DP IB Maths: AI HL

Your notes

4.2 Correlation & Regression

Contents

4.2.1 Bivariate Data

Scatter Diagrams

What does bivariate data mean?

- **Bivariate data** is data which is collected on **two variables** and looks at how one of the factors affects the other
  - Each data value from one variable will be **paired** with a data value from the other variable
  - The two variables are often related, but do not have to be

What is a scatter diagram?

- A **scatter diagram** is a way of graphing bivariate data
  - One variable will be on the *x*-axis and the other will be on the y-axis
  - The variable that can be **controlled** in the data collection is known as the **independent** or **explanatory variable** and is plotted on the *x*-axis
  - The variable that is **measured** or discovered in the data collection is known as the **dependent** or **response variable** and is plotted on the y-axis
- Scatter diagrams can contain **outliers** that do not follow the trend of the data

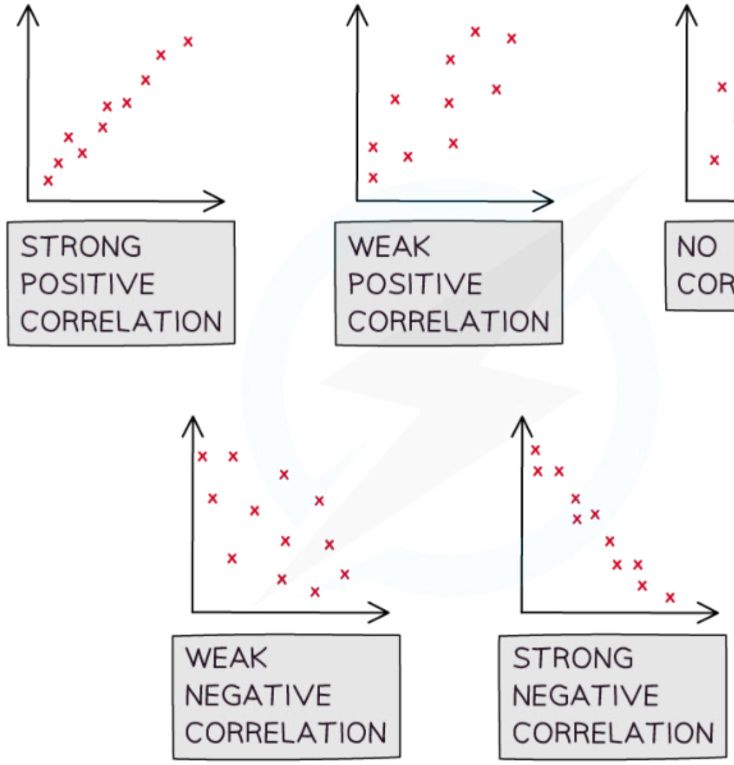
Examiner Tip

- If you use scatter diagrams in your Internal Assessment then be aware that finding outliers for bivariate data is different to finding outliers for univariate data
  - (x, y) could be an outlier for the bivariate data even if x and y are not outliers for their separate univariate data

Correlation

What is correlation?

- **Correlation** is how the **two variables change in relation to each other**
  - Correlation could be the result of a **causal relationship** but this is not always the case
- **Linear correlation** is when the changes are proportional to each other
- **Perfect linear correlation** means that the bivariate data will all lie on a straight line on a scatter diagram
- When describing correlation mention
  - The type of the correlation
    - **Positive correlation** is when an **increase** in one variable results in the other variable **increasing**
    - **Negative correlation** is when an **increase** in one variable results in the other variable **decreasing**
    - **No linear correlation** is when the data points don't appear to follow a trend
  - The strength of the correlation
    - **Strong linear correlation** is when the data points lie **close** to a **straight line**
    - **Weak linear correlation** is when the data points are **not close** to a **straight line**
- If there is **strong linear correlation** you can draw a l**ine of best fit** (by eye)
  - The line of best fit will pass through the mean point $(\overline{x}, \overline{y})$
  - If you are asked to draw a line of best fit
    - Plot the mean point
    - Draw a line going through it that follows the trend of the data


STRONG POSITIVE CORRELATION
WEAK POSITIVE CORRELATION
NO CORRELATION
WEAK NEGATIVE CORRELATION
STRONG NEGATIVE CORRELATION

What is the difference between correlation and causation?

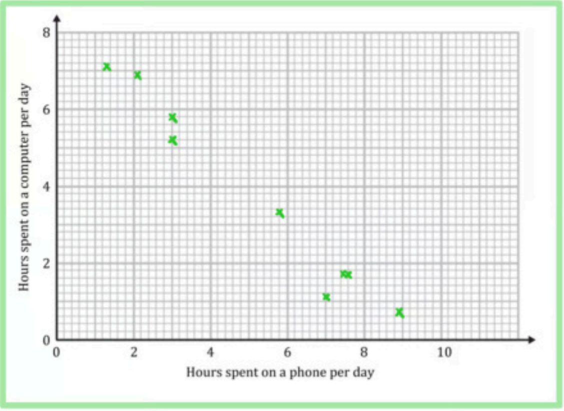
- It is important to be aware that just because correlation exists, it does not mean that the change in one of the variables is **causing** the change in the other variable
  - **Correlation does not imply causation!**
- If a change in one variable **causes** a change in the other then the two variables are said to have a **causal relationship**
  - Observing correlation between two variables does **not always** mean that there is a causal relationship
    - There could be **underlying factors** which is causing the correlation
  - Look at the two variables in question and consider the context of the question to decide if there could be a causal relationship
    - If the two variables are temperature and number of ice creams sold at a park then it is likely to be a causal relationship
    - Correlation may exist between global temperatures and the number of monkeys kept as pets in the UK but they are unlikely to have a causal relationship

Worked example

A teacher is interested in the relationship between the number of hours her students spend on a phone per day and the number of hours they spend on a computer. She takes a sample of nine students and records the results in the table below.

Hours spent on a phone per day	7.6	7.0	8.9	3.0	3.0	7.5	2.1	1.3	5.8
Hours spent on a computer per day	1.7	1.1	0.7	5.8	5.2	1.7	6.9	7.1	3.3

a) Draw a scatter diagram for the data.

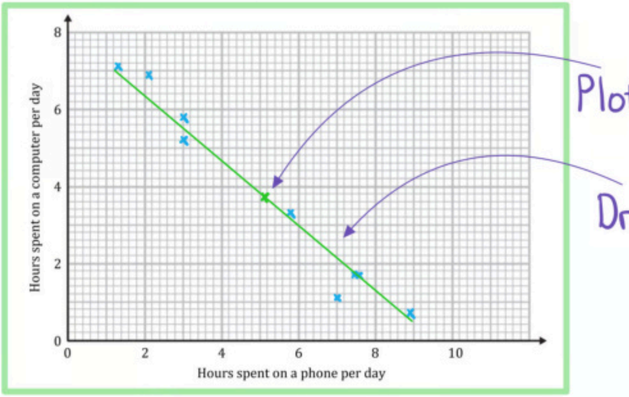


b) Describe the correlation.

Strong negative linear correlation

c) Draw a line of best fit.

Mean point $(\bar{x}, \bar{y}) = (5.133..., 3.722...)$



Plot the mean point

Draw it by eye

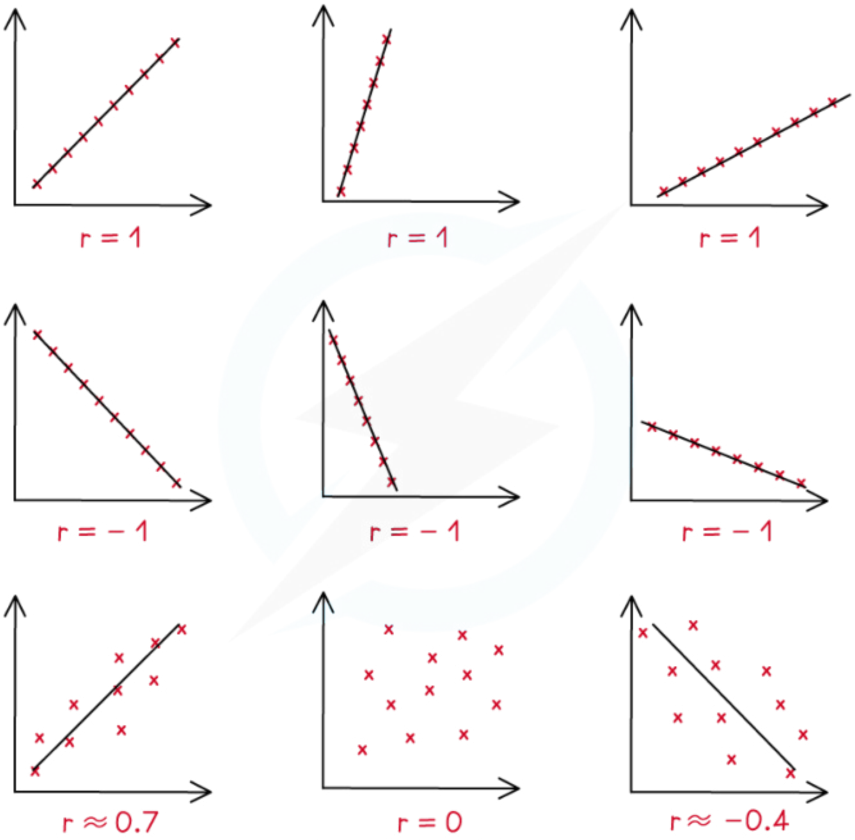
4.2.2 Correlation Coefficients

Your notes

PMCC

What is Pearson's product-moment correlation coefficient?

- Pearson's product-moment correlation coefficient (PMCC) is a way of giving a numerical value to a **linear relationship** of bivariate data
- The PMCC of a sample is denoted by the letter r
  - r can take any value such that $-1 \le r \le 1$
  - A **positive value** of r describes **positive correlation**
  - A **negative value** of r describes **negative correlation**
  - r = 0 means there is **no linear correlation**
  - r = 1 means **perfect positive linear** correlation
  - r = -1 means **perfect negative linear** correlation
  - The closer to 1 or -1 the stronger the correlation



How do I calculate Pearson's product-moment correlation coefficient (PMCC)?

- You will be expected to use the statistics mode on your GDC to calculate the PMCC
- The formula can be useful to deepen your understanding

$$r = \frac{S_{xy}}{S_x S_y}$$

  - $S_{xy} = \sum_{i=1}^{n} x_i y_i - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)$ is linked to the **covariance**
  - $S_x = \sqrt{\sum_{i=1}^{n} x_i^{\,2} - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2}$ and $S_y = \sqrt{\sum_{i=1}^{n} y_i^{\,2} - \frac{1}{n}\left(\sum_{i=1}^{n} y_i\right)^2}$ are linked to the **variances**
  - You **do not need to learn this** as using your GDC will be expected

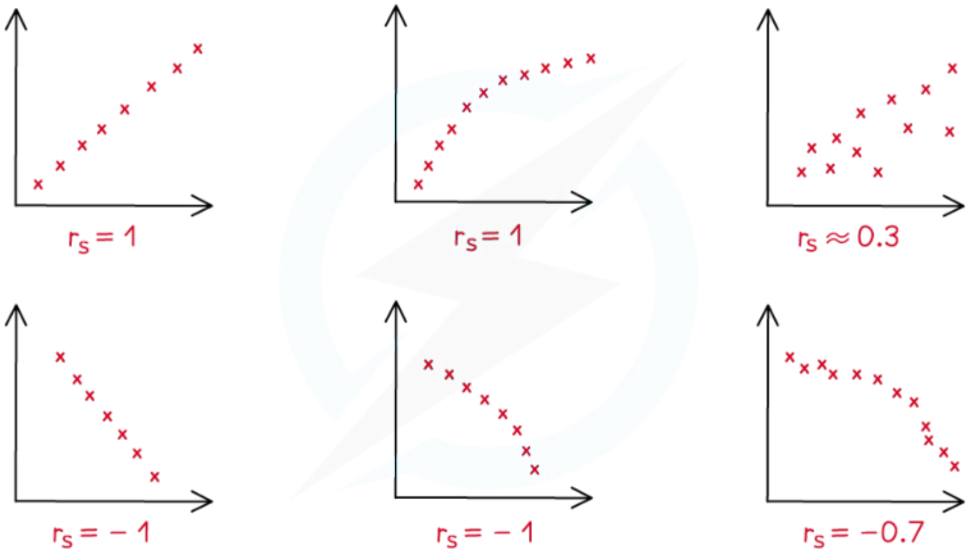
When does the PMCC suggest there is a linear relationship?

- **Critical values** of *r* indicate when the PMCC would suggest there is a linear relationship
  - In your exam you will be given critical values where appropriate
  - Critical values will depend on the size of the sample
- If the **absolute value** of the **PMCC** is **bigger** than the **critical value** then this suggests a linear model is appropriate

Spearman's Rank

What is Spearman's rank correlation coefficient?

- Spearman's rank correlation coefficient is a measure of how well the relationship between two variables can be described using a **monotonic** function
  - **Monotonic** means the points are either always increasing or always decreasing
  - This can be used as a way to **measure correlation in linear models**
  - Though Spearman's Rank correlation coefficient can also be used to assess a non-linear relationship
- Each data is ranked, from biggest to smallest or from smallest to biggest
  - For *n* data values, they are ranked from 1 to *n*
  - It doesn't matter whether variables are ranked from biggest to smallest or smallest to biggest, but they must be ranked in the **same order for both variables**
- Spearman's rank of a sample is denoted by r_s
  - r_s can take any value such that $-1 \le r_s \le 1$
  - A **positive value** of r_s describes a **degree of agreement** between the rankings
  - A **negative value** of r_s describes a **degree of disagreement** between the rankings
  - r_s = 0 means the data shows **no monotonic behaviour**
  - r_s = 1 means the rankings are in complete agreement: the data is **strictly increasing**
    - An increase in one variable means an increase in the other
  - r_s = -1 means the rankings are in complete disagreement: the data is **strictly decreasing**
    - An increase in one variable means a decrease in the other
  - The **closer to 1 or -1** the **stronger the correlation** of the rankings



How do I calculate Spearman's rank correlation coefficient (PMCC)?

- Rank each set of data independently

  - 1 to *n* for the *x*-values
  - 1 to *n* for the *y*-values
- If some values are equal then give each the average of the ranks they would occupy
  - For example: if the 3rd, 4th and 5th highest values are equal then give each the ranking of 4
    - $\frac{3+4+5}{3}=4$
- Calculate the PMCC of the **rankings** using your GDC
  - This value is **Spearman's rank correlation coefficient**

Appropriateness & Limitations

Which correlation coefficient should I use?

- **Pearson's PMCC** tests for a **linear relationship** between two variables
  - It will not tell you if the variables have a non-linear relationship
    - Such as exponential growth
  - Use this if you are interested in a linear relationship
- **Spearman's rank** tests for a **monotonic relationship** (always increasing or always decreasing) between two variables
  - It will not tell you what function can be used to model the relationship
    - Both linear relationships and exponential relationships can be monotonic
  - Use this if you think there is a non-linear monotonic relationship

How are Pearson's and Spearman's correlation coefficients connected?

- If there is **linear correlation** then the relationship is also **monotonic**
  - $r = 1 \Rightarrow r_s = 1$
  - $r = -1 \Rightarrow r_s = -1$
  - However the **converse is not true**
- It is possible for Spearman's rank to be 1 (or -1) but for the PMCC to be different
  - For example: data that follows an **exponential growth model**
    - $r_s = 1$ as the points are always increasing
    - $r < 1$ as the points do not lie on a straight line

Are Pearson's and Spearman's correlation coefficients affected by outliers?

- Pearson's PMCC **is** affected by outliers
  - as it uses the numerical value of each data point
- Spearman's rank is **not usually** affected by outliers
  - as it only uses the ranks of each data point

Examiner Tip

- You can use your GDC to plot the scatter diagram to help you visualise the data

Worked example

The table below shows the scores of eight students for a maths test and an English test.

Maths (x)	7	18	37	52	61	68	75	82
English (y)	5	3	9	12	17	41	49	97

a) Write down the value of Pearson's product-moment correlation coefficient, r.

Enter data into GDC.

$r = 0.79433...$

$\boxed{r = 0.794 \ (3sf)}$

b) Find the value of Spearman's rank correlation coefficient, r_s.

Rank the data

x rank	8	7	6	5	4	3	2	1
y rank	7	8	6	5	4	3	2	1

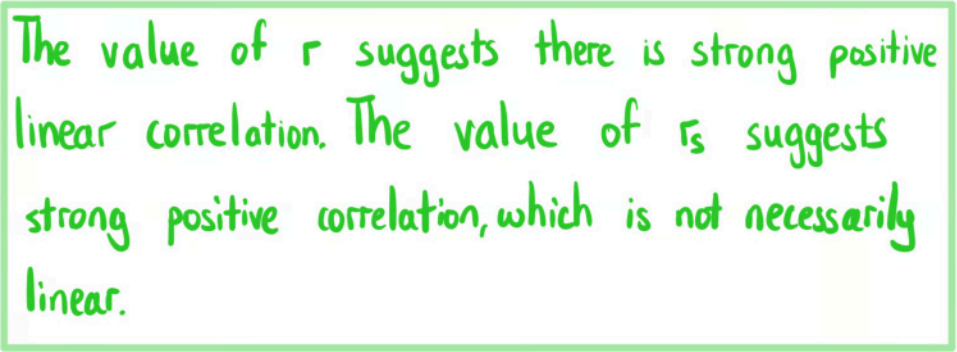
Find PMCC of ranks

$r_s = 0.97619...$

$\boxed{r_s = 0.976 \ (3sf)}$

c) Comment on the values of the two correlation coefficients.

Your notes



The value of r suggests there is strong positive linear correlation. The value of r_s suggests strong positive correlation, which is not necessarily linear.

4.2.3 Linear Regression

Linear Regression

What is linear regression?

- If **strong linear correlation** exists on a scatter diagram then the data can be modelled by a **linear model**
  - Drawing lines of best fit by eye is not the best method as it can be difficult to judge the best position for the line
- The **least squares regression line** is the line of best fit that minimises the **sum of the squares** of the gap between the line and each data value
  - This is usually called the **regression line of *y* on *x***
  - It can be calculated by looking at the vertical distances between the line and the data values
- The **regression line of y on x** is written in the form $y = ax + b$
- *a* is the **gradient** of the line
  - It represents the change in *y* for each individual unit change in *x*
    - If *a* is **positive** this means *y* **increases** by *a* for a unit increase in *x*
    - If *a* is **negative** this means *y* **decreases** by |a| for a unit increase in *x*
- *b* is the ***y* – intercept**
  - It shows the value of *y* when *x* is zero
- You are expected to use your **GDC** to find the equation of the regression line
  - Enter the bivariate data and choose the **model "*ax* + *b*"**
  - Remember the **mean point** $(\overline{x}, \overline{y})$ will lie on the regression line

How do I use a regression line?

- The equation of the regression line can be used to decide what type of correlation there is if there is no scatter diagram
  - If *a* is **positive** then the data set has **positive correlation**
  - If *a* is **negative** then the data set has **negative correlation**
- The equation of the regression line can also be used to **predict** the value of a **dependent variable (*y*)** from an **independent variable (*x*)**
  - The equation should **only be used** to make **predictions for *y***
    - Using a *y* on *x* line to **predict *x* is not always reliable**
  - Making a prediction **within the range** of the given data is called **interpolation**
    - This is usually reliable
    - The stronger the correlation the more reliable the prediction
  - Making a prediction **outside of the range** of the given data is called **extrapolation**
    - This is much less reliable
  - The prediction will be more reliable if the number of data values in the original sample set is bigger

Examiner Tip

- Once you calculate the values of *a* and *b* store then in your GDC
  - This means you can use the full display values rather than the rounded values when using the linear regression equation to predict values
  - This avoids rounding errors

Worked example

Barry is a music teacher. For 7 students, he records the time they spend practising per week (X hours) and their score in a test (y %).

Time (X)	2	5	6	7	10	11	12
Score (y)	11	49	55	75	63	68	82

a) Write down the equation of the regression line of y on X, giving your answer in the form $y = ax + b$ where a and b are constants to be found.

Enter data into GDC

a is the coefficient of x $a = 5.5680...$

b is the constant term $b = 15.4136...$

$$\boxed{y = 5.57x + 15.4}$$

b) Give an interpretation of the value of a.

a = 5.57 means that the model suggests that the score increases by 5.57 % for every extra hour of practice.

c) Another of Barry's students practises for 15 hours a week, estimate their score. Comment on the validity of this prediction.

Substitute $x = 15$

$y = (5.5680...) \times 15 + (15.4136...) = 98.93..$

The model predicts a score of 98.9% but this is unreliable as $x = 15$ is outside the range of data. Therefore extrapolation is being used.