

DP IB Maths: AA HL



2.9 Further Functions & Graphs

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Your notes

2.9.1 Modulus Functions

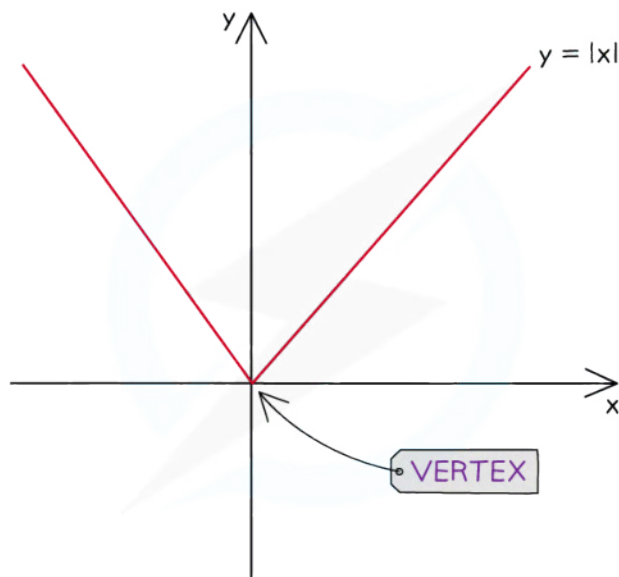
Modulus Functions & Graphs

What is the modulus function?

- The **modulus function** is defined by $f(x) = |x|$
 - $|x| = \sqrt{x^2}$
 - Equivalently it can be defined $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$
- Its **domain** is the set of **all real values**
- Its **range** is the set of **all real non-negative values**
- The modulus function gives the **distance** between 0 and x
 - This is also called the **absolute value** of x

What are the key features of the modulus graph: $y = |x|$?

- The graph has a **y-intercept** at $(0, 0)$
- The graph has **one root** at $(0, 0)$
- The graph has a **vertex** at $(0, 0)$
- The graph is **symmetrical** about the **y-axis**
- At the **origin**
 - The function is **continuous**
 - The function is **not differentiable**



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Your notes

What are the key features of the modulus graph: $y = a|x + p| + q$?

- Every **modulus graph** which is formed by **linear transformations** can be written in this form using key features of the modulus function
 - $|ax| = |a||x|$
 - For example: $|2x + 1| = 2\left|x + \frac{1}{2}\right|$
 - $|p - x| = |x - p|$
 - For example: $|4 - x| = |x - 4|$
- The graph has a **y-intercept** when $x = 0$
- The graph can have 0, 1 or 2 **roots**
 - If a and q have the **same sign** then there will be **0 roots**
 - If $q = 0$ then there will be **1 root** at $(-p, 0)$
 - If a and q have **different signs** then there will be **2 roots** at $\left(-p \pm \frac{q}{a}, 0\right)$
- The graph has a **vertex** at $(-p, q)$
- The graph is **symmetrical** about the line $x = -p$
- The value of a determines the **shape** and the **steepness** of the graph
 - If a is **positive** the graph looks like ∇
 - If a is **negative** the graph looks like \wedge
 - The **larger** the value of $|a|$ the **steeper** the lines
- At the **vertex**
 - The function is **continuous**
 - The function is **not differentiable**



Your notes

2.9.2 Modulus Transformations

Modulus Transformations

How do I sketch the graph of the modulus of a function: $y = |f(x)|$?

- **STEP 1:** Keep the parts of the graph of $y = f(x)$ that are **on or above the x-axis**
- **STEP 2:** Any parts of the **graph below the x-axis** get **reflected** in the x-axis

How do I sketch the graph of a function of a modulus: $y = f(|x|)$?

- **STEP 1:** Keep the graph of $y = f(x)$ **only for $x \geq 0$**
- **STEP 2:** **Reflect** this in the **y-axis**

What is the difference between $y = |f(x)|$ and $y = f(|x|)$?

- The graph of $y = |f(x)|$ **never goes below the x-axis**
 - It does not have to have any lines of symmetry
- The graph of $y = f(|x|)$ is **always symmetrical about the x-axis**
 - It can go below the x-axis

When multiple transformations are involved how do I determine the order?

- The transformations **outside the function** follow the **same order** as the **order of operations**
 - $y = |af(x) + b|$
 - Deal with the a then the b then the modulus
 - $y = a|f(x)| + b$
 - Deal with the modulus then the a then the b
- The transformations **inside the function** are in the **reverse order** to the **order of operations**
 - $y = f(|ax + b|)$
 - Deal with the modulus then the b then the a
 - $y = f(a|x| + b)$
 - Deal with the b then the a then the modulus

Examiner Tip

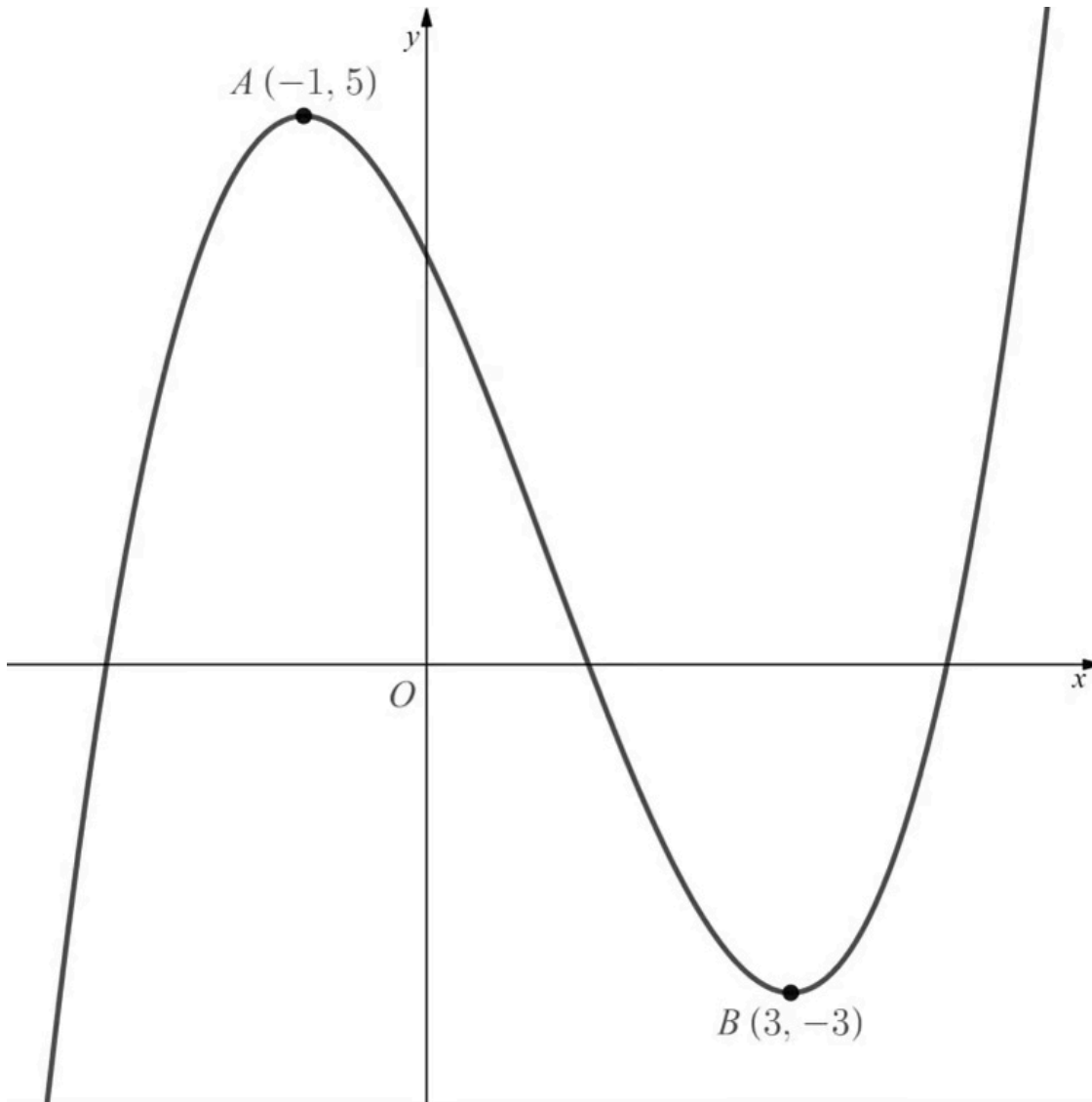
- When sketching one of these transformations in an exam question make sure that the graphs do not look smooth at the points where the original graph have been reflected
 - For $y = |f(x)|$ the graph should look "sharp" at the points where it has been reflected on the x-axis
 - For $y = f(|x|)$ the graph should look "sharp" at the point where it has been reflected on the y-axis



Your notes

 **Worked example**

The diagram below shows the graph of $y = f(x)$.



(a) Sketch the graph of $y = |f(x)|$.



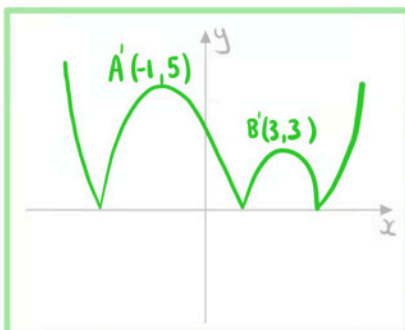
Your notes

If the graph is on or above the x -axis then it stays the same

If the graph is below the x -axis then it is reflected in the x -axis

A stays the same $(-1, 5)$

B becomes $(3, 3)$



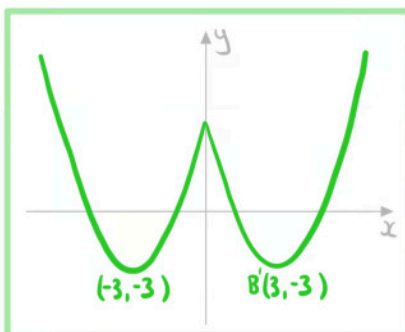
(b) Sketch the graph of $y = f(|x|)$.

keep the graph for $x \geq 0$

Reflect this in the y -axis

A disappears

B stays the same $(3, -3)$





Your notes

2.9.3 Modulus Equations & Inequalities

Modulus Equations

How do I find the modulus of a function?

- The **modulus of a function** $f(x)$ is
 - $|f(x)| = \begin{cases} f(x) & f(x) \geq 0 \\ -f(x) & f(x) < 0 \end{cases}$ or
 - $|f(x)| = \sqrt{[f(x)]^2}$

How do I solve modulus equations graphically?

- To solve $|f(x)| = g(x)$ graphically
 - Draw $y = |f(x)|$ and $y = g(x)$ into your GDC
 - Find the x -coordinates of the **points of intersection**

How do I solve modulus equations analytically?

- To solve $|f(x)| = g(x)$ analytically
 - Form **two equations**
 - $f(x) = g(x)$
 - $f(x) = -g(x)$
 - Solve both equations
 - **Check solutions** work in the original equation
 - For example: $x - 2 = 2x - 3$ has solution $x = 1$
 - But $|(1) - 2| = 1$ and $2(1) - 3 = -1$
 - So $x = 1$ is not a solution to $|x - 2| = 2x - 3$



Your notes

Worked example

 Solve for x :

$$\text{a) } \left| \frac{2x+3}{2-x} \right| = 5$$

Analytically
Split into two equations

$$\frac{2x+3}{2-x} = \pm 5$$

Solve individually

$$\frac{2x+3}{2-x} = 5$$

$$2x+3 = 10-5x$$

$$7x = 7$$

$$x = 1$$

$$\frac{2x+3}{2-x} = -5$$

$$2x+3 = 5x-10$$

$$13 = 3x$$

$$x = \frac{13}{3}$$

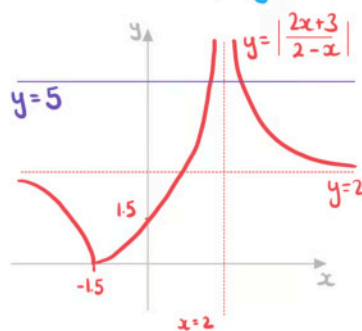
Check:

$$\left| \frac{2(1)+3}{2-(1)} \right| = 5 \checkmark$$

$$\left| \frac{2(\frac{13}{3})+3}{2-(\frac{13}{3})} \right| = 5 \checkmark$$

$$x = 1 \text{ or } x = \frac{13}{3}$$

Graphically
Sketch the two graphs



Find the points of intersection

$$(1, 5) \quad (4.33, 5)$$

Choose the x -coordinates

$$x = 1 \text{ or } x = 4.33 \text{ (3sf)}$$

$$\text{b) } |3x-1| = 5x-11.$$



Your notes

Analytically

Split into two equations

$$3x - 1 = \pm(5x - 11)$$

Solve individually

$$3x - 1 = 5x - 11$$

$$10 = 2x$$

$$x = 5$$

$$3x - 1 = 11 - 5x$$

$$8x = 12$$

$$x = 1.5$$

Check:

$$|3(5) - 1| = 14 \quad \checkmark$$

$$5(5) - 11 = 14 \quad \checkmark$$

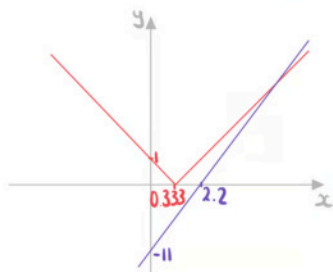
$$|3(1.5) - 1| = 3.5$$

$$5(1.5) - 11 = -3.5 \quad \times$$

$$x = 5$$

Graphically

Sketch the two graphs



Find the points of intersection

$$(5, 14)$$

Choose the x-coordinates

$$x = 5$$

Modulus Inequalities

How do I solve modulus inequalities analytically?

- To solve **any** modulus inequality
 - First solve the corresponding modulus equation
 - Remembering to **check whether solutions are valid**
 - Then use a graphical method or a sign table to find the intervals that satisfy the inequality
- Another method is to solve **two pairs of inequalities**
 - For $|f(x)| < g(x)$ solve:
 - $f(x) < g(x)$ when $f(x) \geq 0$
 - $f(x) > -g(x)$ when $f(x) \leq 0$
 - For $|f(x)| > g(x)$ solve:
 - $f(x) > g(x)$ when $f(x) \geq 0$
 - $f(x) < -g(x)$ when $f(x) \leq 0$

Examiner Tip

- If a question on this appears on a calculator paper then use the same ideas as solving other inequalities
 - Sketch the graphs and find the intersections



Your notes



Your notes

 **Worked example**

Solve the following inequalities for x .

a) $|2x - 1| < 4$

Solve for $2x - 1 \geq 0$

For $x \geq \frac{1}{2}$: $2x - 1 < 4 \Rightarrow x < \frac{5}{2} \quad \therefore \frac{1}{2} \leq x < \frac{5}{2}$

Solve for $2x - 1 < 0$

For $x < \frac{1}{2}$: $2x - 1 > -4 \Rightarrow x > -\frac{3}{2} \quad \therefore -\frac{3}{2} < x < \frac{1}{2}$

Combine inequalities $\boxed{-\frac{3}{2} < x < \frac{5}{2}}$

b) $|x + 1| < |2x + 3|$

Solve the corresponding equation

$|x + 1| = |2x + 3| \Rightarrow x + 1 = \pm(2x + 3)$

Solve	$x + 1 = 2x + 3$	$x + 1 = -2x - 3$
	$x = -2$	$x = -\frac{4}{3}$

Check	$ (-2) + 1 = 1$	$ (-\frac{4}{3}) + 1 = \frac{1}{3}$
	$ 2(-2) + 3 = 1 \quad \checkmark$	$ 2(-\frac{4}{3}) + 3 = \frac{1}{3} \quad \checkmark$

Use a sign table

Check $x = -3$	Check $x = 1.5$	Check $x = 0$
$ (-3) + 1 < 2(-3) + 3 $	$ 1.5 + 1 < 2(1.5) + 3 $	$ 0 + 1 < 2(0) + 3 $
$2 < 3$	$0.5 < 0$	$1 < 3$
True	False	True
\checkmark	\times	\checkmark

$\xrightarrow{\hspace{10em}}$
 $\quad \quad \quad -2 \quad \quad \quad -\frac{4}{3}$

Write solution $\boxed{x < -2 \text{ or } x > -\frac{4}{3}}$



Your notes

2.9.4 Reciprocal & Square Transformations

Reciprocal Transformations

What effects do reciprocal transformations have on the graphs?

- The **x-coordinates stay the same**
- The **y-coordinates change**
 - Their values become their **reciprocals**
- The coordinates (x, y) become $\left(x, \frac{1}{y}\right)$ where $y \neq 0$
 - If $y = 0$ then a vertical asymptote goes through the original coordinate
 - Points that lie on the line **$y = 1$** or the line **$y = -1$** stay the same

How do I sketch the graph of the reciprocal of a function: $y = 1/f(x)$?

- Sketch the **reciprocal transformation** by considering the **different features** of the original graph
- Consider key points on the original graph
 - If (x_1, y_1) is a point on $y = f(x)$ where $y_1 \neq 0$
 - $\left(x_1, \frac{1}{y_1}\right)$ is a point on $y = \frac{1}{f(x)}$
 - If $|y_1| < 1$ then the point gets **further away from the x-axis**
 - If $|y_1| > 1$ then the point gets **closer to the x-axis**
 - If $y = f(x)$ has a **y-intercept** at $(0, c)$ where $c \neq 0$
 - The reciprocal graph $y = \frac{1}{f(x)}$ has a **y-intercept** at $\left(0, \frac{1}{c}\right)$
 - If $y = f(x)$ has a **root** at $(a, 0)$
 - The reciprocal graph $y = \frac{1}{f(x)}$ has a **vertical asymptote** at $x = a$
 - If $y = f(x)$ has a **vertical asymptote** at $x = a$
 - The reciprocal graph $y = \frac{1}{f(x)}$ has a **discontinuity** at $(a, 0)$
 - The **discontinuity** will look like a **root**



Your notes

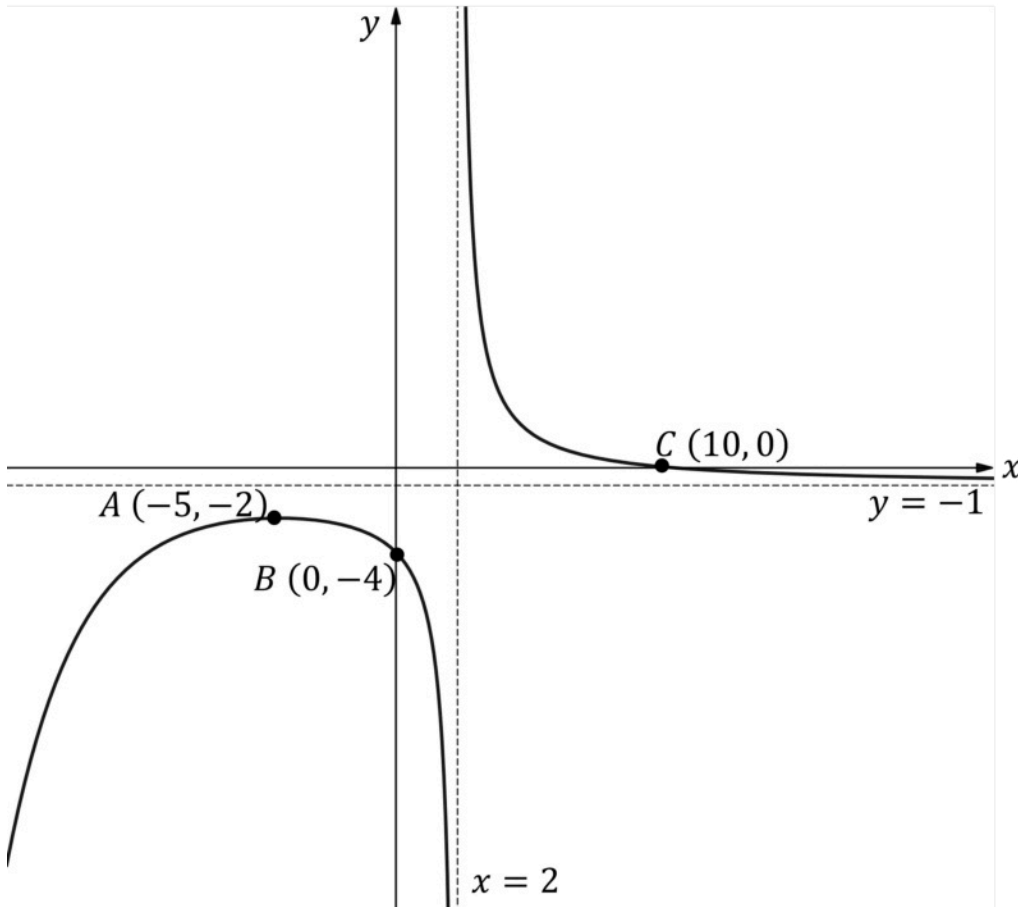
- If $y = f(x)$ has a **local maximum** at (x_1, y_1) where $y_1 \neq 0$
 - The reciprocal graph $y = \frac{1}{f(x)}$ has a **local minimum** at $\left(x_1, \frac{1}{y_1}\right)$
- If $y = f(x)$ has a **local minimum** at (x_1, y_1) where $y_1 \neq 0$
 - The reciprocal graph $y = \frac{1}{f(x)}$ has a **local maximum** at $\left(x_1, \frac{1}{y_1}\right)$
- Consider key regions on the original graph
 - If $y = f(x)$ is **positive** then $y = \frac{1}{f(x)}$ is **positive**
 - If $y = f(x)$ is **negative** then $y = \frac{1}{f(x)}$ is **negative**
 - If $y = f(x)$ is **increasing** then $y = \frac{1}{f(x)}$ is **decreasing**
 - If $y = f(x)$ is **decreasing** then $y = \frac{1}{f(x)}$ is **increasing**
 - If $y = f(x)$ has a **horizontal asymptote** at $y = k$
 - $y = \frac{1}{f(x)}$ has a **horizontal asymptote** at $y = \frac{1}{k}$ if $k \neq 0$
 - $y = \frac{1}{f(x)}$ **tends to $\pm \infty$** if $k = 0$
 - If $y = f(x)$ **tends to $\pm \infty$** as x tends to $+\infty$ or $-\infty$
 - $y = \frac{1}{f(x)}$ has a **horizontal asymptote** at $y = 0$



Your notes

Worked example

The diagram below shows the graph of $y = f(x)$ which has a local maximum at the point A.

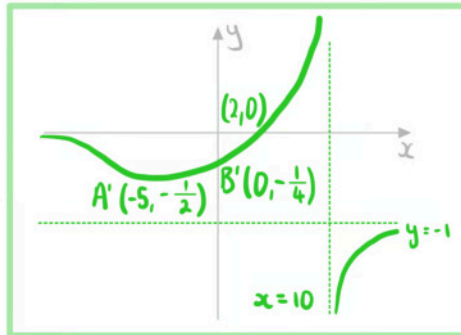


Sketch the graph of $y = \frac{1}{f(x)}$.



Your notes

A becomes local minimum $(-5, -\frac{1}{2})$
 Vertical asymptote becomes root $(2, 0)$
 B becomes $(0, -\frac{1}{4})$
 C becomes vertical asymptote $x=10$
 Horizontal asymptote $y=-1$ remains





Your notes

Square Transformations

What effects do square transformations have on the graphs?

- The effects are **similar to** the transformation $y = |f(x)|$
 - The parts **below the x-axis are reflected**
 - The **vertical distance** between a point and the x-axis is **squared**
 - This has the effect of **smoothing the curve** at the x-axis
- $y = [f(x)]^2$ is **never below the x-axis**
- The **x-coordinates stay the same**
- The **y-coordinates change**
 - Their values are **squared**
- The coordinates (x, y) become (x, y^2)
 - Points that lie on the **x-axis** or the line **$y = 1$** stay the same

How do I sketch the graph of the square of a function: $y = [f(x)]^2$?

- Sketch the **square transformation** by considering the **different features** of the original graph
- Consider key points on the original graph
 - If (x_1, y_1) is a point on $y = f(x)$
 - (x_1, y_1^2) is a point on $y = [f(x)]^2$
 - If $|y_1| < 1$ then the point gets **closer to the x-axis**
 - If $|y_1| > 1$ then the point gets **further away from the x-axis**
 - If $y = f(x)$ has a **y-intercept** at $(0, c)$
 - The square graph $y = [f(x)]^2$ has a **y-intercept** at $(0, c^2)$
 - If $y = f(x)$ has a **root** at $(a, 0)$
 - The square graph $y = [f(x)]^2$ has a **root and turning point** at $(a, 0)$
 - If $y = f(x)$ has a **vertical asymptote** at $X = a$
 - The square graph $y = [f(x)]^2$ has a **vertical asymptote** at $X = a$
 - If $y = f(x)$ has a **local maximum** at (x_1, y_1)
 - The square graph $y = [f(x)]^2$ has a **local maximum** at (x_1, y_1^2) if $y_1 > 0$
 - The square graph $y = [f(x)]^2$ has a **local minimum** at (x_1, y_1^2) if $y_1 < 0$
 - If $y = f(x)$ has a **local minimum** at (x_1, y_1)
 - The square graph $y = [f(x)]^2$ has a **local minimum** at (x_1, y_1^2) if $y_1 > 0$
 - The square graph $y = [f(x)]^2$ has a **local maximum** at (x_1, y_1^2) if $y_1 < 0$

Examiner Tip

- In an exam question when sketching $y = [f(x)]^2$ make it clear that the points where the new graph touches the x-axis are smooth
 - This will make it clear to the examiner that you understand the difference between the roots of the graphs $y = |f(x)|$ and $y = [f(x)]^2$



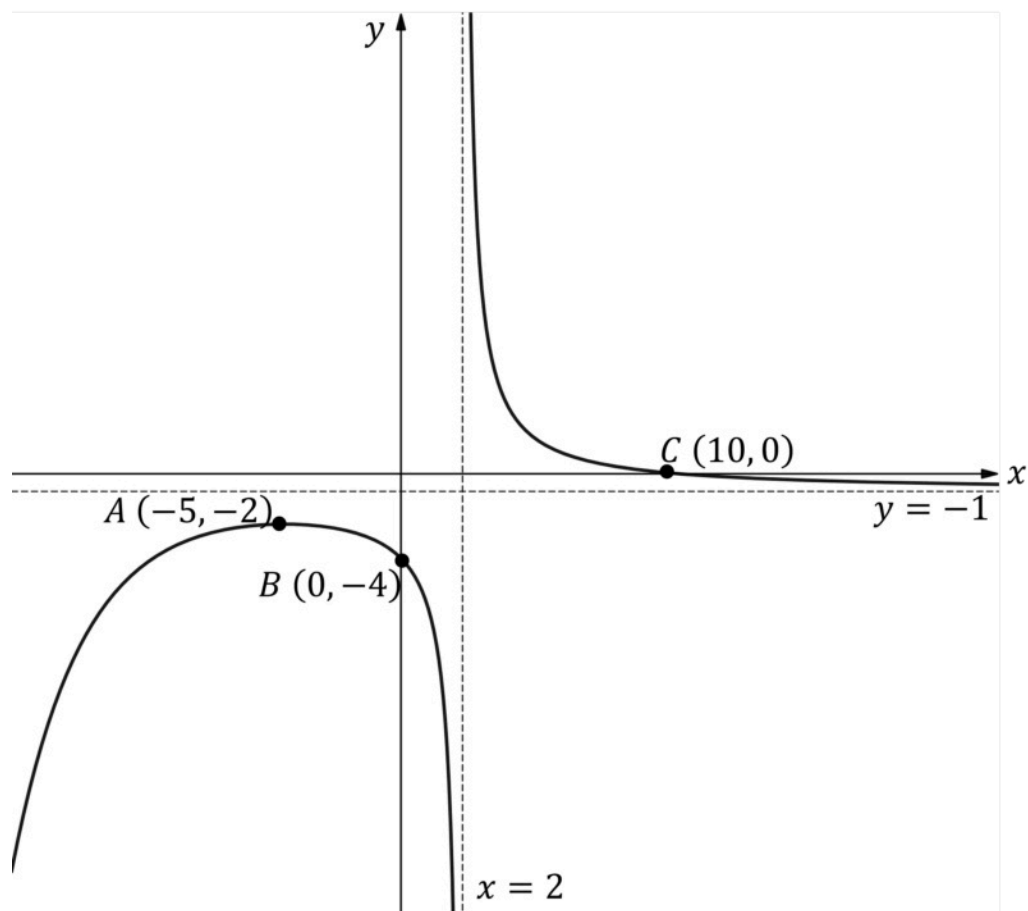
Your notes



Your notes

Worked example

The diagram below shows the graph of $y = f(x)$ which has a local maximum at the point A.



Sketch the graph of $y = [f(x)]^2$.

- A becomes local minimum $(-5, 4)$
- Vertical asymptote $x = 2$ remains
- B becomes $(0, 16)$
- C becomes local minimum
- Horizontal asymptote becomes $y = 1$

