

DP IB Maths: AI SL



Your notes

2.2 Further Functions & Graphs

Contents

- * 2.2.1 Functions
- * 2.2.2 Graphing Functions
- * 2.2.3 Properties of Graphs



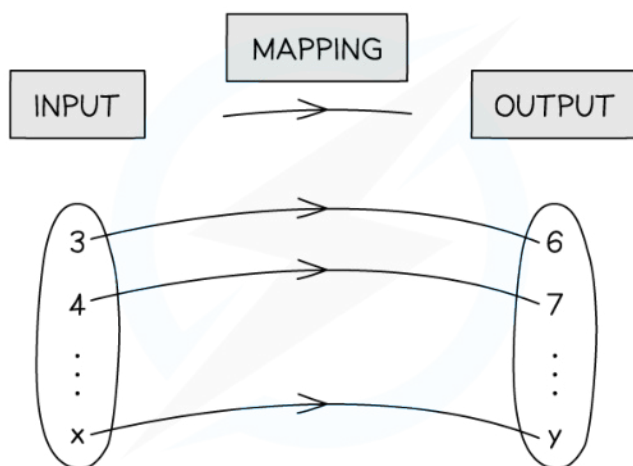
Your notes

2.2.1 Functions

Language of Functions

What is a mapping?

- A **mapping transforms** one set of values (**inputs**) into another set of values (**outputs**)
- Mappings can be:
 - **One-to-one**
 - Each input gets mapped to **exactly one unique** output
 - No two inputs are mapped to the same output
 - For example: A mapping that cubes the input
 - **Many-to-one**
 - Each input gets mapped to **exactly one** output
 - Multiple inputs can be mapped to the same output
 - For example: A mapping that squares the input
 - **One-to-many**
 - An input can be mapped to **more than one** output
 - No two inputs are mapped to the same output
 - For example: A mapping that gives the numbers which when squared equal the input
 - **Many-to-many**
 - An input can be mapped to **more than one** output
 - Multiple inputs can be mapped to the same output
 - For example: A mapping that gives the factors of the input



Copyright © Save My Exams. All Rights Reserved

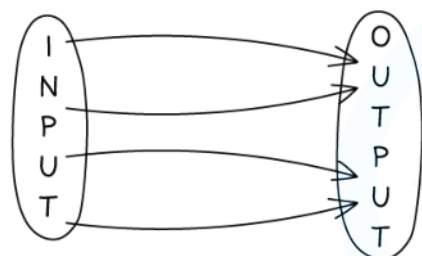


What is a function?

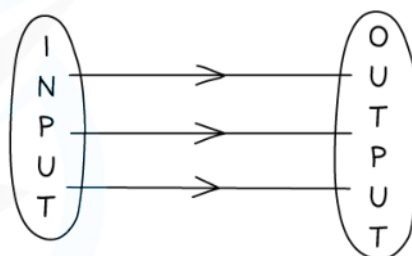


Your notes

- A **function** is a mapping between two sets of numbers where **each input** gets mapped to **exactly one output**
 - The output does not need to be unique
- **One-to-one** and **many-to-one** mappings are functions
- A mapping is a function if its graph passes the **vertical line test**
 - Any **vertical line** will intersect with the graph **at most once**



MANY-TO-ONE MAPPINGS
ARE FUNCTIONS



ONE-TO-ONE MAPPINGS
ARE FUNCTIONS

Copyright © Save My Exams. All Rights Reserved



What notation is used for functions?

- Functions are denoted using letters (such as f , v , g , etc)
 - A function is followed by a variable in a bracket
 - This shows the input for the function
 - The letter f is used most commonly for functions and will be used for the remainder of this revision note
- $f(x)$ represents an expression for the value of the function f when evaluated for the variable x
- Function notation gets rid of the need for words which makes it **universal**
 - $f = 5$ when $x = 2$ can simply be written as $f(2) = 5$

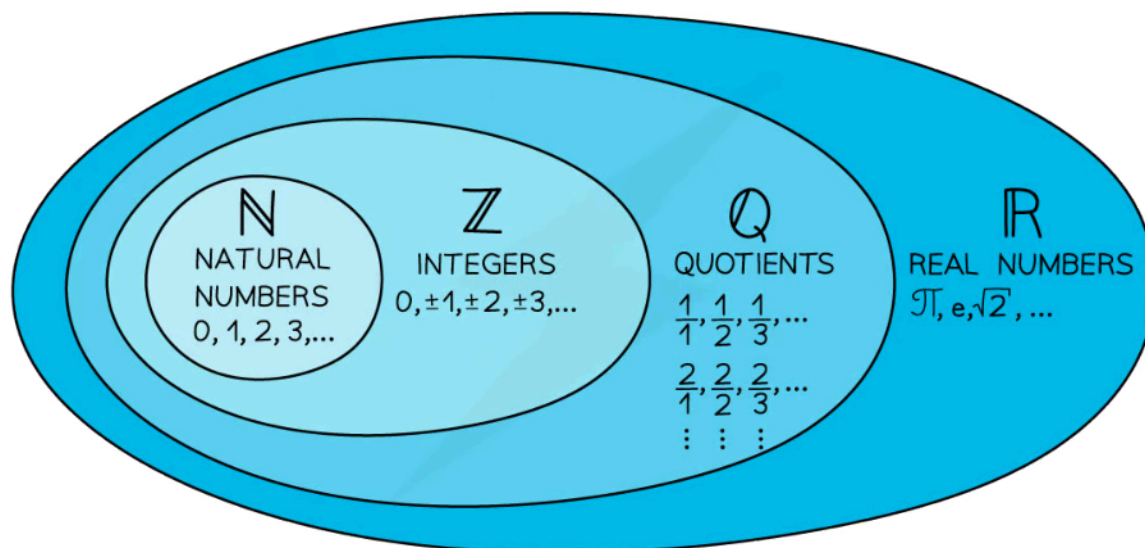
What are the domain and range of a function?

- The **domain** of a function is the set of values that are used as **inputs**
- A domain should be stated with a function
 - If a domain is not stated then it is assumed the domain is all the real values which would work as inputs for the function
 - Domains are expressed in terms of the input
 - $x \leq 2$
- The **range** of a function is the set of values that are given as **outputs**
 - The range depends on the domain
 - Ranges are expressed in terms of the output



Your notes

- $f(x) \geq 0$
- To graph a function we use the **inputs as the x-coordinates** and the **outputs as the y-coordinates**
 - $f(2) = 5$ corresponds to the coordinates (2, 5)
- Graphing the function can help you visualise the range
- Common sets of numbers have special symbols:
 - \mathbb{R} represents all the real numbers that can be placed on a number line
 - $x \in \mathbb{R}$ means x is a real number
 - \mathbb{Q} represents all the rational numbers $\frac{a}{b}$ where a and b are integers and $b \neq 0$
 - \mathbb{Z} represents all the integers (positive, negative and zero)
 - \mathbb{Z}^+ represents positive integers
 - \mathbb{N} represents the natural numbers (0,1,2,3...)



Copyright © Save My Exams. All Rights Reserved

Examiner Tip

- Questions may refer to "the largest possible domain"
 - this would usually be $x \in \mathbb{R}$ unless natural numbers, integers or quotients has already been stated
 - there are usually some exceptions
 - e.g. square roots; $x \geq 0$ for a function involving \sqrt{x}
 - e.g. reciprocal functions; $x \neq 2$ for a function with denominator $(x - 2)$



Your notes

 **Worked example**For the function $f(x) = x^3 + 1$, $2 \leq x \leq 10$:

- a) write down the value of
- $f(7)$
- .

Substitute $x = 7$

$$f(7) = 7^3 + 1$$

$$f(7) = 344$$

- b) find the range of
- $f(x)$
- .

Find the values of $x^3 + 1$ when $2 \leq x \leq 10$

$$2 \leq x \leq 10$$

$$8 \leq x^3 \leq 1000$$

$$9 \leq x^3 + 1 \leq 1001$$

$$9 \leq f(x) \leq 1001$$

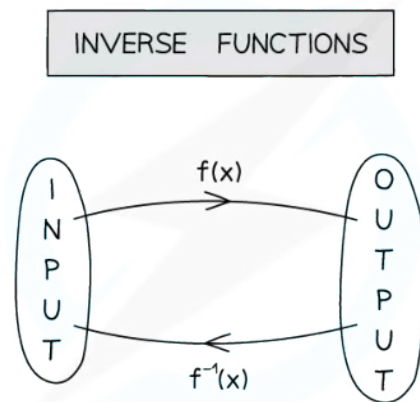


Your notes

Inverse Functions

What is an inverse function?

- Only **one-to-one** functions have inverses
- A function has an inverse if its graph passes the **horizontal line test**
 - Any **horizontal line** will intersect with the graph **at most once**
- Given a function $f(x)$ we denote the inverse function as $f^{-1}(x)$
- An inverse function **reverses the effect** of a function
 - $f(2) = 5$ means $f^{-1}(5) = 2$
- Inverse functions are used to solve equations
 - The solution of $f(x) = 5$ is $x = f^{-1}(5)$



Copyright © Save My Exams. All Rights Reserved

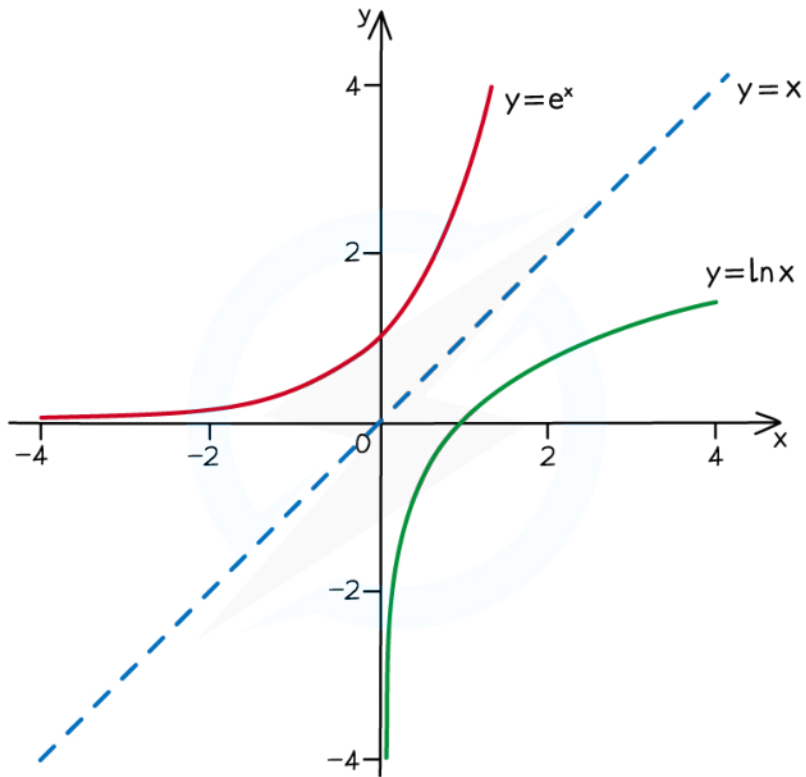


What are the connections between a function and its inverse function?

- The **domain of a function** becomes the **range of its inverse**
- The **range of a function** becomes the **domain of its inverse**
- The graph of $y = f^{-1}(x)$ is a **reflection** of the graph $y = f(x)$ in the line $y = x$
 - Therefore solutions to $f(x) = x$ or $f^{-1}(x) = x$ will also be solutions to $f(x) = f^{-1}(x)$
 - There could be other solutions to $f(x) = f^{-1}(x)$ that don't lie on the line $y = x$



Your notes



Copyright © Save My Exams. All Rights Reserved



Examiner Tip

- Remember that, in general, $f^{-1}(x) \neq \frac{1}{f(x)}$



Your notes

 **Worked example**

For the function $f(x) = x^3 + 1$, $2 \leq x \leq 10$:

- a) write down the range of the inverse function, $f^{-1}(x)$.

The range of $f^{-1}(x)$ is the domain of $f(x)$

$$2 \leq f^{-1}(x) \leq 10$$

- b) find the value of $f^{-1}(217)$.

If $x = f^{-1}(a)$ then $f(x) = a$

$$x = f^{-1}(217)$$

$$f(x) = 217$$

$$x^3 + 1 = 217 \quad \leftarrow \text{Use definition of function}$$

$$x^3 = 216 \quad \leftarrow \text{Subtract 1}$$

$$x = 6 \quad \leftarrow \text{Cube root}$$

$$f^{-1}(217) = 6$$



Your notes

Piecewise Functions

What are piecewise functions?

- **Piecewise functions** are defined by different functions depending on which interval the input is in
 - E.g. $f(x) = \begin{cases} x+1 & x \leq 5 \\ 2x-4 & 5 < x < 10 \end{cases}$
- The intervals for the individual functions **cannot overlap**
- To evaluate a piecewise function for a particular value $x = k$
 - Find which interval includes k
 - Substitute $x = k$ into the corresponding function

Worked example

For the piecewise function

$$f(x) = \begin{cases} 2x - 5 & -10 \leq x \leq 10 \\ 3x + 1 & x > 10 \end{cases}$$

- a) find the values of $f(0)$, $f(10)$, $f(20)$.

Identify the correct function to use

$$x=0 \text{ is in } -10 \leq x \leq 10 \Rightarrow f(0) = 2(0) - 5 = -5$$

$$x=10 \text{ is in } -10 \leq x \leq 10 \Rightarrow f(10) = 2(10) - 5 = 15$$

$$x=20 \text{ is in } x > 10 \Rightarrow f(20) = 3(20) + 1 = 61$$

$$f(0) = -5 \quad f(10) = 15 \quad f(20) = 61$$

- b) state the domain.

Domain is the set of inputs

$$-10 \leq x \leq 10 \text{ and } x > 10$$

$$x \geq -10$$



Your notes

2.2.2 Graphing Functions

Graphing Functions

How do I graph the function $y = f(x)$?

- A point (a, b) lies on the graph $y = f(x)$ if $f(a) = b$
- The **horizontal axis** is used for the **domain**
- The **vertical axis** is used for the **range**
- You will be able to graph some functions by hand
- For some functions you will need to use your GDC
- You might be asked to graph the **sum** or **difference** of two functions
 - Use your GDC to graph $y = f(x) + g(x)$ or $y = f(x) - g(x)$
 - Just type the functions into the graphing mode

What is the difference between “draw” and “sketch”?

- If asked to sketch you should:
 - Show the general shape
 - Label any key points such as the intersections with the axes
 - Label the axes
- If asked to draw you should:
 - Use a pencil and ruler
 - Draw to scale
 - Plot any points **accurately**
 - Join points with a straight line or smooth curve
 - Label any key points such as the intersections with the axes
 - Label the axes

How can my GDC help me sketch/draw a graph?

- You use your GDC to plot the graph
 - Check the scales on the graph to make sure you see the full shape
- Use your GDC to find any key points
- Use your GDC to check specific points to help you plot the graph

Key Features of Graphs

What are the key features of graphs?

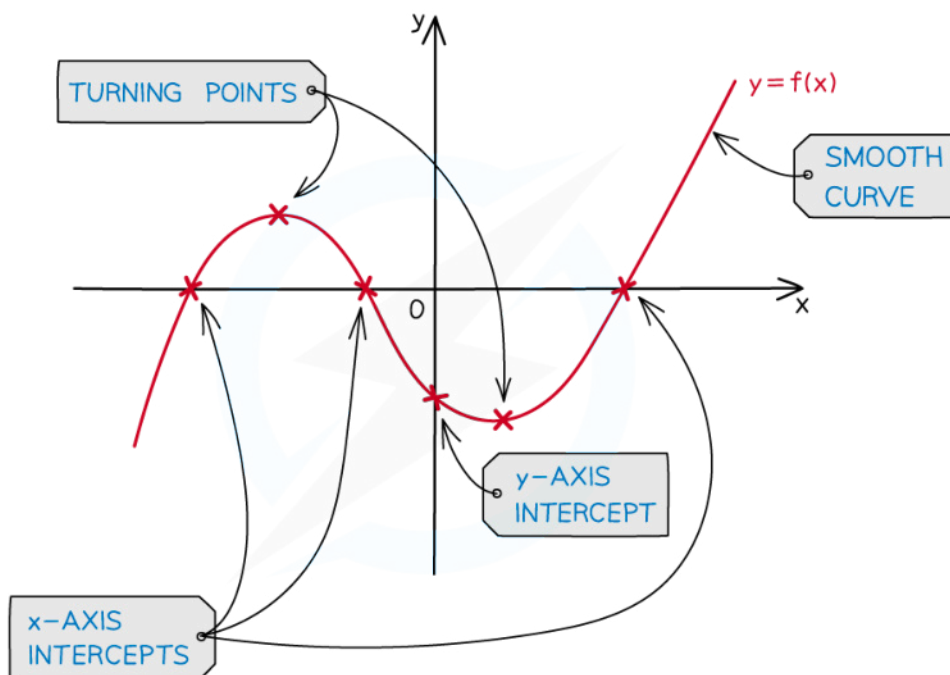
- You should be familiar with the following key features and know how to use your GDC to find them
- Local minimums/maximums
 - These are points where the graph has a minimum/maximum for a small region
 - They are also called **turning points**
 - This is where the graph changes its direction between upwards and downwards directions
 - A graph can have multiple local minimums/maximums
 - A local minimum/maximum is not necessarily the minimum/maximum of the whole graph
 - This would be called the **global** minimum/maximum
 - For quadratic graphs the minimum/maximum is called the **vertex**
- Intercepts
 - y - intercepts are where the graph crosses the y -axis
 - At these points $x = 0$
 - x - intercepts are where the graph crosses the x -axis
 - At these points $y = 0$
 - These points are also called the **zeros of the function** or **roots of the equation**
- Symmetry
 - Some graphs have lines of symmetry
 - A quadratic will have a vertical line of symmetry
- Asymptotes
 - These are lines which the graph will get closer to but not cross
 - These can be horizontal or vertical
 - Exponential graphs have horizontal asymptotes
 - Graphs of variables which vary inversely can have vertical and horizontal asymptotes



Your notes



Your notes



Copyright © Save My Exams. All Rights Reserved



Examiner Tip

- Most GDC makes/models will not plot/show asymptotes just from inputting a function
 - Add the asymptotes as additional graphs for your GDC to plot
 - You can then check the equations of your asymptotes visually
 - You may have to zoom in or change the viewing window options to confirm an asymptote
- Even if using your GDC to plot graphs and solve problems sketching them as part of your working is good exam technique
 - Label the key features of the graph and anything else relevant to the question on your sketch



Your notes

Worked example

Two functions are defined by

$$f(x) = x^2 - 4x - 5 \text{ and } g(x) = 2 + \frac{1}{x+1}.$$

a) Draw the graph $y = f(x)$.

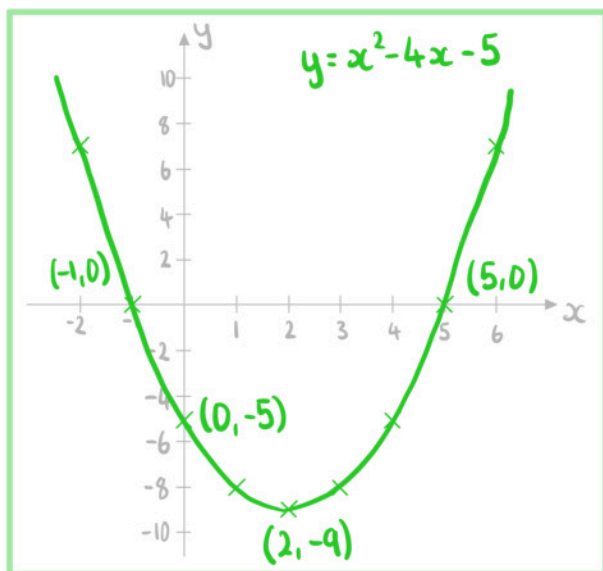
Draw means accurately

Use GDC to find vertex, roots and y-intercepts

Vertex = $(2, -9)$

Roots = $(-1, 0)$ and $(5, 0)$

y-intercept = $(0, -5)$



b) Sketch the graph $y = g(x)$.



Your notes

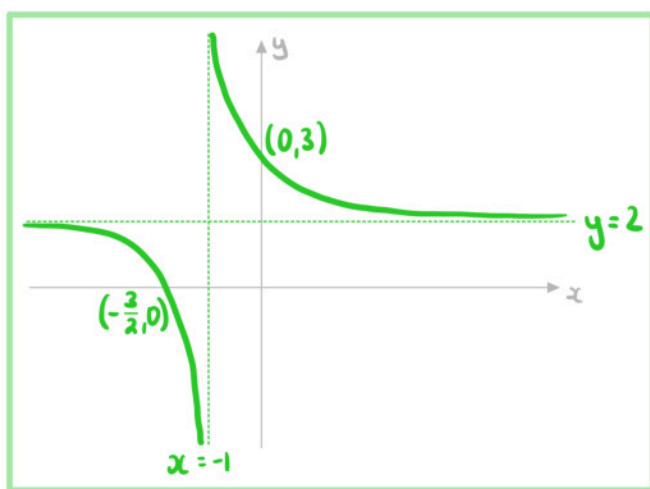
Sketch means rough but showing key points

Use GDC to find x and y -intercepts and asymptotes

$$x\text{-intercept} = \left(-\frac{3}{2}, 0\right)$$

$$y\text{-intercept} = (0, 3)$$

Asymptotes : $x = -1$ and $y = 2$



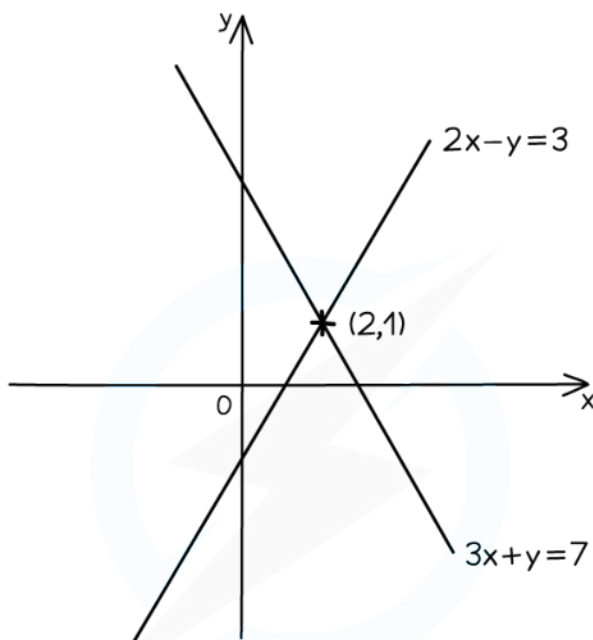


Your notes

Intersecting Graphs

How do I find where two graphs intersect?

- Plot both graphs on your GDC
- Use the intersect function to find the intersections
- Check if there is more than one point of intersection



- LINES INTERSECT AT (2,1)
- SOLVING $2x - y = 3$ AND $3x + y = 7$ SIMULTANEOUSLY IS $x = 2$, $y = 1$

Copyright © Save My Exams. All Rights Reserved



How can I use graphs to solve equations?

- One method to solve equations is to use graphs
- To solve $f(x) = a$
 - Plot the two graphs $y = f(x)$ and $y = a$ on your GDC
 - Find the points of intersections
 - The **x-coordinates** are the **solutions** of the equation
- To solve $f(x) = g(x)$

- Plot the two graphs $y = f(x)$ and $y = g(x)$ on your GDC
- Find the points of intersections
- The **x-coordinates** are the **solutions** of the equation
- Using graphs makes it easier to see **how many solutions** an equation will have

Examiner Tip

- You can use graphs to solve equations
 - Questions will not necessarily ask for a drawing/sketch or make reference to graphs
 - Use your GDC to plot the equations and find the intersections between the graphs



Your notes



Your notes

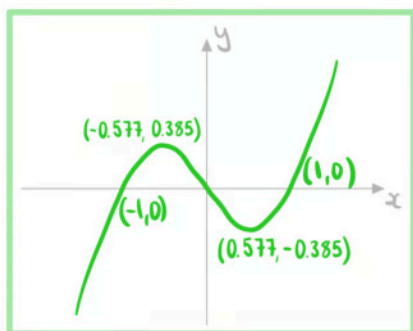
Worked example

Two functions are defined by

$$f(x) = x^3 - x \text{ and } g(x) = \frac{4}{x}$$

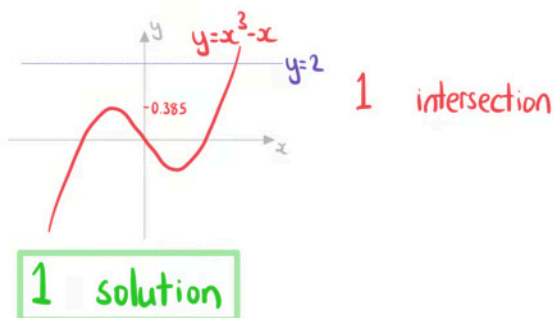
- a) Sketch the graph $y = f(x)$.

Use GDC to find max, min, intercepts



- b) Write down the number of real solutions to the equation $x^3 - x = 2$.

Identify the number of intersections between $y = x^3 - x$ and $y = 2$

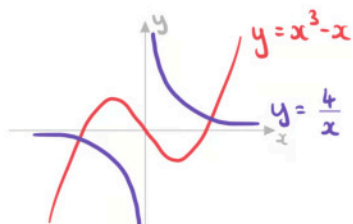


- c) Find the coordinates of the points where $y = f(x)$ and $y = g(x)$ intersect.



Your notes

Use GDC to sketch both graphs



$(-1.60, -2.50)$ and $(1.60, 2.50)$

d)

Write down the solutions to the equation $x^3 - x = \frac{4}{x}$.

Solutions to $x^3 - x = \frac{4}{x}$ are the x coordinates of the points of intersection.

$x = -1.60$ and $x = 1.60$



Your notes

2.2.3 Properties of Graphs

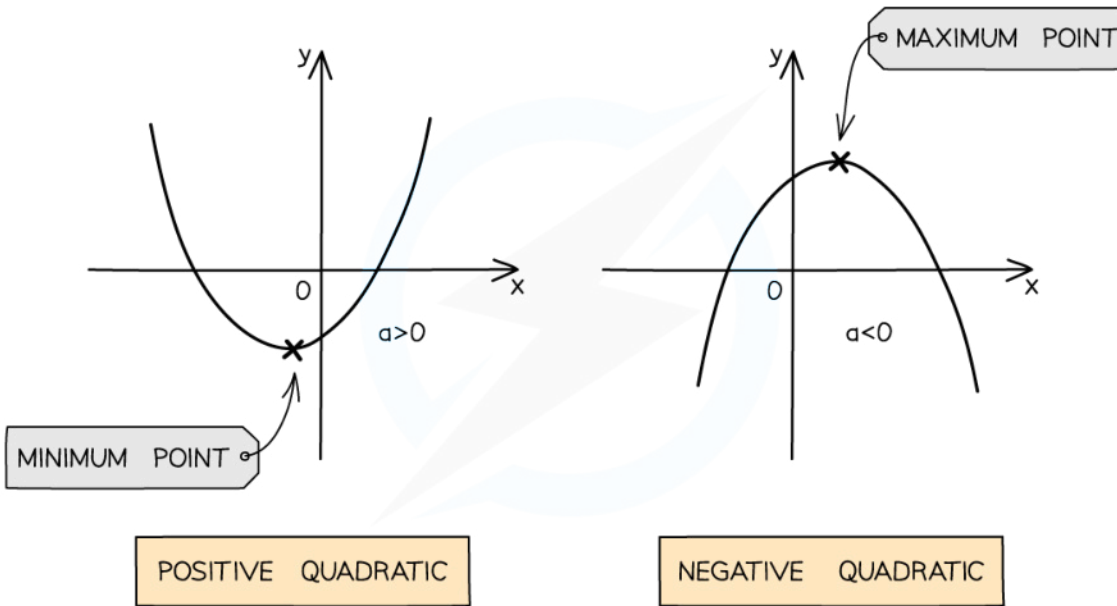
Quadratic Functions & Graphs

What are the key features of quadratic graphs?

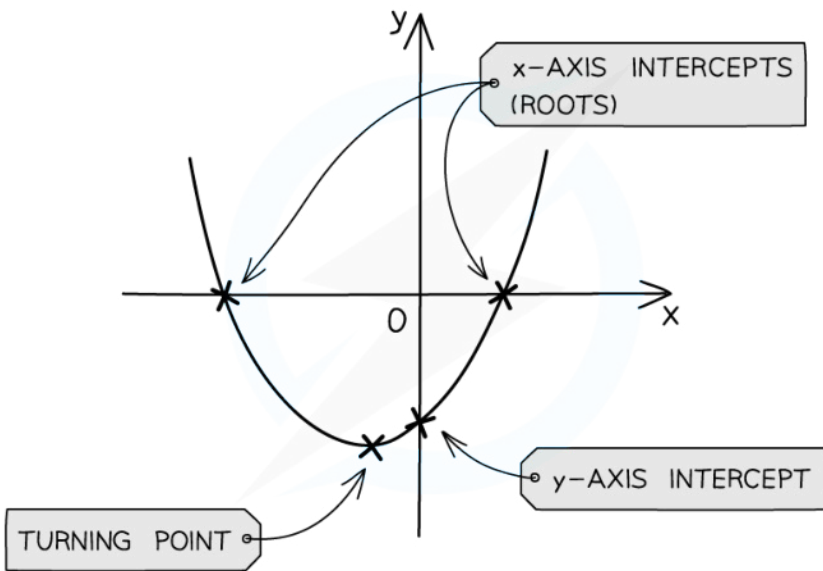
- A **quadratic** graph is of the form $y = ax^2 + bx + c$ where $a \neq 0$.
- The value of a affects the shape of the curve
 - If a is positive the shape is \cup
 - If a is negative the shape is \cap
- The **y-intercept** is at the point $(0, c)$
- The **zeros or roots** are the solutions to $ax^2 + bx + c = 0$
 - These can be found using your GDC or the quadratic formula
 - These are also called the x-intercepts
 - There can be 0, 1 or 2 x-intercepts
- There is an **axis of symmetry** at $x = -\frac{b}{2a}$
 - This is given in your **formula booklet**
 - If there are two x-intercepts then the axis of symmetry goes through the midpoint of them
- The **vertex** lies on the axis of symmetry
 - The x-coordinate is $-\frac{b}{2a}$
 - The y-coordinate can be found using the GDC or by calculating y when $x = -\frac{b}{2a}$
 - If a is **positive** then the vertex is the **minimum** point
 - If a is **negative** then the vertex is the **maximum** point



Your notes



Copyright © Save My Exams. All Rights Reserved



Copyright © Save My Exams. All Rights Reserved





Your notes

Examiner Tip

- Use your GDC to find the roots and the turning point of a quadratic function
 - You do not need to factorise or complete the square
 - It is good exam technique to sketch the graph from your GDC as part of your working

Worked example

- a) Write down the equation of the axis of symmetry for the graph $y = 4x^2 - 4x - 3$.

Formula booklet

Axis of symmetry of the graph of a quadratic function

$$f(x) = ax^2 + bx + c \Rightarrow \text{axis of symmetry is } x = -\frac{b}{2a}$$

$$a = 4 \quad b = -4 \quad c = -3$$

$$x = -\frac{-4}{2(4)}$$

$$x = \frac{1}{2}$$

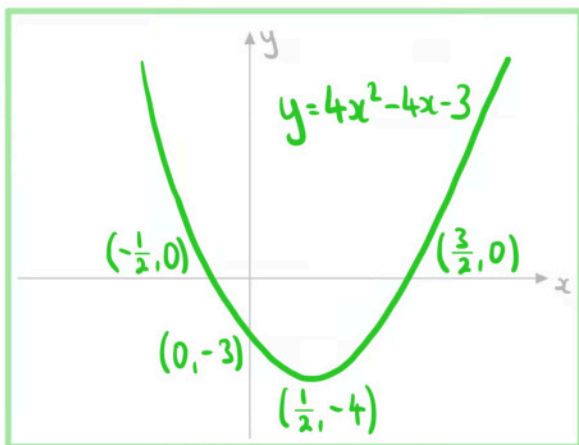
- b) Sketch the graph $y = 4x^2 - 4x - 3$.

Use GDC to find vertex, roots and y-intercepts

$$\text{Vertex} = \left(\frac{1}{2}, -4\right)$$

$$\text{Roots} = \left(-\frac{1}{2}, 0\right) \text{ and } \left(\frac{3}{2}, 0\right)$$

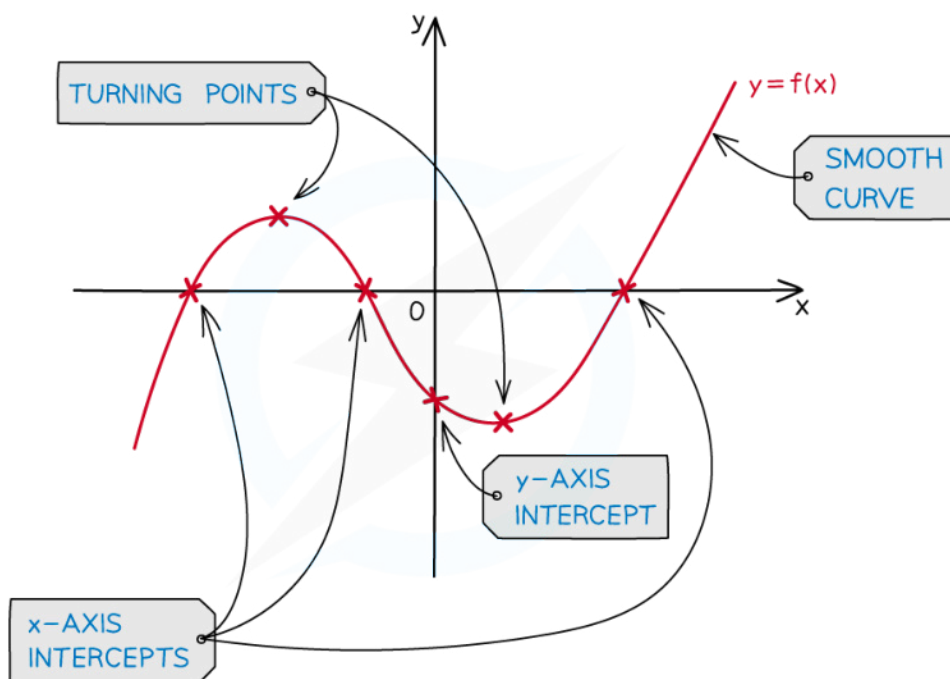
$$\text{y-intercept} = (0, -3)$$



Cubic Functions & Graphs

What are the key features of cubic graphs?

- A **cubic** graph is of the form $y = ax^3 + bx^2 + cx + d$ where $a \neq 0$.
- The value of a affects the shape of the curve
 - If a is **positive** the graph goes from **bottom left to top right**
 - If a is **negative** the graph goes from **top left to bottom right**
- The **y-intercept** is at the point $(0, d)$
- The **zeros or roots** are the solutions to $ax^3 + bx^2 + cx + d = 0$
 - These can be found using your GDC
 - These are also called the x-intercepts
 - There can be 1, 2 or 3 x-intercepts
 - There is always at least 1
- There are either **0 or 2 local minimums/maximums**
 - If there are 0 then the curve is **monotonic** (always increasing or always decreasing)
 - If there are 2 then one is a local minimum and one is a local maximum



Copyright © Save My Exams. All Rights Reserved



Your notes



Your notes

Examiner Tip

- You can use your GDC to find the roots, the local maximum and local minimum of a cubic function
- When drawing/sketching the graph of a cubic function be sure to label all the key features
 - x and y axes intercepts
 - the local maximum point
 - the local minimum point

Worked example

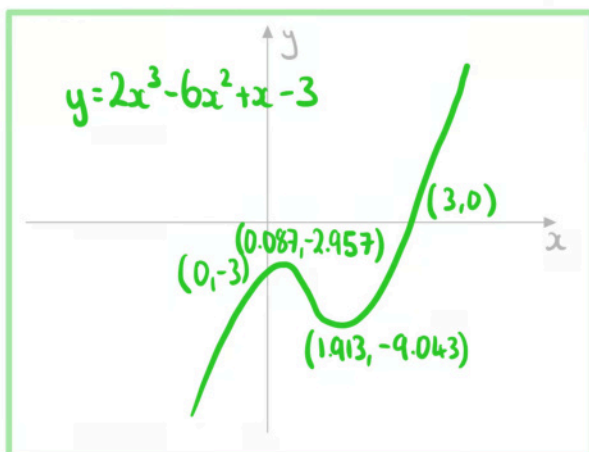
Sketch the graph $y = 2x^3 - 6x^2 + x - 3$.

Use GDC to find min, max, roots and y-intercept

$$\text{Min} = (1.913, -9.043) \quad \text{Max} = (0.087, -2.957)$$

$$\text{Root} = (3, 0)$$

$$\text{y-intercept} = (0, -3)$$





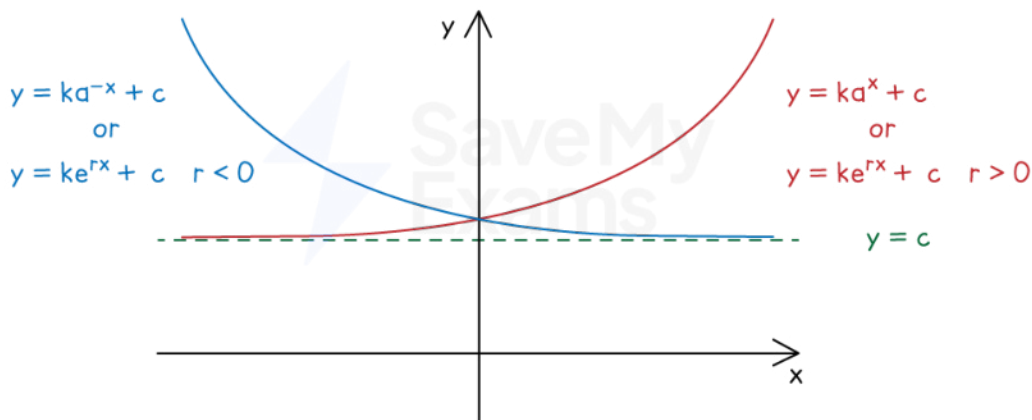
Your notes

Exponential Functions & Graphs

What are the key features of exponential graphs?

- An **exponential** graph is of the form
 - $y = ka^x + c$ or $y = ka^{-x} + c$ where $a > 0$
 - $y = ke^{rx} + c$
 - Where e is the mathematical constant 2.718...
- The **y-intercept** is at the point $(0, k + c)$
- There is a **horizontal asymptote** at $y = c$
- The value of k determines whether the graph is **above or below the asymptote**
 - If **k is positive** the graph is **above the asymptote**
 - So the range is $y > c$
 - If **k is negative** the graph is **below the asymptote**
 - So the range is $y < c$
- The coefficient of x and the constant k determine whether the graph is **increasing or decreasing**
 - If the coefficient of x and k have the **same sign** then **graph is increasing**
 - If the coefficient of x and k have **different signs** then the **graph is decreasing**
- There is at **most 1 root**
 - It can be found using your GDC

IN BOTH CURVES, $k > 0$ AND $a > 1$



Copyright © Save My Exams. All Rights Reserved

Examiner Tip

- You may have to change the viewing window settings on your GDC to make asymptotes clear
 - A small scale can make it look as though the curve and an asymptote intersect
- Be careful about how two exponential graphs drawn on the same axes look
 - Particularly which one is "on top" either side of the y -axis



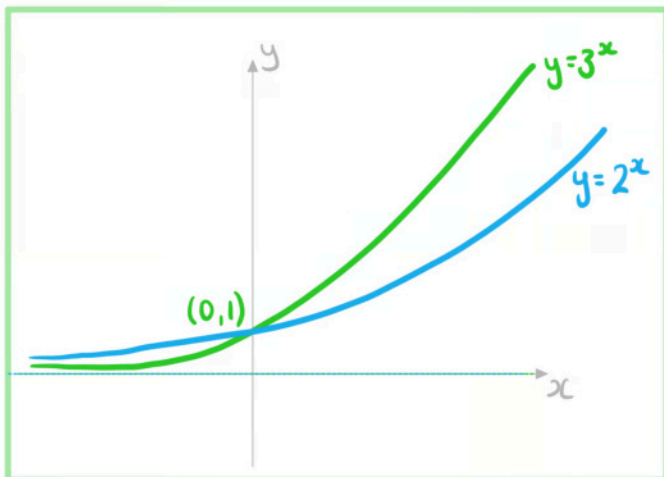
Your notes



Your notes

 **Worked example**

- a) On the same set of axes sketch the graphs $y = 2^x$ and $y = 3^x$. Clearly label each graph.

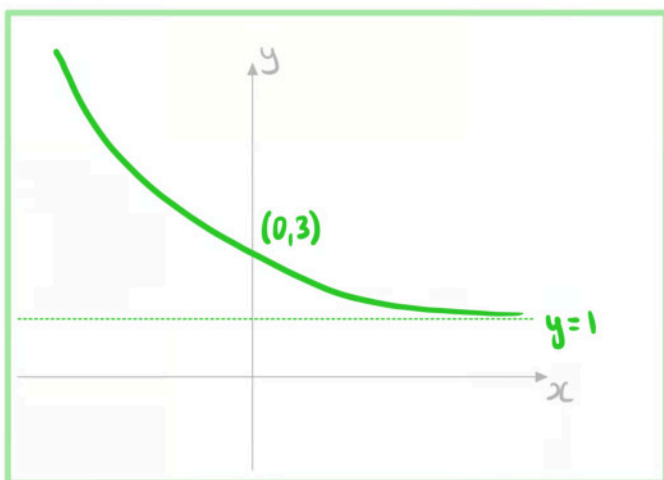


- b) Sketch the graph $y = 2e^{-3x} + 1$.

Use GDC to find intercept and asymptote

y-intercept = (0, 3)

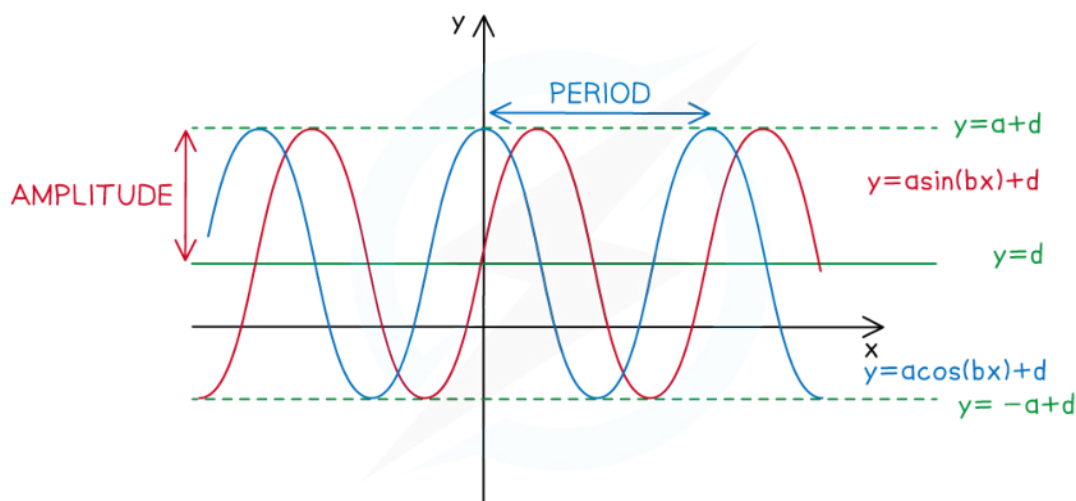
Asymptote: $y = 1$



Sinusoidal Functions & Graphs

What are the key features of sinusoidal graphs?

- A **sinusoidal** graph is of the form
 - $y = a\sin(bx) + d$
 - $y = a\cos(bx) + d$
- The **y-intercept** is at the point
 - $(0, d)$ for $y = a\sin(bx) + d$
 - $(0, a + d)$ for $y = a\cos(bx) + d$
- The **period** of the graph is the length of the interval of a full cycle
 - This is $\frac{360^\circ}{b}$
- The **maximum value** is $y = a + d$
- The **minimum value** is $y = -a + d$
- The **principal axis** is the horizontal line halfway between the maximum and minimum values
 - This is $y = d$
- The **amplitude** is the vertical distance from the principal axis to the maximum value
 - This is a



Copyright © Save My Exams. All Rights Reserved

Examiner Tip

- Pay careful attention to the angles between which you are required to use or draw/sketch a sinusoidal graph
 - e.g. $0^\circ \leq x \leq 360^\circ$



Your notes

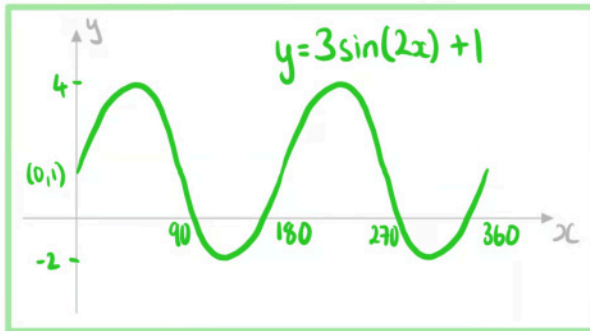


Your notes

Worked example

- a) Sketch the graph $y = 3\sin(2x) + 1$ for the values $0 \leq x \leq 360$.

Use GDC to find max and min



- b) State the equation of the principal axis of the curve.

Principal axis is in middle of maximum and minimum points

$$\frac{4 + (-2)}{2} = 1$$

$$y = 1$$

- c) State the period and amplitude.

Period is how often it repeats

$$\frac{360}{2} = 180$$

$$\text{Period} = 180^\circ$$

Amplitude is distance from principal axis to maximum or minimum

$$4 - 1 = 1 - (-2) = 3$$

$$\text{Amplitude} = 3$$