

DP IB Maths: AA HL



Your notes

2.2 Quadratic Functions & Graphs

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2.2.1 Quadratic Functions

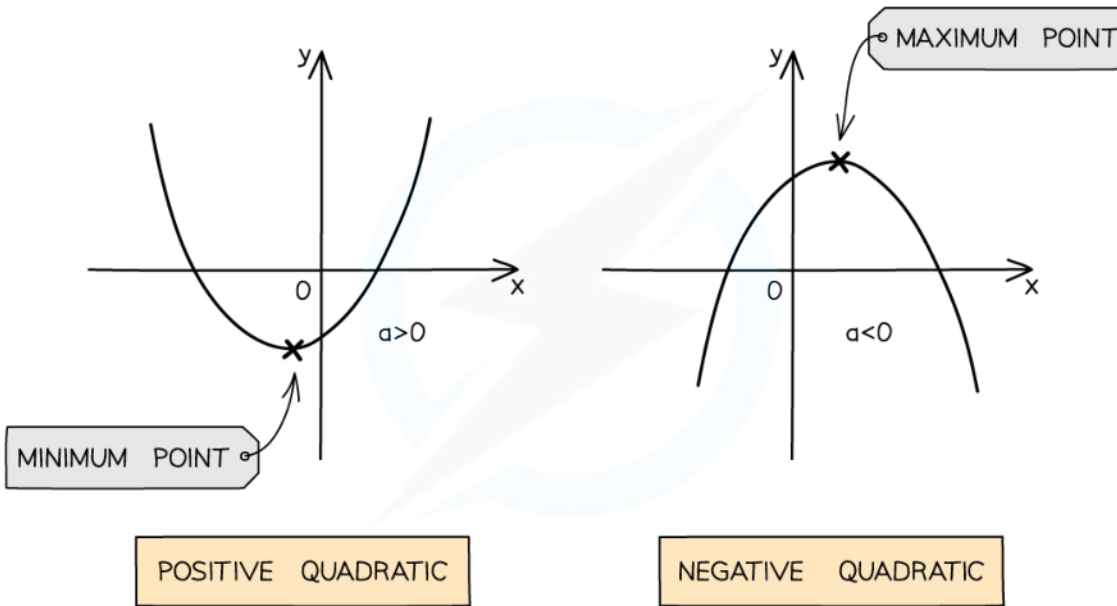
Quadratic Functions & Graphs

What are the key features of quadratic graphs?

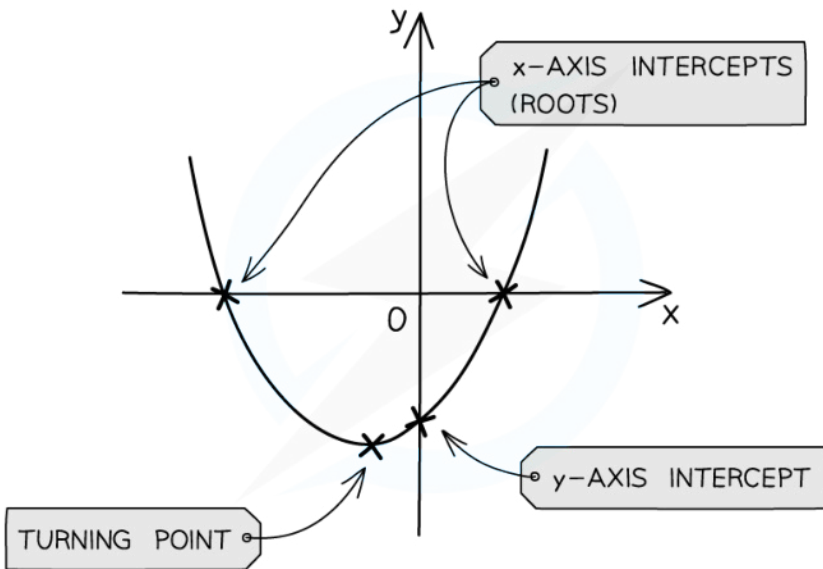
- A **quadratic** graph can be written in the form $y = ax^2 + bx + c$ where $a \neq 0$
- The value of a affects the shape of the curve
 - If a is **positive** the shape is **concave up** \cup
 - If a is **negative** the shape is **concave down** \cap
- The **y-intercept** is at the point $(0, c)$
- The **zeros or roots** are the solutions to $ax^2 + bx + c = 0$
 - These can be found by
 - Factorising
 - Quadratic formula
 - Using your GDC
 - These are also called the x-intercepts
 - There can be 0, 1 or 2 x-intercepts
 - This is determined by the value of the **discriminant**
- There is an **axis of symmetry** at $X = -\frac{b}{2a}$
 - This is given in your **formula booklet**
 - If there are two x-intercepts then the axis of symmetry goes through the midpoint of them
- The **vertex** lies on the axis of symmetry
 - It can be found by **completing the square**
 - The x-coordinate is $X = -\frac{b}{2a}$
 - The y-coordinate can be found using the GDC or by calculating y when $X = -\frac{b}{2a}$
 - If a is **positive** then the vertex is the **minimum point**
 - If a is **negative** then the vertex is the **maximum point**



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What are the equations of a quadratic function?



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- $f(x) = ax^2 + bx + c$
 - This is the **general form**
 - It clearly shows the y-intercept (0, c)
 - You can find the axis of symmetry by $x = -\frac{b}{2a}$
 - This is given in the formula booklet
- $f(x) = a(x - p)(x - q)$
 - This is the **factorised form**
 - It clearly shows the roots (p, 0) & (q, 0)
 - You can find the axis of symmetry by $x = \frac{p + q}{2}$
- $f(x) = a(x - h)^2 + k$
 - This is the **vertex form**
 - It clearly shows the vertex (h, k)
 - The axis of symmetry is therefore $x = h$
 - It clearly shows how the function can be transformed from the graph $y = x^2$
 - Vertical stretch by scale factor a
 - Translation by vector $\begin{pmatrix} h \\ k \end{pmatrix}$

How do I find an equation of a quadratic?

- If you have the **roots** $x = p$ and $x = q$...
 - Write in **factorised form** $y = a(x - p)(x - q)$
 - You will need a third point to find the value of a
- If you have the **vertex** (h, k) then...
 - Write in **vertex form** $y = a(x - h)^2 + k$
 - You will need a second point to find the value of a
- If you have **three random points** (x_1, y_1) , (x_2, y_2) & (x_3, y_3) then...
 - Write in the **general form** $y = ax^2 + bx + c$
 - Substitute the three points into the equation
 - Form and solve a system of three linear equations to find the values of a, b & c

Examiner Tip

- Use your GDC to find the roots and the turning point of a quadratic function
 - You do not need to factorise or complete the square
 - It is good exam technique to sketch the graph from your GDC as part of your working

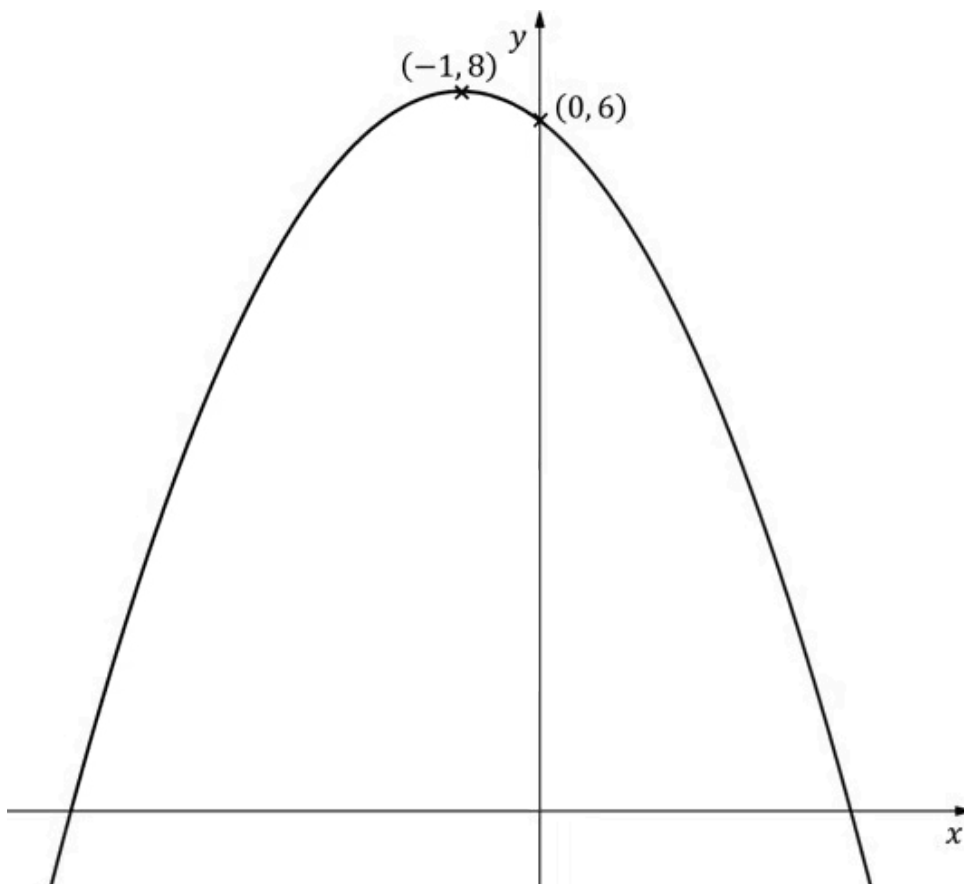


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 **Worked example**

The diagram below shows the graph of $y = f(x)$, where $f(x)$ is a quadratic function.

The intercept with the y -axis and the vertex have been labelled.



Write down an expression for $y = f(x)$.



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We have the vertex so use $y = a(x-h)^2 + k$

$$\text{Vertex } (-1, 8) : y = a(x - (-1))^2 + 8$$

$$y = a(x + 1)^2 + 8$$

Substitute the second point

$$x = 0, y = 6 : 6 = a(0 + 1)^2 + 8$$

$$6 = a + 8$$

$$a = -2$$

$$y = -2(x + 1)^2 + 8$$



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2.2.2 Factorising & Completing the Square

Factorising Quadratics

Why is factorising quadratics useful?

- Factorising gives **roots (zeroes or solutions)** of a quadratic
- It gives the **x-intercepts** when drawing the graph

How do I factorise a monic quadratic of the form $x^2 + bx + c$?

- A monic quadratic is a quadratic where the coefficient of the x^2 term is 1
- You might be able to spot the factors by **inspection**
 - Especially if c is a **prime number**
- Otherwise find two numbers m and n ..
 - A sum equal to b
 - $p + q = b$
 - A product equal to c
 - $pq = c$
- Rewrite bx as $mx + nx$
- Use this to factorise $x^2 + mx + nx + c$
- A shortcut is to write:
 - $(x + p)(x + q)$

How do I factorise a non-monic quadratic of the form $ax^2 + bx + c$?

- A non-monic quadratic is a quadratic where the coefficient of the x^2 term is not equal to 1
- If a , b & c have a common factor then first factorise that out to leave a quadratic with coefficients that have **no common factors**
- You might be able to spot the factors by **inspection**
 - Especially if a and/or c are **prime numbers**
- Otherwise find two numbers m and n ..
 - A sum equal to b
 - $m + n = b$
 - A product equal to ac
 - $mn = ac$
- Rewrite bx as $mx + nx$
- Use this to factorise $ax^2 + mx + nx + c$
- A shortcut is to write:
 - $$\frac{(ax + m)(ax + n)}{a}$$
 - Then factorise common factors from numerator to cancel with the a on the denominator

How do I use the difference of two squares to factorise a quadratic of the form $a^2x^2 - c^2$?



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- The **difference of two squares** can be used when...
 - There is **no x term**
 - The **constant term is a negative**
- Square root the two terms a^2x^2 and c^2
- The two factors are the **sum of square roots** and the **difference of the square roots**
- A shortcut is to write:
 - $(ax + c)(ax - c)$

Examiner Tip

- You can deduce the factors of a quadratic function by using your GDC to find the solutions of a quadratic equation

- Using your GDC, the quadratic equation $6x^2 + x - 2 = 0$ has solutions $x = -\frac{2}{3}$ and

$$x = \frac{1}{2}$$

- Therefore the factors would be $(3x + 2)$ and $(2x - 1)$
- i.e. $6x^2 + x - 2 = (3x + 2)(2x - 1)$



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Worked example

Factorise fully:

a) $x^2 - 7x + 12$.

 Find two numbers m and n such that

$$m+n=b=-7 \quad mn=c=12$$

$$-4 + -3 = -7 \quad -4 \times -3 = 12$$

 Split $-7x$ up and factorise

$$x^2 - 4x - 3x + 12$$

$$x(x-4) - 3(x-4)$$

$$(x-3)(x-4)$$

Shortcut

$$(x+m)(x+n)$$

$$(x-3)(x-4)$$

b) $4x^2 + 4x - 15$.

 Find two numbers m and n such that

$$m+n=b=4 \quad mn=ac=4 \times -15 = -60$$

$$10 + -6 = 4 \quad 10 \times -6 = -60$$

 Split $4x$ up and factorise

$$4x^2 + 10x - 6x - 15$$

$$2x(2x+5) - 3(2x+5)$$

$$(2x-3)(2x+5)$$

Shortcut

$$\frac{(ax+m)(ax+n)}{a}$$

$$\frac{(4x+10)(4x-6)}{4}$$

$$\frac{2(2x+5) \cdot 2(2x-3)}{4}$$

$$(2x-3)(2x+5)$$

c) $18 - 50x^2$.

Factorise the common factor

$$2(9 - 25x^2)$$

Use difference of two squares

$$2(3 - 5x)(3 + 5x)$$



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Completing the Square

Why is completing the square for quadratics useful?

- Completing the square gives the **maximum/minimum** of a quadratic function
 - This can be used to define the **range** of the function
- It gives the **vertex** when drawing the graph
- It can be used to **solve quadratic equations**
- It can be used to derive the **quadratic formula**

How do I complete the square for a monic quadratic of the form $x^2 + bx + c$?

- Half the value of b** and write $\left(x + \frac{b}{2}\right)^2$
 - This is because $\left(x + \frac{b}{2}\right)^2 = x^2 + bx + \frac{b^2}{4}$
- Subtract the unwanted $\frac{b^2}{4}$ term and add on the constant c**
 - $\left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} + c$

How do I complete the square for a non-monic quadratic of the form $ax^2 + bx + c$?

- Factorise out the a** from the terms involving x
 - $a\left(x^2 + \frac{b}{a}x\right) + c$
 - Leaving the c alone will **avoid working with lots of fractions**
- Complete the square** on the quadratic term
 - Half $\frac{b}{a}$** and write $\left(x + \frac{b}{2a}\right)^2$
 - This is because $\left(x + \frac{b}{2a}\right)^2 = x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$
 - Subtract the unwanted $\frac{b^2}{4a^2}$ term**
- Multiply by a and add the constant c**
 - $a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right] + c$
 - $a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$



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💡 Examiner Tip

- Some questions may not use the phrase "completing the square" so ensure you can recognise a quadratic expression or equation written in this form
 - $a(x - h)^2 + k (= 0)$

✍️ Worked example

Complete the square:

a) $x^2 - 8x + 3$.

Half b and subtract its square

$$(x - 4)^2 - 4^2 + 3$$

$$(x - 4)^2 - 13$$

b) $3x^2 + 12x - 5$.

Factorise the 3 from the x terms

$$3(x^2 + 4x) - 5$$

Complete the square on $x^2 + 4x$

$$3((x+2)^2 - 2^2) - 5$$

Simplify

$$3((x+2)^2 - 4) - 5$$

$$3(x+2)^2 - 12 - 5$$

$$3(x+2)^2 - 17$$



Your notes

2.2.3 Solving Quadratics

Solving Quadratic Equations

How do I decide the best method to solve a quadratic equation?

- A quadratic equation is of the form $ax^2 + bx + c = 0$
- If it is a calculator paper then use your GDC to solve the quadratic
- If it is a non-calculator paper then...
 - you can always use the **quadratic formula**
 - you can **factorise** if it can be factorised with integers
 - you can always **complete the square**

How do I solve a quadratic equation by the quadratic formula?

- If necessary **rewrite** in the form $ax^2 + bx + c = 0$
- **Clearly identify** the values of a , b & c
- **Substitute** the values into the formula
 - $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
 - This is given in the **formula booklet**
- **Simplify** the solutions as much as possible

How do I solve a quadratic equation by factorising?

- **Factorise** to rewrite the quadratic equation in the form $a(x - p)(x - q) = 0$
- Set each factor to zero and **solve**
 - $x - p = 0 \Rightarrow x = p$
 - $x - q = 0 \Rightarrow x = q$

How do I solve a quadratic equation by completing the square?

- **Complete the square** to rewrite the quadratic equation in the form $a(x - h)^2 + k = 0$
- Get the squared term by itself
 - $(x - h)^2 = -\frac{k}{a}$
- If $\left(-\frac{k}{a}\right)$ is **negative** then there will be **no solutions**
- If $\left(-\frac{k}{a}\right)$ is **positive** then there will be **two values** for $x - h$

- $x - h = \pm \sqrt{-\frac{k}{a}}$

- **Solve** for x

- $x = h \pm \sqrt{-\frac{k}{a}}$



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Examiner Tip

- When using the quadratic formula with awkward values or fractions you may find it easier to deal with the " $b^2 - 4ac$ " (**discriminant**) first
 - This can help avoid numerical and negative errors, improving accuracy



Your notes

 **Worked example**

Solve the equations:

a) $4x^2 + 4x - 15 = 0$.

This can be factorised

$$(2x + 5)(2x - 3) = 0$$

$$2x + 5 = 0 \quad \text{or} \quad 2x - 3 = 0$$

$$x = -\frac{5}{2} \quad \text{or} \quad x = \frac{3}{2}$$

b) $3x^2 + 12x - 5 = 0$.

This can not be factorised but $3x^2$ and $12x$ have a common factor so complete the square

$$3(x+2)^2 - 17 = 0$$

$$(x+2)^2 = \frac{17}{3} \quad \leftarrow \text{Rearrange}$$

$$x + 2 = \pm \sqrt{\frac{17}{3}} \quad \leftarrow \text{Remember } \pm$$

$$x = -2 \pm \sqrt{\frac{17}{3}}$$

c) $7 - 3x - 5x^2 = 0$.

This can not be factorised so use formula

Formula booklet

Solutions of a quadratic equation	$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, a \neq 0$
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$$a = -5 \quad b = -3 \quad c = 7$$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(-5)(7)}}{2(-5)}$$

$$= \frac{3 \pm \sqrt{9 + 140}}{-10}$$

$$x = -\frac{3 \pm \sqrt{149}}{10}$$



Your notes



Your notes

2.2.4 Quadratic Inequalities

Quadratic Inequalities

What affects the inequality sign when rearranging a quadratic inequality?

- The inequality sign is **unchanged** by...
 - **Adding/subtracting** a term to both sides
 - **Multiplying/dividing** both sides by a **positive term**
- The inequality sign **flips** ($<$ changes to $>$) when...
 - **Multiplying/dividing** both sides by a **negative term**

How do I solve a quadratic inequality?

- **STEP 1: Rearrange** the inequality into quadratic form with a **positive squared term**
 - $ax^2 + bx + c > 0$
 - $ax^2 + bx + c \geq 0$
 - $ax^2 + bx + c < 0$
 - $ax^2 + bx + c \leq 0$
- **STEP 2:** Find the **roots** of the quadratic equation
 - Solve $ax^2 + bx + c = 0$ to get x_1 and x_2 where $x_1 < x_2$
- **STEP 3: Sketch** a graph of the quadratic and label the roots
 - As the squared term is positive it will be **concave up** so "U" shaped
- **STEP 4: Identify** the **region** that satisfies the inequality
 - If you want the graph to be **above the x-axis** then choose the region to be the **two intervals outside** of the two roots
 - If you want the graph to be **below the x-axis** then choose the region to be the **interval between** the two roots
 - For $ax^2 + bx + c > 0$
 - The solution is $x < x_1$ or $x > x_2$
 - For $ax^2 + bx + c \geq 0$
 - The solution is $x \leq x_1$ or $x \geq x_2$
 - For $ax^2 + bx + c < 0$
 - The solution is $x_1 < x < x_2$
 - For $ax^2 + bx + c \leq 0$
 - The solution is $x_1 \leq x \leq x_2$

How do I solve a quadratic inequality of the form $(x - h)^2 < n$ or $(x - h)^2 > n$?

- The safest way is by following the steps above
 - Expand and rearrange
- A **common mistake** is writing $x - h < \pm \sqrt{n}$ or $x - h > \pm \sqrt{n}$
 - This is **NOT correct!**
- The correct solution to $(x - h)^2 < n$ is

- $|x - h| < \sqrt{n}$ which can be written as $-\sqrt{n} < x - h < \sqrt{n}$
- The **final solution** is $h - \sqrt{n} < x < h + \sqrt{n}$
- The correct solution to $(x - h)^2 > n$ is
 - $|x - h| > \sqrt{n}$ which can be written as $x - h < -\sqrt{n}$ or $x - h > \sqrt{n}$
 - The **final solution** is $x < h - \sqrt{n}$ or $x > h + \sqrt{n}$



Your notes

Examiner Tip

- It is easiest to sketch the graph of a quadratic when it has a positive x^2 term, so rearrange first if necessary
- Use your GDC to help select the correct region(s) for the inequality
- Some makes/models of GDC may have the ability to solve inequalities directly
 - However unconventional notation may be used to display the answer (e.g. $6 > x > 3$ rather than $3 < x < 6$)
 - The safest method is to **always** sketch the graph



Your notes

Worked example

Find the set of values which satisfy $3x^2 + 2x - 6 > x^2 + 4x - 2$.

STEP 1: Rearrange

$$(3x^2 + 2x - 6) - (x^2 + 4x - 2) > 0$$

This way gives $a > 0$

$$2x^2 - 2x - 4 > 0$$

$$x^2 - x - 2 > 0$$

Divide by factor of 2

STEP 2: Find the roots

$$x^2 - x - 2 = 0$$

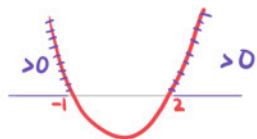
$$(x - 2)(x + 1) = 0$$

$$x = 2 \text{ or } x = -1$$

STEP 3: Sketch



STEP 4: Identify region



$$x < -1 \text{ or } x > 2$$



Your notes

2.2.5 Discriminants

Discriminants

What is the discriminant of a quadratic function?

- The discriminant of a quadratic is denoted by the Greek letter Δ (upper case delta)
- For the quadratic function the discriminant is given by
 - $\Delta = b^2 - 4ac$
 - This is given in the **formula booklet**
- The discriminant is the expression that is square rooted in the **quadratic formula**

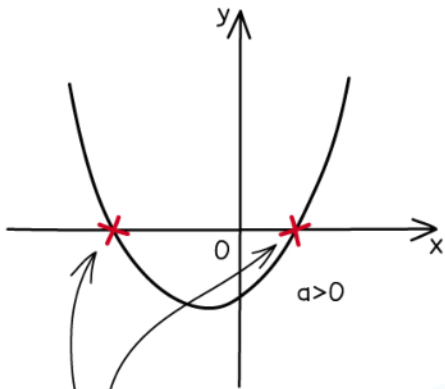
How does the discriminant of a quadratic function affect its graph and roots?

- If $\Delta > 0$ then $\sqrt{b^2 - 4ac}$ and $-\sqrt{b^2 - 4ac}$ are **two distinct values**
 - The equation $ax^2 + bx + c = 0$ has **two distinct real solutions**
 - The graph of $y = ax^2 + bx + c$ has **two distinct real roots**
 - This means the graph **crosses** the x-axis **twice**
- If $\Delta = 0$ then $\sqrt{b^2 - 4ac}$ and $-\sqrt{b^2 - 4ac}$ are **both zero**
 - The equation $ax^2 + bx + c = 0$ has **one repeated real solution**
 - The graph of $y = ax^2 + bx + c$ has **one repeated real root**
 - This means the graph **touches** the x-axis at **exactly one point**
 - This means that the **x-axis** is a **tangent** to the graph
- If $\Delta < 0$ then $\sqrt{b^2 - 4ac}$ and $-\sqrt{b^2 - 4ac}$ are **both undefined**
 - The equation $ax^2 + bx + c = 0$ has **no real solutions**
 - The graph of $y = ax^2 + bx + c$ has **no real roots**
 - This means the graph **never touches** the x-axis
 - This means that graph is **wholly above** (or **below**) the x-axis



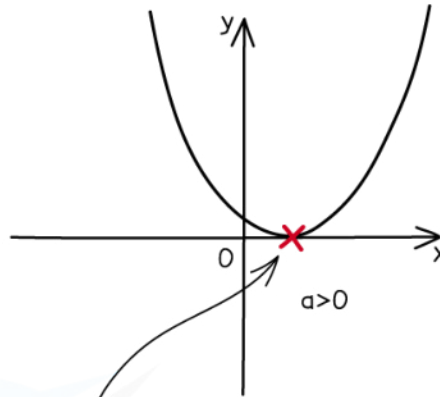
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IF $b^2 - 4ac > 0$



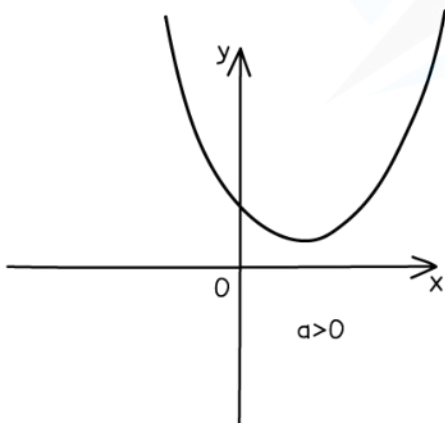
TWO DISTINCT
REAL ROOTS

IF $b^2 - 4ac = 0$



ONE REAL ROOT
(REPEATED ROOTS)

IF $b^2 - 4ac < 0$



NO REAL ROOTS

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Forming equations and inequalities using the discriminant

- Often at least one of the coefficients of a quadratic is **unknown**
 - Questions usually use the letter k for the unknown constant
- You will be given a fact about the quadratic such as:
 - The **number of solutions** of the equation
 - The **number of roots** of the graph

- To find the **value or range of values** of k
 - Find an **expression for the discriminant**
 - Use $\Delta = b^2 - 4ac$
 - Decide whether $\Delta > 0$, $\Delta = 0$ or $\Delta < 0$
 - If the question says there are **real roots** but does not specify how many then use $\Delta \geq 0$
 - **Solve** the resulting equation or inequality



Your notes

Examiner Tip

- Questions will rarely use the word discriminant so it is important to recognise when its use is required
 - Look for
 - a number of roots or solutions being stated
 - whether and/or how often the graph of a quadratic function intercepts the **X**-axis
- Be careful setting up inequalities that concern "two real roots" ($\Delta \geq 0$) as opposed to "two real distinct roots" ($\Delta > 0$)



Your notes

Worked example

A function is given by $f(x) = 2kx^2 + kx - k + 2$, where k is a constant. The graph of $y = f(x)$ has two distinct real roots.

a) Show that $9k^2 - 16k > 0$.

Two distinct real roots $\Rightarrow \Delta > 0$

Formula booklet

Discriminant	$\Delta = b^2 - 4ac$
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$$a = 2k \quad b = k \quad c = (-k + 2)$$

$$\Delta = k^2 - 4(2k)(-k + 2)$$

$$= k^2 + 8k^2 - 16k$$

$$= 9k^2 - 16k$$

$$\Delta > 0 \Rightarrow 9k^2 - 16k > 0$$

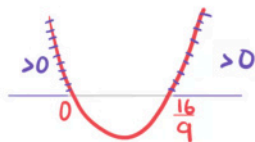
b) Hence find the set of possible values of k .

Solve the inequality

$$9k^2 - 16k = 0$$

$$k(9k - 16) = 0$$

$$k = 0 \text{ or } k = \frac{16}{9}$$



$$k < 0 \text{ or } k > \frac{16}{9}$$