DP IB Maths: AI HL

Your notes

5.5 Kinematics

Contents

5.5.1 Kinematics Toolkit

Displacement, Velocity & Acceleration

What is kinematics?

- **Kinematics** is the branch of mathematics that models and analyses the **motion** of objects
- Common words such as **distance, speed** and **acceleration** are used in kinematics but are used according to their technical definition

What definitions do I need to be aware of?

- Firstly, only motion of an object in a **straight line** is considered
  - this could be a **horizontal** straight line
    - the **positive** direction would be to the **right**
  - or this could be a **vertical** straight line
    - the **positive** direction would be **upwards**

Particle

- A **particle** is the general term for an **object**
  - some questions may use a **specific** object such as a **car** or a **ball**

Time t seconds

- **Displacement**, **velocity** and **acceleration** are all **functions** of **time** t
- **Initially** time is zero $t=0$

Displacement s m

- The **displacement** of a particle is its **distance relative** to a **fixed point**
  - the fixed point is often (but not always) the particle's **initial position**
- **Displacement** will be **zero** $s=0$ if the object is at or has returned to its initial position
- **Displacement** will be negative if its **position relative** to the **fixed point** is in the **negative direction** (left or down)

Distance d m

- Use of the word **distance** needs to be considered carefully and could refer to
  - the distance **travelled** by a particle
  - the **(straight line)** distance the particle is from a **particular point**
- Be careful not to confuse **displacement** with **distance**
  - if a bus route starts and ends at a bus depot, when the bus has returned to the depot, its **displacement** will be **zero** but the distance the bus has travelled will be the length of the route
- **Distance** is always **positive**

Velocity v m s^{-1}

- The **velocity** of a particle is the **rate of change** of its **displacement** at time t

- **Velocity** will be **negative** if the **particle** is moving in the **negative direction**
- A **velocity** of **zero** means the particle is **stationary** $v = 0$

Speed $|v|$ m s^{-1}

- **Speed** is the **magnitude** (a.k.a. absolute value or modulus) of **velocity**
  - as the particle is **moving** in a **straight line, speed** is the **velocity ignoring** the **direction**
    - if $v = 4, \ |v| = 4$
    - if $v = -6, \ |v| = 6$

Acceleration a m s^{-2}

- The **acceleration** of a particle is the **rate of change** of its **velocity** at time t
- Acceleration can be **negative** but this alone cannot fully describe the particle's motion
  - if **velocity** and **acceleration** have the **same** sign the particle is **accelerating** (speeding up)
  - if **velocity** and **acceleration** have **different** signs then the particle is **decelerating** (slowing down)
  - if **acceleration** is **zero** $a = 0$ the particle is moving with **constant** velocity
  - in all cases the **direction** of **motion** is determined by the **sign** of **velocity**

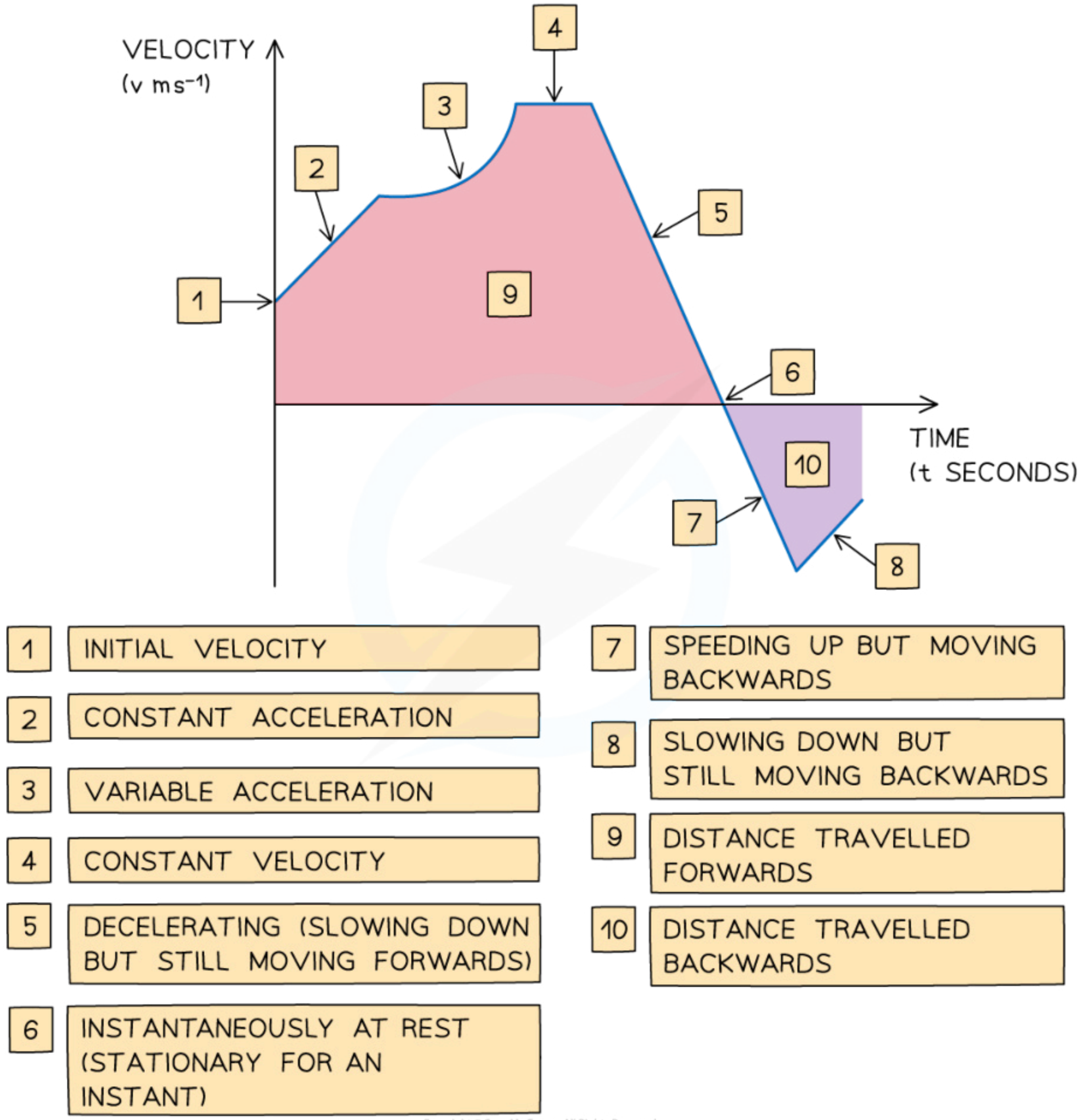
Are there any other words or phrases in kinematics I should know?

- Certain words and phrases can imply values or directions in kinematics
  - a particle described as "at **rest**" means that its velocity is zero, $\boldsymbol{v = 0}$
  - a particle described as moving "**due east**" or "**right**" or would be moving in the **positive horizontal** direction
    - this also means that $\boldsymbol{v > 0}$
  - a particle "**dropped from the top of a cliff**" or "**down**" would be moving in the **negative vertical** direction
    - this also means that $\boldsymbol{v < 0}$

What are the key features of a velocity-time graph?

- The **gradient** of the graph equals the **acceleration** of an object
- A **straight line** shows that the object is **accelerating** at a **constant rate**
- A **horizontal** line shows that the object is moving at a **constant velocity**
- The **area** between graph and the x-axis tells us the **change in displacement** of the object
  - Graph **above** the x-axis means the object is moving **forwards**
  - Graph **below** the x-axis means the object is moving **backwards**

- The **total displacement** of the object from its starting point is the sum of the **areas above** the x-axis **minus** the sum of the **areas below** the x-axis
- The **total distance travelled** by the object is the sum of **all** the **areas**
- If the graph **touches** the **x-axis** then the object is **stationary** at that time
- If the graph is **above** the **x-axis** then the object has positive velocity and is **travelling forwards**
- If the graph is **below** the **x-axis** then the object has negative velocity and is **travelling backwards**



Examiner Tip

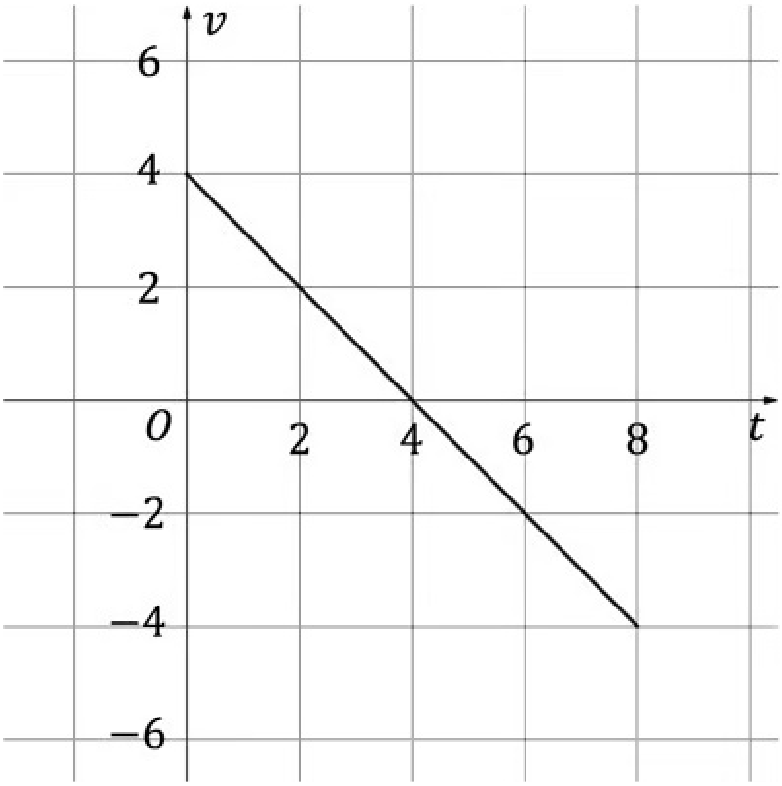
- In an exam if you are given an expression for the velocity then sketching a velocity-time graph can help visualise the problem

Your notes

Worked example

A particle is projected vertically upwards from ground level, taking 8 seconds to return to the ground.

The velocity-time graph below illustrates the motion of the particle for these 8 seconds.



i) How many seconds does the particle take to reach its maximum height?
Give a reason for your answer.

ii) State, with a reason, whether the particle is accelerating or decelerating at time $t = 3$.

i. At maximum height, velocity is zero

$v = 0$ at $t = 4$

$\therefore$ The particle takes 4 seconds to reach its maximum height. This is because its velocity is 0 m s^{-1} at 4 seconds.

ii. At $t=3$, velocity is POSITIVE

Acceleration is the gradient of velocity

At $t=3$, acceleration is NEGATIVE

$\therefore$ At 3 seconds the particle is decelerating as its velocity and acceleration have different signs.

5.5.2 Calculus for Kinematics

Differentiation for Kinematics

How is differentiation used in kinematics?

- **Displacement**, **velocity** and **acceleration** are related by calculus
- In terms of differentiation and derivatives
  - **velocity** is the **rate** of **change** of **displacement**
    - $v = \frac{\mathrm{d}s}{\mathrm{d}t}$ or $v(t) = s'(t)$
  - **acceleration** is the **rate** of **change** of **velocity**
    - $a = \frac{\mathrm{d}v}{\mathrm{d}t}$ or $a(t) = v'(t)$
  - so **acceleration** is also the **second derivative** of **displacement**
    - $a = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$ or $a(t) = s''(t)$
  - Sometimes **velocity** may be a **function** of **displacement** rather than time
    - $v(s)$ rather than $v(t)$
    - in such circumstances, **acceleration** is $a = v\frac{\mathrm{d}v}{\mathrm{d}s}$
    - this result is derived from the **chain rule**
  - All acceleration formulae are given in the **formula booklet**
- Even if a motion graph is given, if possible, use your GDC to draw one
  - you can then use your GDC's graphing features to find **gradients**
    - **velocity** is the **gradient** on a **displacement** (-time) graph
    - **acceleration** is the **gradient** on a **velocity** (-time) graph
- **Dot notation** is often used to indicate time derivatives
  - x is sometimes used as displacement (rather than s) in such circumstances
  - $\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t}$, so $\dot{x}$ is **velocity**
  - $\overset{''}{x} = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}$, so $\overset{''}{x}$ is **acceleration**

Worked example

a) The displacement, x m, of a particle at t seconds, is modelled by the function $x(t) = 2t^3 - 27t^2 + 84t$.

Find expressions for $\dot{x}$ and $\ddot{x}$.

$$x = 2t^3 - 27t^2 + 84t$$

$$\dot{x} = \frac{dx}{dt} \qquad \therefore \dot{x} = 6t^2 - 54t + 84$$

$$\dot{x} = 6(t^2 - 9t + 14)$$

$$\boxed{\dot{x} = 6(t-2)(t-7)}$$

It is not essential to factorise answers

$$\ddot{x} = \frac{d^2x}{dt^2} \qquad \therefore \ddot{x} = 12t - 54$$

$$\boxed{\ddot{x} = 6(2t-9)}$$

b) The velocity, v m s^{-1}, of a particle is given as $v(s) = 6s - 5s^2 - 4$, where s m is the displacement of the particle.

Find an expression, in terms of s, for the acceleration of the particle.

$$v = 6s - 5s^2 - 4$$

$$a = v\frac{dv}{ds} \qquad \therefore a = \underset{\overset{\uparrow}{v(s)}}{(6s - 5s^2 - 4)}\underset{\overset{\uparrow}{\frac{dv}{ds}}}{(6 - 10s)}$$

$$\boxed{a = 2(3-5s)(6s-5s^2-4)}$$

Integration for Kinematics

How is integration used in kinematics?

- Since **velocity** is the **derivative** of **displacement** ($v=\frac{\mathrm{d}s}{\mathrm{d}t}$) it follows that

$$s=\int v\,\mathrm{d}t$$

- Similarly, **velocity** will be an **antiderivative** of **acceleration**

$$v=\int a\,\mathrm{d}t$$

- You might be given the **acceleration** in terms of the **velocity and/or** the **displacement**
  - In this case you can solve a differential equation to find an **expression for the velocity in terms of the displacement**

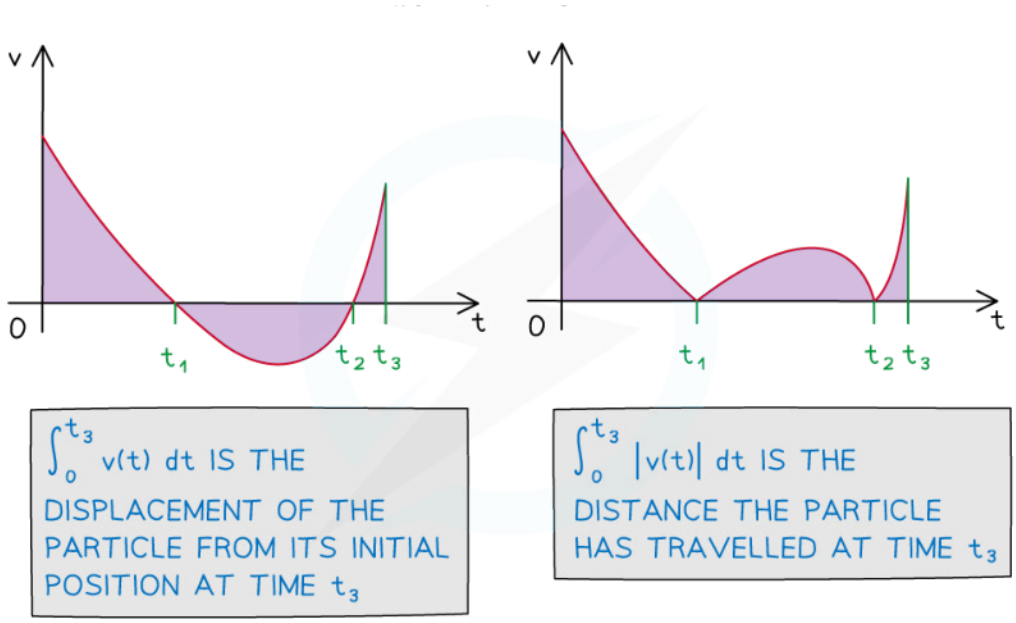
$$a=v\frac{\mathrm{d}v}{\mathrm{d}s}$$

How would I find the constant of integration in kinematics problems?

- A **boundary** or **initial** condition would need to be known
  - phrases involving the word **"initial"**, or **"initially"** are referring to **time** being **zero**, i.e. $t=0$
  - you might also be given information about the object at some other time (this is called a **boundary condition**)
  - **substituting** the values in from the **initial or boundary condition** would allow the **constant** of **integration** to be found

How are definite integrals used in kinematics?

- Definite integrals can be used to find the displacement of a particle between two points in time
  - $\int_{t_1}^{t_2} v(t)\,\mathrm{d}t$ would give the **displacement** of the particle **between** the times $t=t_1$ and $t=t_2$
    - This can be found using a velocity-time graph by **subtracting** the **total area below** the horizontal axis from the **total area above**
  - $\int_{t_1}^{t_2} |v(t)|\,\mathrm{d}t$ gives the **distance** a particle has **travelled** between the times $t=t_1$ and $t=t_2$
    - This can be found using a velocity velocity-time graph by **adding** the **total area below** the horizontal axis to the **total area above**
    - Use a GDC to plot the modulus graph $y=|v(t)|$



$\int_0^{t_3} v(t)\,dt$ IS THE DISPLACEMENT OF THE PARTICLE FROM ITS INITIAL POSITION AT TIME t_3

$\int_0^{t_3} |v(t)|\,dt$ IS THE DISTANCE THE PARTICLE HAS TRAVELLED AT TIME t_3

Examiner Tip

- Sketching the velocity-time graph can help you visualise the distances travelled using areas between the graph and the horizontal axis

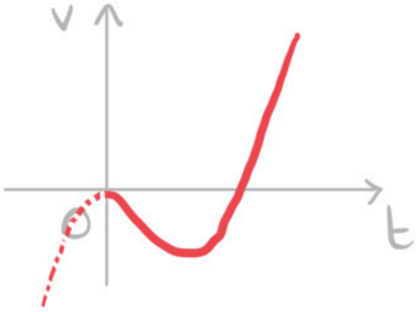
Your notes

Worked example

A particle moving in a straight horizontal line has velocity ($v\ \mathrm{m\ s}^{-2}$) at time t seconds modelled by $v(t) = 8t^3 - 12t^2 - 2t$.

i. Given that the initial position of the particle is at the origin, find an expression for its displacement from the origin at time t seconds.
ii. Find the displacement of the particle from the origin in the first five seconds of its motion.
iii. Find the distance travelled by the particle in the first five seconds of its motion.

Use your GDC to sketch a velocity(-time) graph and use it to check to see if your answers are sensible.



i. "initial" — $t=0$, "origin" — $s=0$

$$s(t) = \int v(t)\,dt = \int (8t^3-12t^2-2t)\,dt$$

$$s(t) = 2t^4-4t^3-t^2+c$$ where c is a constant

at $t=0$, $s=0$, $\therefore c=0$

$$\boxed{\therefore s(t) = 2t^4-4t^3-t^2}$$

ii. "first five seconds" — $t_1=0$, $t_2=5$

Using a GDC this would be

$$s = \int_0^5 (8t^3-12t^2-2t)\,dt$$

$$\boxed{s = 725\text{ m}}$$

iii. Using a GDC this would be

$$d = \int_0^5 \left|8t^3-12t^2-2t\right|\,dt$$ d for distance

$d = 736.734\,020\ldots$

$$\boxed{\therefore d = 737\text{ m } (3\text{ s.f.})}$$