terms and asked to find both the first term and the common ratio
  - Find the common ratio by dividing a term by the one before it
  - Substitute this and one of the terms into the formula to find the first term
- Sometimes you may be given a term and the formula for the  $n^{\text{th}}$  term and asked to find the value of  $n$ 
  - You can solve these using **logarithms** on your GDC

### Examiner Tip

- You will sometimes need to use logarithms to answer geometric sequences questions
  - Make sure you are confident doing this
  - Practice using your GDC for different types of questions



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### Worked example

The sixth term,  $u_6$ , of a geometric sequence is 486 and the seventh term,  $u_7$ , is 1458.

Find,

- i) the common ratio,  $r$ , of the sequence,

$$u_6 = 486, \quad u_7 = 1458$$

The common ratio,  $r$ , is given by

$$r = \frac{u_2}{u_1} = \frac{u_3}{u_2} = \dots = \frac{u_{n+1}}{u_n}$$

$$\text{Sub in } u_6 = 486, \quad u_7 = 1458$$

$$r = \frac{u_7}{u_6} = \frac{1458}{486} = 3$$

$$r = 3$$

- ii) the first term of the sequence,  $u_1$ .

Formula for  $n^{\text{th}}$  term of a geometric series:

$$u_n = u_1 r^{n-1}$$

Sub in  $r=3$  and either  $u_6 = 486$  or  $u_7 = 1458$

$$u_6 = u_1(3)^{6-1} = 486$$

$$\text{Solve: } 243 u_1 = 486$$

$$u_1 = 2$$

$$u_1 = 2$$



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## Geometric Series

### How do I find the sum of a geometric series?

- A **geometric series** is the sum of a certain number of terms in a **geometric sequence**
  - For the geometric sequence 2, 6, 18, 54, ... the geometric series is  $2 + 6 + 18 + 54 + \dots$
- The following formulae will let you find the sum of the first  $n$  terms of a geometric series:

$$S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}$$

- $u_1$  is the first term
- $r$  is the common ratio
- Both formulae are given in the formula booklet, you do not need to know how to derive them
- You can use whichever formula is more convenient for a given question
  - The first version of the formula is more convenient if  $r > 1$  and the second is more convenient if  $r < 1$
- A question will often give you the sum of a certain number of terms and ask you to find the value of the first term, the common ratio, or the number of terms within the sequence
  - Substitute the information into the formula and solve the equation
    - You could use your GDC for this

#### Examiner Tip

- The geometric series formulae are in the formulae booklet, you don't need to memorise them
  - Make sure you can locate them quickly in the formula booklet



Your notes





Your notes

### Worked example

A geometric sequence has  $u_1 = 25$  and  $r = 0.8$ . Find the value of  $u_5$  and  $S_5$ .

$$u_1 = 25, \quad r = 0.8$$

Formula for  $n^{\text{th}}$  term of a geometric series:

$$u_n = u_1 r^{n-1}$$

Sub in  $u_1 = 25, \quad r = 0.8$

$$u_5 = 25(0.8)^4 = 10.24$$

Formula for the sum of a geometric series:

$$S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}$$

$r < 1$  so this version is easier to use.

Sub in  $u_1 = 25, \quad r = 0.8$

$$S_5 = \frac{u_1(1 - r^5)}{1 - r} = \frac{25(1 - 0.8^5)}{1 - 0.8} = 84.04$$

$$u_5 = 10.24$$

$$S_5 = 84.04$$



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## Sum to Infinity

### What is the sum to infinity of a geometric series?

- A geometric sequence will either increase or decrease away from zero or the terms will get progressively closer to zero
  - Terms will get closer to zero if the common ratio,  $r$ , is between 1 and -1
- If the terms are getting closer to zero then the series is said to **converge**
  - This means that the sum of the series will approach a limiting value
  - As the number of terms increase, the sum of the terms will get closer to the limiting value

### How do we calculate the sum to infinity?

- If asked to find out if a geometric sequence converges find the value of  $r$ 
  - If  $|r| < 1$  then the sequence converges
  - If  $|r| \geq 1$  then the sequence does not converge and the sum to infinity cannot be calculated
  - $|r| < 1$  means  $-1 < r < 1$
- If  $|r| < 1$ , then the geometric series **converges** to a finite value given by the formula

$$S_{\infty} = \frac{u_1}{1-r}, \quad |r| < 1$$

- $u_1$  is the first term
- $r$  is the common ratio
- This is **in the formula book**, you do not need to remember it

#### Examiner Tip

- Learn and remember the conditions for when a sum to infinity can be calculated



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### Worked example

The first three terms of a geometric sequence are  $6$ ,  $2$ ,  $\frac{2}{3}$ . Explain why the series converges and find the sum to infinity.

$$u_1 = 6, \quad u_2 = 2, \quad u_3 = \frac{2}{3}$$

$$\text{Find the value of } r: \quad r = \frac{u_2}{u_1}$$

$$r = \frac{u_2}{u_1} = \frac{2}{6} = \frac{1}{3}$$

$|r| < 1$  so the series converges

$$\text{Find the sum to infinity: } S_\infty = \frac{u_1}{1-r}$$

$$S_\infty = \frac{u_1}{1-r} = \frac{6}{1-\frac{1}{3}} = \frac{6}{\frac{2}{3}} = 9$$

$$S_\infty = 9$$



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## 1.3.4 Applications of Sequences & Series

### Applications of Arithmetic Sequences & Series

Many real-life situations can be modelled using sequences and series, including but not limited to: patterns made when tiling floors; seating people around a table; the rate of change of a population; the spread of a virus and many more.

#### What do I need to know about applications of arithmetic sequences and series?

- If a quantity is changing repeatedly by having a fixed amount **added to** or **subtracted from** it then the use of **arithmetic sequences** and **arithmetic series** is appropriate to **model** the situation
  - If a sequence seems to fit the pattern of an arithmetic sequence it can be said to be **modelled** by an arithmetic sequence
  - The scenario can be **modelled** using the given information and the formulae from the formula booklet
- A common application of arithmetic sequences and series is **simple interest**
  - Simple interest is when an initial investment is made and then a percentage of the initial investment is added to this amount on a regular basis (usually per year)
- Arithmetic sequences can be used to make estimations about how something will change in the future

#### Examiner Tip

- Exam questions won't always tell you to use sequences and series methods, practice spotting them by looking for clues in the question
- If a given amount is repeated periodically then it is likely the question is on arithmetic sequences or series



Your notes

### Worked example

Jasper is saving for a new car. He puts USD \$100 into his savings account and then each month he puts in USD \$10 more than the month before. Jasper needs USD \$1200 for the car. Assuming no interest is added, find,

- i) the amount Jasper has saved after four months,

Identify the arithmetic sequence :

$$u_1 = 100, \quad d = 10$$

After 4 months Jasper will have saved:

$$u_1 + u_2 + u_3 + u_4 = S_4$$

Formula for the sum of an arithmetic series :

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$

$$S_4 = \frac{4}{2}(2u_1 + (4-1)d)$$

Sub in  $u_1 = 100$  and  $d = 10$

$$S_4 = \frac{4}{2}(2(100) + (4-1)(10))$$

$$= 2(200 + 30)$$

$$= 2(230)$$

$$S_4 = \$460$$

- ii) the month in which Jasper reaches his goal of USD \$1200.



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Sub  $S_n = 1200$ ,  $u_1 = 100$ ,  $d = 10$  into formula:

$$1200 = \frac{n}{2}(2(100) + (n-1)(10))$$

Solve using algebraic solver on GDC:

$$n = 8.67... \text{ or } n = -27.67...$$

↑ disregard as  $n$  cannot be negative.

$$\therefore S_8 < 1200$$

$S_9 > 1200$  reaches total in 9<sup>th</sup> month

Jasper will reach USD \$1200  
in the 9<sup>th</sup> month.

## Applications of Geometric Sequences & Series

### What do I need to know about applications of geometric sequences and series?

- If a quantity is changing repeatedly by a fixed **percentage**, or by being **multiplied** repeatedly by a fixed amount, then the use of **geometric sequences** and **geometric series** is appropriate to **model** the situation
  - If a sequence seems to fit the pattern of a geometric sequence it can be said to be **modelled** by a geometric sequence
  - The scenario can be **modelled** using the given information and the formulae from the formula booklet
- A common application of geometric sequences and series is **compound interest**
  - Compound interest is when an initial investment is made and then interest is paid on the initial amount **and on the interest already earned** on a regular basis (usually every year)
- Geometric sequences can be used to make estimations about how something will change in the future
- The questions won't always tell you to use sequences and series methods, so be prepared to spot 'hidden' sequences and series questions
  - Look out for questions on savings accounts, salaries, sales commissions, profits, population growth and decay, spread of bacteria etc

#### **Examiner Tip**

- Exam questions won't always tell you to use sequences and series methods, practice spotting them by looking for clues in the question
- If a given amount is changing by a percentage or multiple then it is likely the question is on geometric sequences or series



Your notes



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### Worked example

A new virus is circulating on a remote island. On day one there were 10 people infected, with the number of new infections increasing at a rate of 40% per day.

- a) Find the expected number of people newly infected on the 7<sup>th</sup> day.

Identify the geometric sequence:

$$u_1 = 10, \quad r = 1.4$$

↖ 40% increase so 140%  
of the day before

New infections :  $u_7$

Formula for  $n^{\text{th}}$  term of a geometric series :

$$u_n = u_1 r^{n-1}$$

Sub in  $u_1 = 10, \quad r = 1.4$

$$u_7 = 10(1.4)^6 = 75.29\dots$$

Expected number of new infections = 75

- b) Find the expected number of infected people after one week (7 days), assuming no one has recovered yet.





Your notes

Total infections:  $S_7$ 

Formula for the sum of a geometric series:

$$S_n = \frac{u_1(r^n - 1)}{r - 1} \leftarrow r > 1 \text{ so this version is easier to use.}$$

Sub in  $u_1 = 10$ ,  $r = 1.4$ 

$$S_7 = \frac{10(1.4^7 - 1)}{1.4 - 1} = 238.53\dots$$

Expected number of total infections = 239