

# DP IB Maths: AI HL



## 2.5 Transformations of Graphs

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## 2.5.1 Translations of Graphs



Your notes

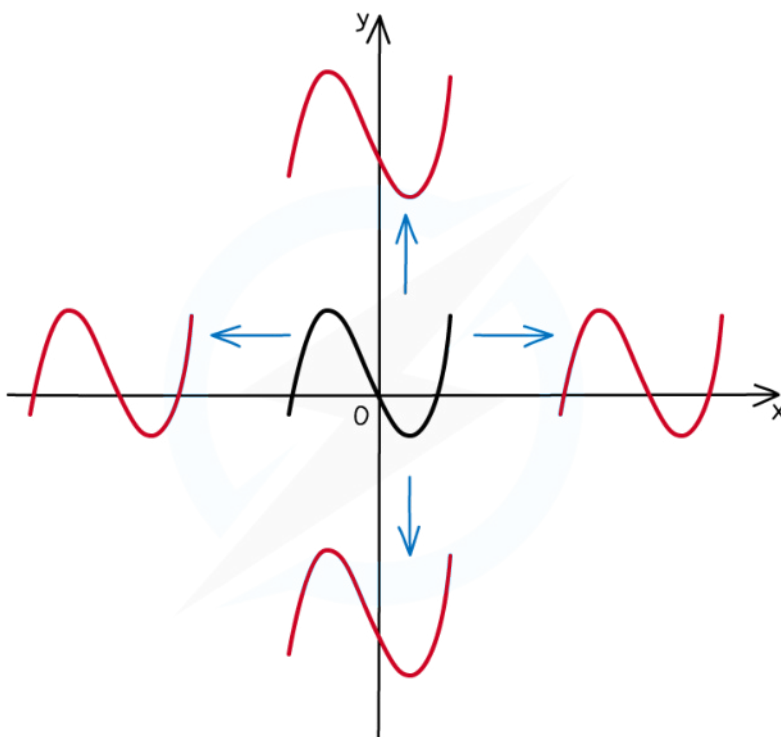
### Translations of Graphs

#### What are translations of graphs?

- When you alter a function in certain ways, the effects on the graph of the function can be described by **geometrical transformations**
- For a **translation**:
  - the graph is **moved** (up or down, left or right) in the xy plane
    - Its position **changes**
    - the shape, size, and orientation of the graph remain **unchanged**
- A particular translation (how far left/right, how far up/down) is specified by a **translation vector**  $\begin{pmatrix} x \\ y \end{pmatrix}$ :
  - $x$  is the **horizontal** displacement
    - **Positive** moves **right**
    - **Negative** moves **left**
  - $y$  is the **vertical** displacement
    - **Positive** moves **up**
    - **Negative** moves **down**



Your notes



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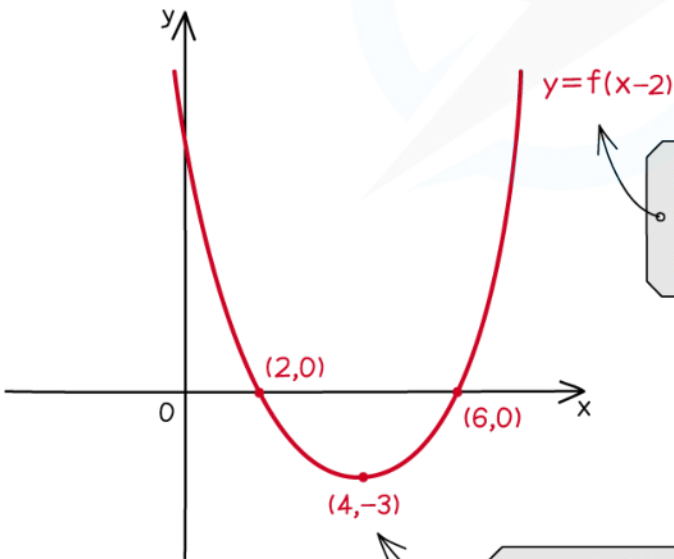
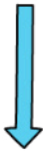
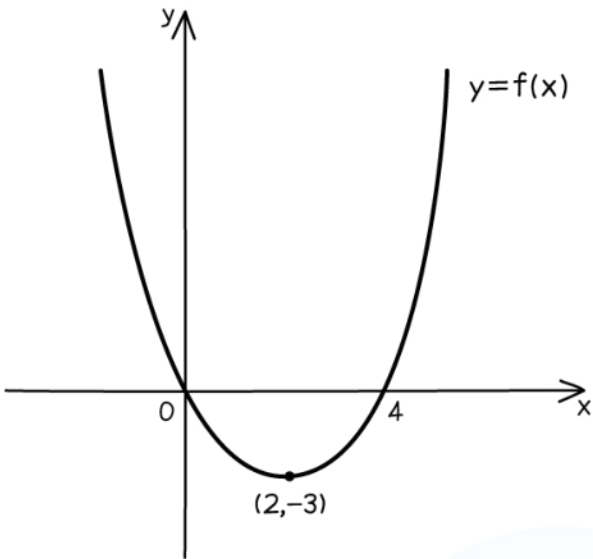


### What effects do horizontal translations have on the graphs and functions?

- A **horizontal translation** of the graph  $y = f(x)$  by the vector  $\begin{pmatrix} a \\ 0 \end{pmatrix}$  is represented by
  - $y = f(x - a)$
- The **x-coordinates change**
  - The value  $a$  is **subtracted** from them
- The **y-coordinates stay the same**
- The coordinates  $(x, y)$  become  $(x + a, y)$
- **Horizontal asymptotes stay the same**
- **Vertical asymptotes change**
  - $x = k$  becomes  $x = k + a$



Your notes



HORIZONTAL  
TRANSLATION  
BY THE VECTOR  $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

x-COORDINATES CHANGE,  
y-COORDINATES STAY  
THE SAME

## What effects do vertical translations have on the graphs and functions?

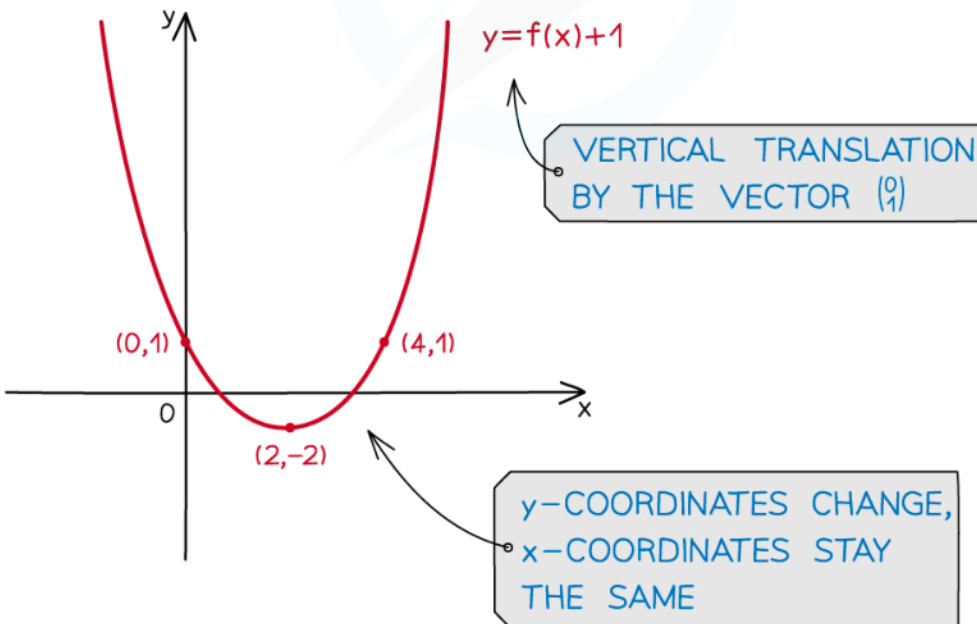
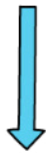
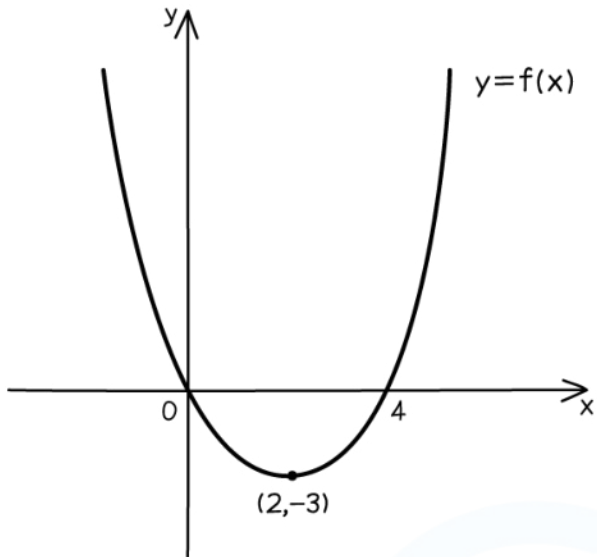
- A **vertical translation** of the graph  $y = f(x)$  by the vector  $\begin{pmatrix} 0 \\ b \end{pmatrix}$  is represented by
  - $y - b = f(x)$ 
    - This is often rearranged to  $y = f(x) + b$
  - The **x-coordinates stay the same**
  - The **y-coordinates change**
    - The value  $b$  is **added** to them
  - The coordinates  $(x, y)$  become  $(x, y + b)$
  - **Horizontal asymptotes change**
    - $y = k$  becomes  $y = k + b$
  - **Vertical asymptotes stay the same**



Your notes



Your notes



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### Examiner Tip

- To get full marks in an exam make sure you use correct mathematical terminology
  - For example: Translate by the vector  $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$



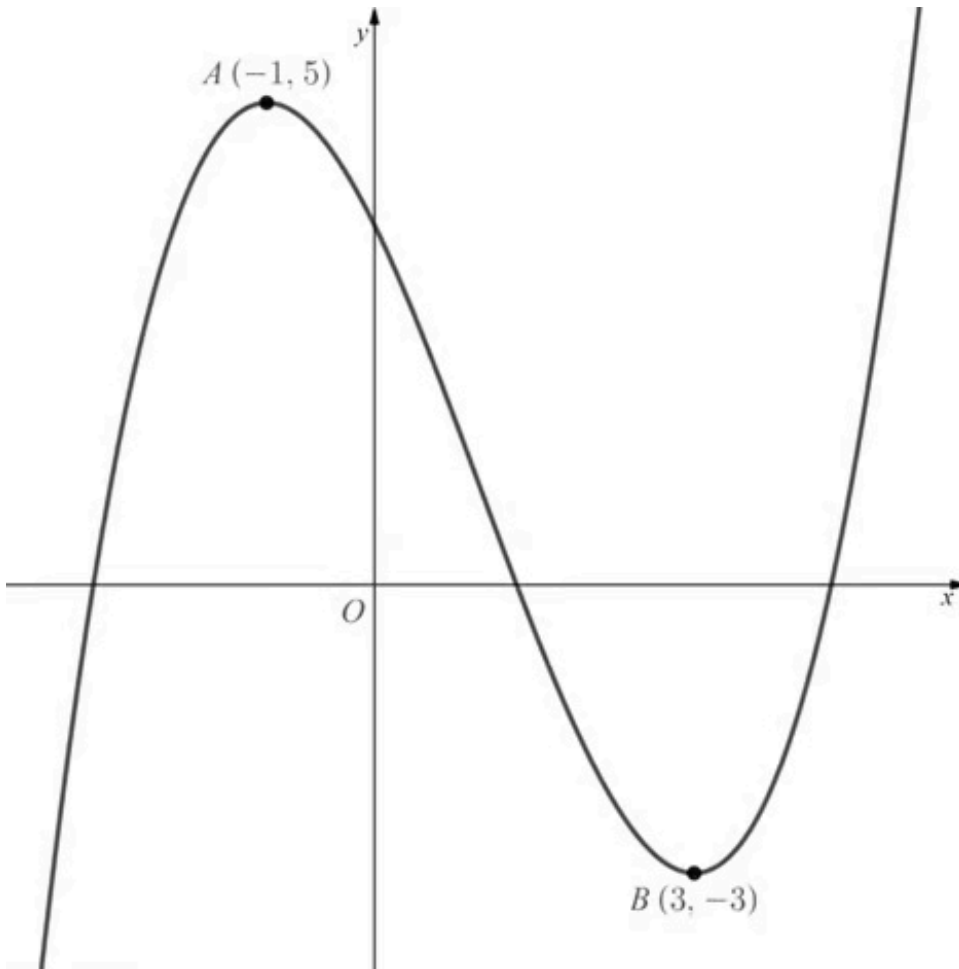
Your notes



Your notes

 **Worked example**

The diagram below shows the graph of  $y = f(x)$ .



- a) Sketch the graph of  $y = f(x + 3)$ .





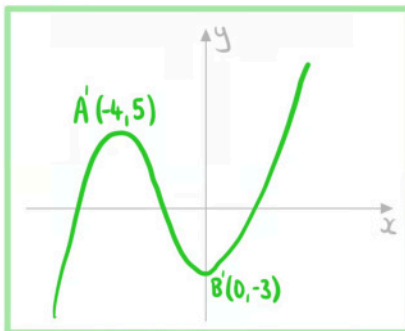
Your notes

$y = f(x+k)$  translation by  $\begin{pmatrix} -k \\ 0 \end{pmatrix}$

Translate  $y = f(x)$  by  $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$

A becomes  $(-4, 5)$

B becomes  $(0, -3)$



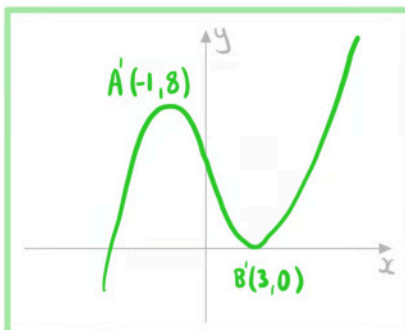
b) Sketch the graph of  $y = f(x) + 3$ .

$y = f(x) + k$  translation by  $\begin{pmatrix} 0 \\ k \end{pmatrix}$

Translate  $y = f(x)$  by  $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$

A becomes  $(-1, 8)$

B becomes  $(3, 0)$





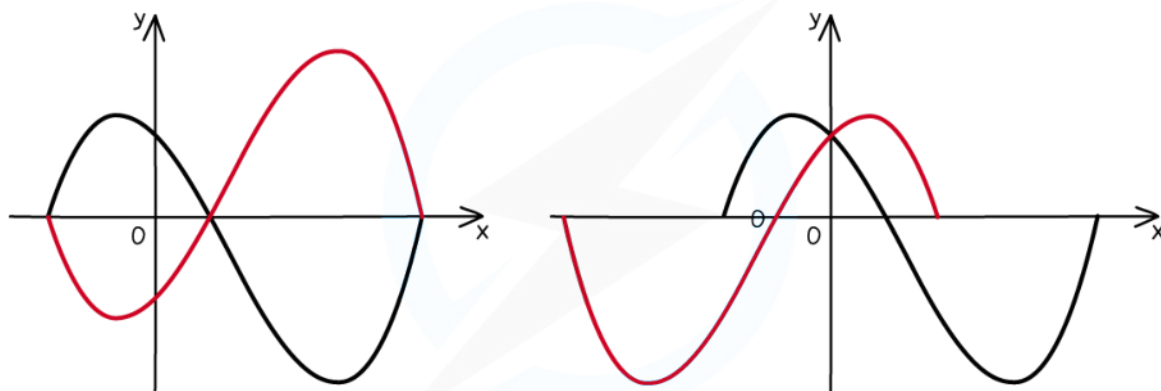
Your notes

## 2.5.2 Reflections of Graphs

### Reflections of Graphs

#### What are reflections of graphs?

- When you alter a function in certain ways, the effects on the graph of the function can be described by **geometrical transformations**
- For a **reflection**:
  - the graph is **flipped** about one of the coordinate axes
    - Its orientation **changes**
    - the size of the graph remains **unchanged**
- A particular reflection is specified by an **axis of symmetry**:
  - $y = 0$ 
    - This is the x-axis
  - $x = 0$ 
    - This is the y-axis



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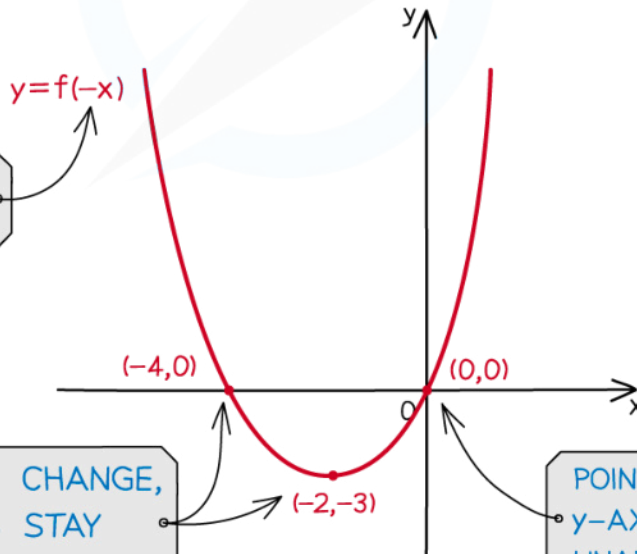
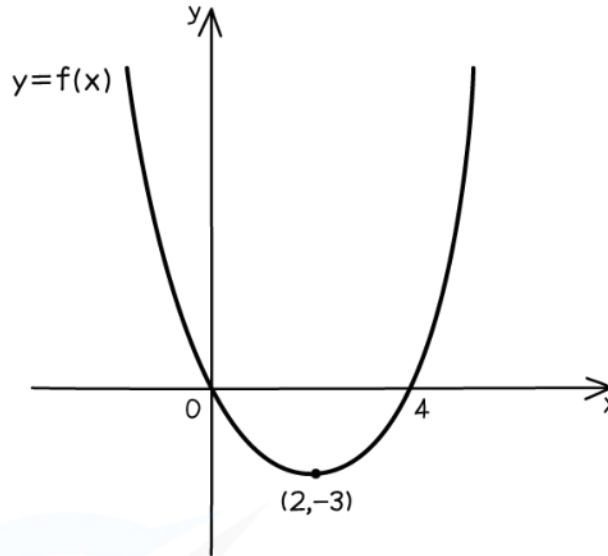
#### What effects do horizontal reflections have on the graphs and functions?

- A **horizontal reflection** of the graph  $y = f(x)$  about the y-axis is represented by
  - $y = f(-x)$
- The **x-coordinates change**
  - Their **sign** changes
- The **y-coordinates stay the same**
- The coordinates  $(x, y)$  become  $(-x, y)$
- **Horizontal** asymptotes **stay the same**
- **Vertical** asymptotes **change**

- $x = k$  becomes  $x = -k$



Your notes



REFLECTION IN THE  $y$ -AXIS

$x$ -COORDINATES CHANGE,  $y$ -COORDINATES STAY THE SAME

POINTS ON THE  $y$ -AXIS ARE UNAFFECTED

### What effects do vertical reflections have on the graphs and functions?

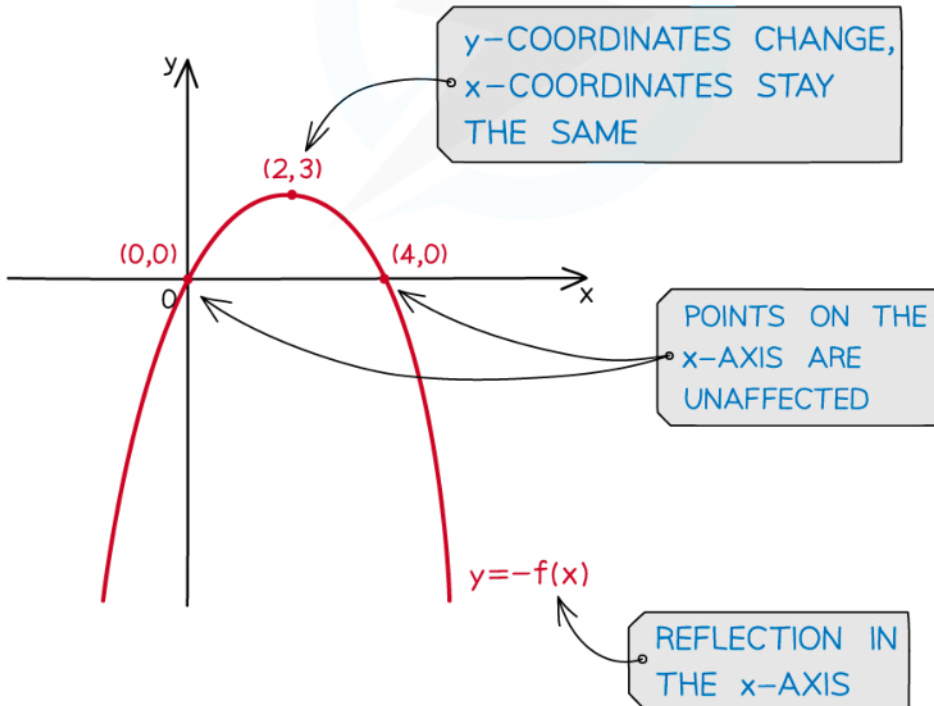
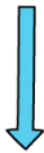
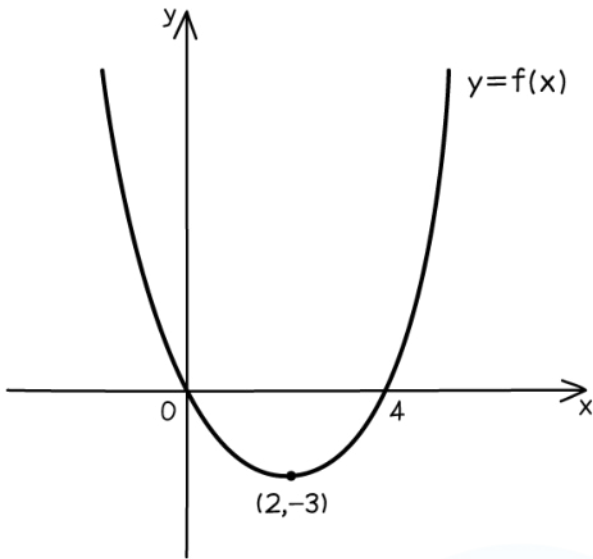
- A **vertical reflection** of the graph  $y = f(x)$  about the x-axis is represented by
  - $-y = f(x)$
  - This is often rearranged to  $y = -f(x)$
- The **x-coordinates stay the same**
- The **y-coordinates change**
  - Their **sign** changes
- The coordinates  $(x, y)$  become  $(x, -y)$
- **Horizontal asymptotes change**
  - $y = k$  becomes  $y = -k$
- **Vertical asymptotes stay the same**



Your notes



Your notes



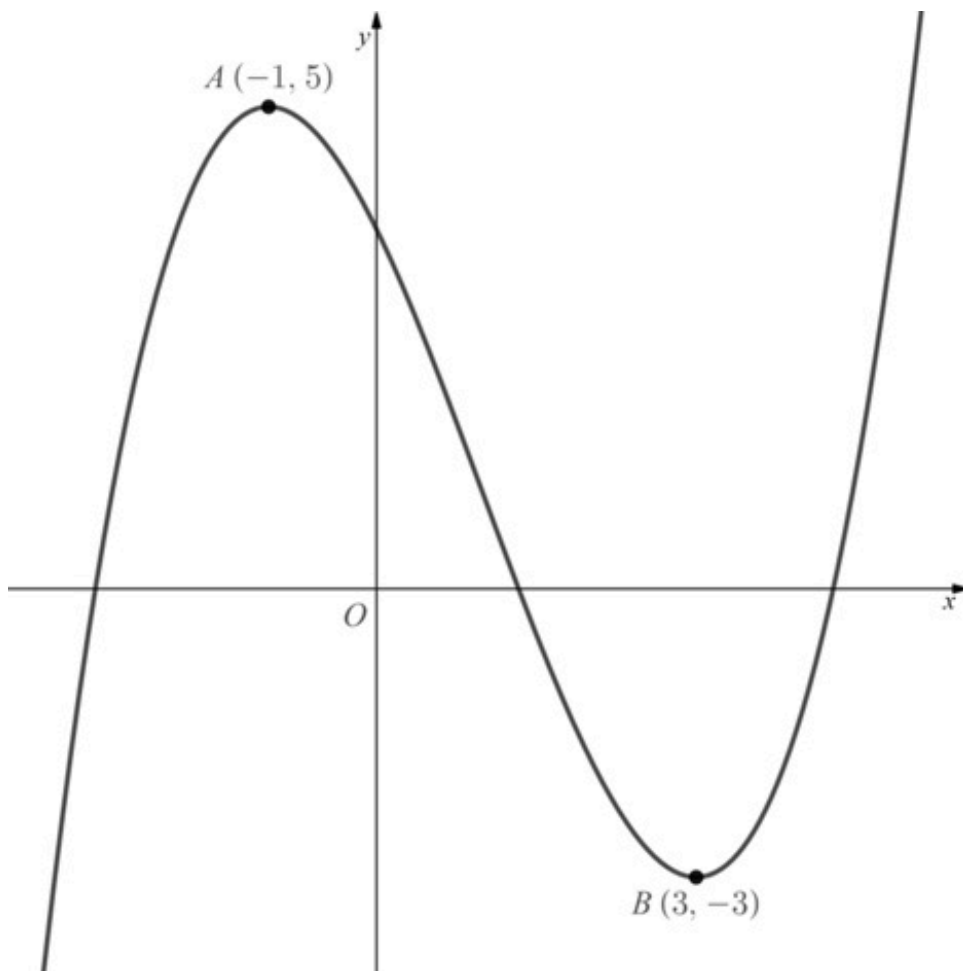
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Your notes

 **Worked example**

The diagram below shows the graph of  $y = f(x)$ .



- a) Sketch the graph of  $y = -f(x)$ .

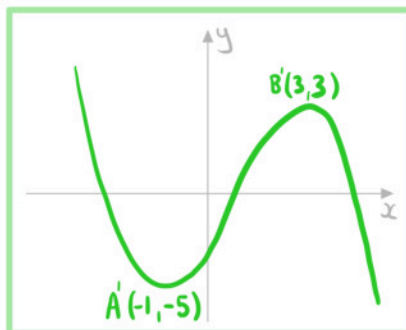


Your notes

$y = -f(x)$  reflection in  $x$ -axis

A becomes  $(-1, -5)$

B becomes  $(3, 3)$

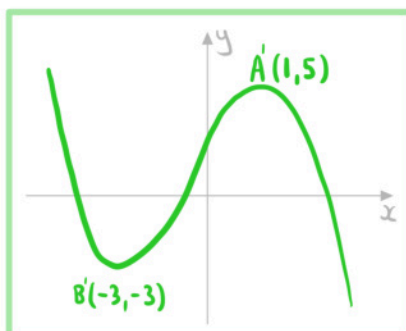


b) Sketch the graph of  $y = f(-x)$ .

$y = f(-x)$  reflection in  $y$ -axis

A becomes  $(1, 5)$

B becomes  $(-3, -3)$





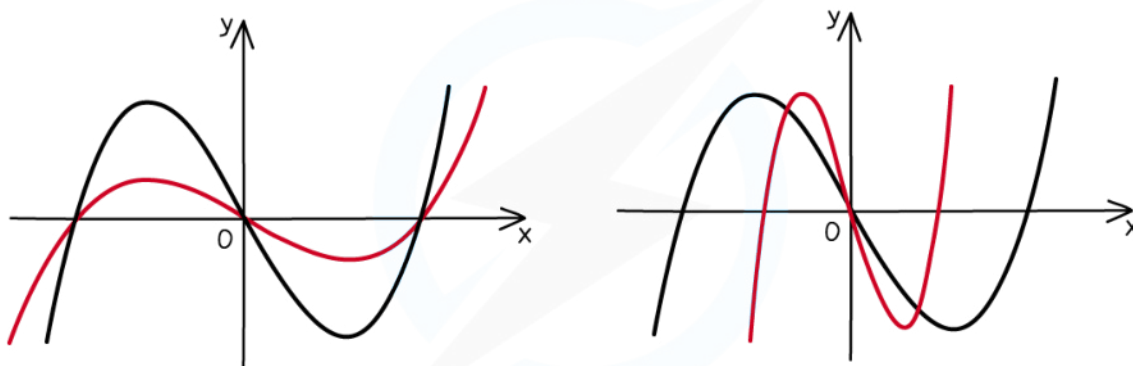
Your notes

## 2.5.3 Stretches of Graphs

### Stretches of Graphs

#### What are stretches of graphs?

- When you alter a function in certain ways, the effects on the graph of the function can be described by **geometrical transformations**
- For a **stretch**:
  - the graph is **stretched** about one of the coordinate axes by a scale factor
    - Its size **changes**
    - the orientation of the graph remains **unchanged**
- A particular stretch is specified by a **coordinate axis** and a **scale factor**:
  - The **distance** between a **point** on the graph and the **specified coordinate axis** is **multiplied** by the **constant scale factor**
  - The graph is stretched in the **direction** which is **parallel** to the **other coordinate axis**
  - For scale factors **bigger than 1**
    - the points on the graph get **further away** from the **specified coordinate axis**
  - For scale factors **between 0 and 1**
    - the points on the graph get **closer** to the **specified coordinate axis**
    - This is also sometimes called a **compression** but in your exam you must use the term **stretch** with the appropriate scale factor



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#### What effects do horizontal stretches have on the graphs and functions?

- A **horizontal stretch** of the graph  $y = f(x)$  by a scale factor  $q$  centred about the  $y$ -axis is represented by



$$y = f\left(\frac{x}{q}\right)$$

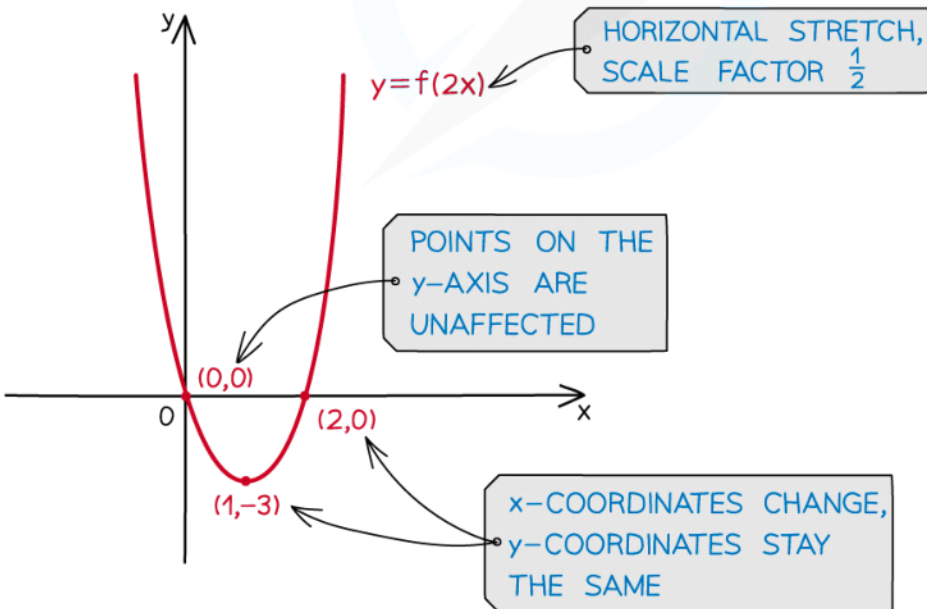
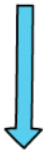
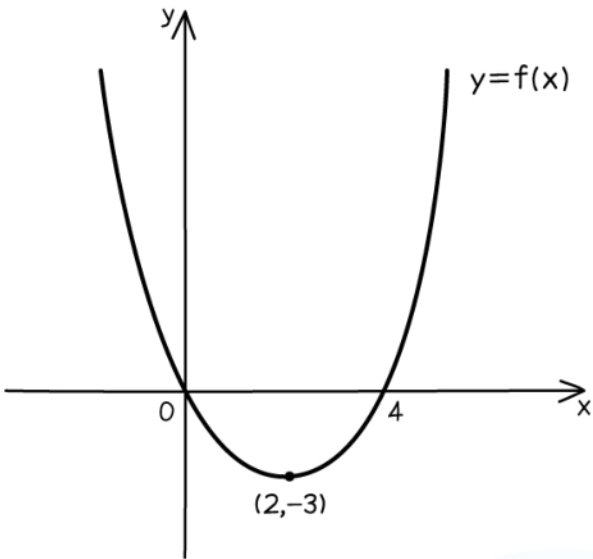
- The **x-coordinates change**
  - They are **divided** by  $q$
- The **y-coordinates stay the same**
- The coordinates  $(x, y)$  become  $(qx, y)$
- **Horizontal** asymptotes **stay the same**
- **Vertical** asymptotes **change**
  - $x = k$  becomes  $x = qk$



Your notes



Your notes



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## What effects do vertical stretches have on the graphs and functions?

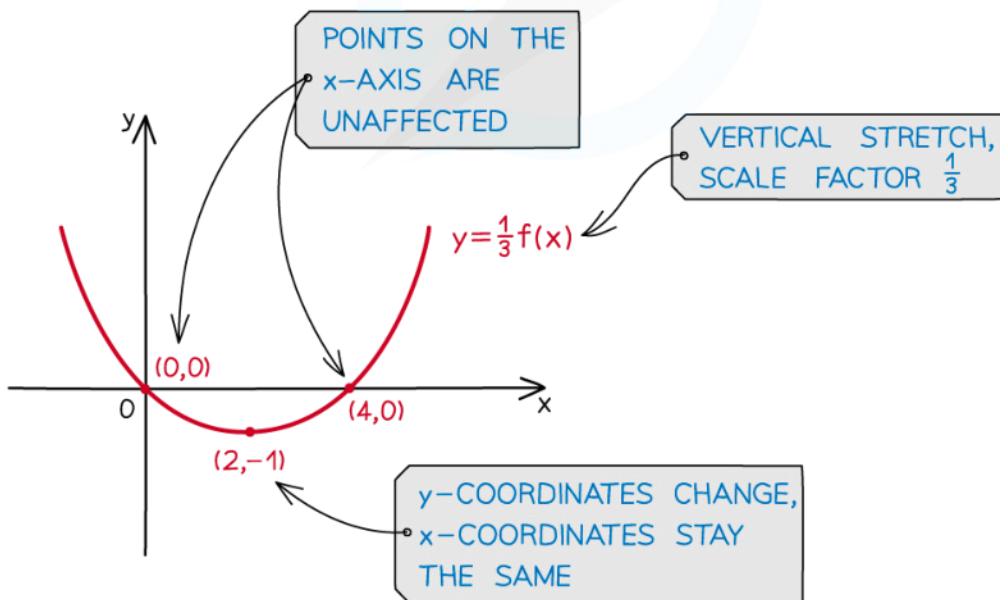
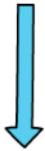
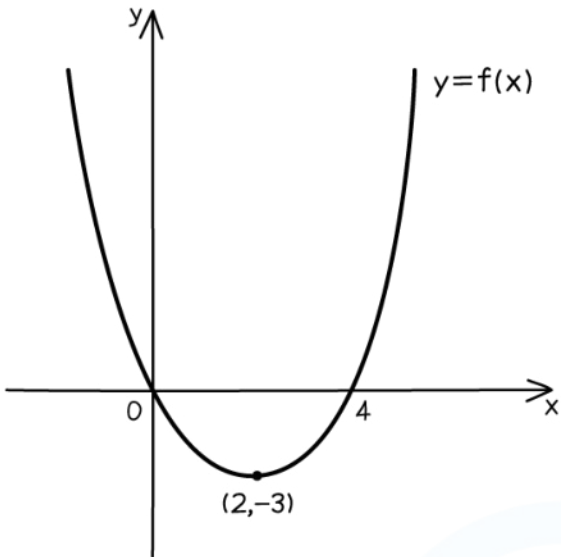
- A **vertical stretch** of the graph  $y = f(x)$  by a scale factor  $p$  centred about the  $x$ -axis is represented by
  - $\frac{y}{p} = f(x)$ 
    - This is often rearranged to  $y = pf(x)$
  - The  **$x$ -coordinates stay the same**
  - The  **$y$ -coordinates change**
    - They are **multiplied** by  $p$
  - The coordinates  $(x, y)$  become  $(x, py)$
  - **Horizontal asymptotes change**
    - $y = k$  becomes  $y = pk$
  - **Vertical asymptotes stay the same**



Your notes



Your notes



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### Examiner Tip

- To get full marks in an exam make sure you use correct mathematical terminology
  - For example: Stretch vertically by scale factor  $\frac{1}{2}$
  - Do not use the word "compress" in your exam



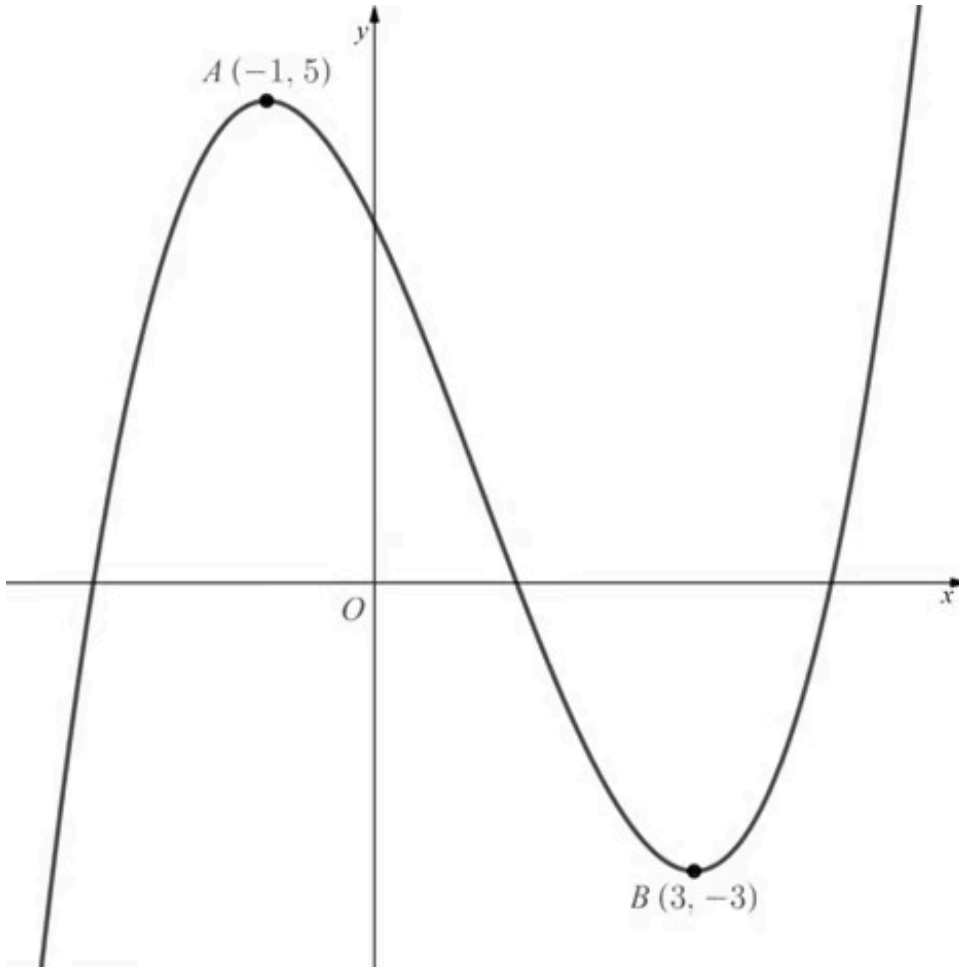
Your notes



Your notes

 **Worked example**

The diagram below shows the graph of  $y = f(x)$ .



- a) Sketch the graph of  $y = 2f(x)$ .



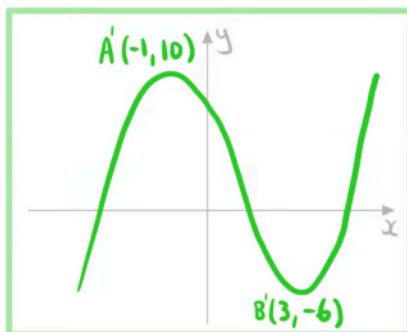
Your notes

$y = kf(x)$  vertical stretch scale factor  $k$

Stretch  $y = f(x)$  vertically  
scale factor 2

A becomes  $(-1, 10)$

B becomes  $(3, -6)$



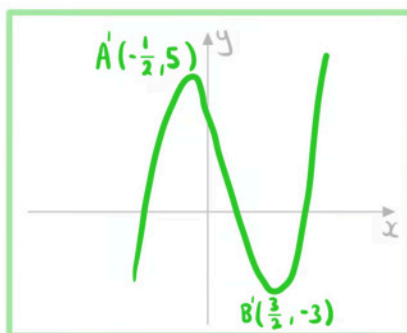
b) Sketch the graph of  $y = f(2x)$ .

$y = f(kx)$  horizontal stretch scale factor  $\frac{1}{k}$

Stretch  $y = f(x)$  horizontally  
scale factor  $\frac{1}{2}$

A becomes  $(-\frac{1}{2}, 5)$

B becomes  $(\frac{3}{2}, -3)$





Your notes

## 2.5.4 Composite Transformations of Graphs

### Composite Transformations of Graphs

#### What transformations do I need to know?

- $y = f(x + k)$  is **horizontal translation** by vector  $\begin{pmatrix} -k \\ 0 \end{pmatrix}$ 
  - If  $k$  is **positive** then the graph moves **left**
  - If  $k$  is **negative** then the graph moves **right**
- $y = f(x) + k$  is **vertical translation** by vector  $\begin{pmatrix} 0 \\ k \end{pmatrix}$ 
  - If  $k$  is **positive** then the graph moves **up**
  - If  $k$  is **negative** then the graph moves **down**
- $y = f(kx)$  is a **horizontal stretch** by scale factor  $\frac{1}{k}$  centred about the  $y$ -axis
  - If  $k > 1$  then the graph gets **closer** to the  $y$ -axis
  - If  $0 < k < 1$  then the graph gets **further** from the  $y$ -axis
- $y = kf(x)$  is a **vertical stretch** by scale factor  $k$  centred about the  $x$ -axis
  - If  $k > 1$  then the graph gets **further** from the  $x$ -axis
  - If  $0 < k < 1$  then the graph gets **closer** to the  $x$ -axis
- $y = f(-x)$  is a **horizontal reflection** about the  $y$ -axis
  - A **horizontal reflection** can be viewed as a special case of a **horizontal stretch**
- $y = -f(x)$  is a **vertical reflection** about the  $x$ -axis
  - A **vertical reflection** can be viewed as a special case of a **vertical stretch**

#### How do horizontal and vertical transformations affect each other?

- **Horizontal and vertical transformations** are **independent** of each other
  - The horizontal transformations involved will need to be applied in their correct order
  - The vertical transformations involved will need to be applied in their correct order
- Suppose there are **two horizontal** transformation  $H_1$  then  $H_2$  and **two vertical** transformations  $V_1$  then  $V_2$  then they can be applied in the following orders:
  - Horizontal then vertical:
    - $H_1 H_2 V_1 V_2$
  - Vertical then horizontal:
    - $V_1 V_2 H_1 H_2$
  - Mixed up (provided that  $H_1$  comes before  $H_2$  and  $V_1$  comes before  $V_2$ ):
    - $H_1 V_1 H_2 V_2$
    - $H_1 V_1 V_2 H_2$
    - $V_1 H_1 V_2 H_2$



- $V_1H_1H_2V_2$

 **Examiner Tip**

- In an exam you are more likely to get the correct solution if you deal with one transformation at a time and sketch the graph after each transformation



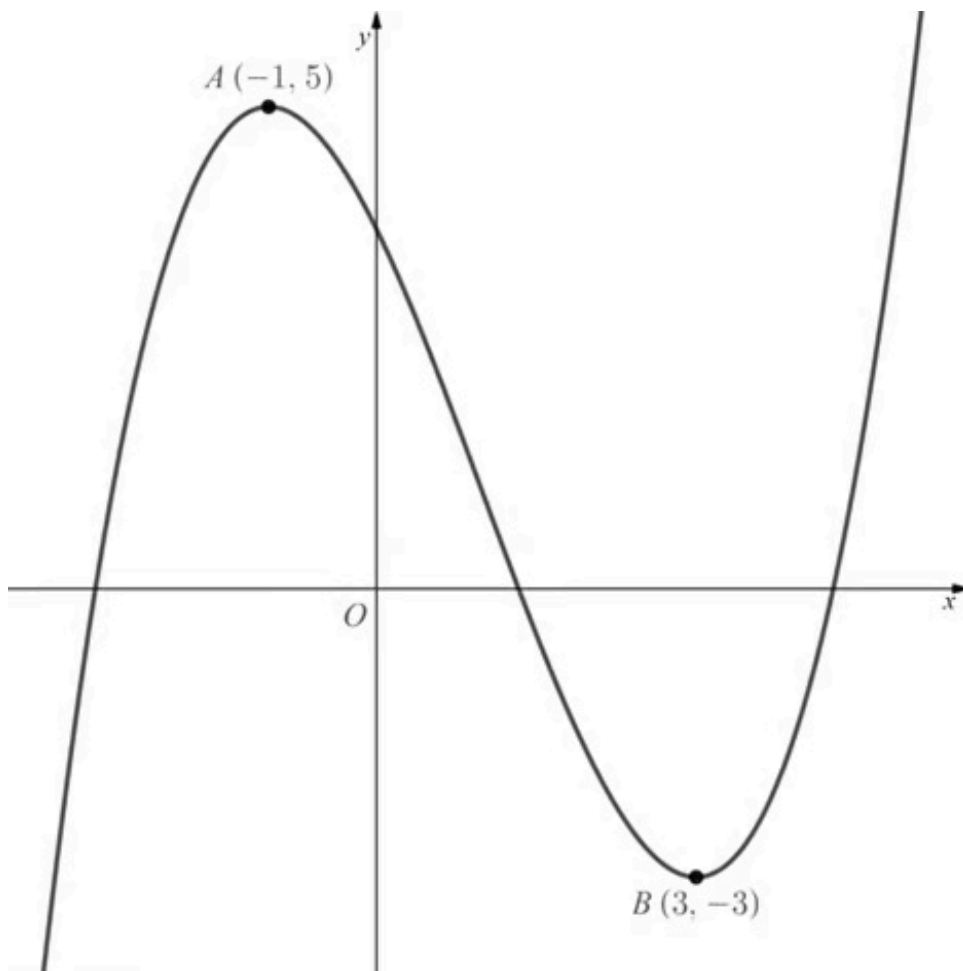
Your notes



Your notes

**Worked example**

The diagram below shows the graph of  $y = f(x)$ .



Sketch the graph of  $y = \frac{1}{2}f\left(\frac{x}{2}\right)$ .



Your notes

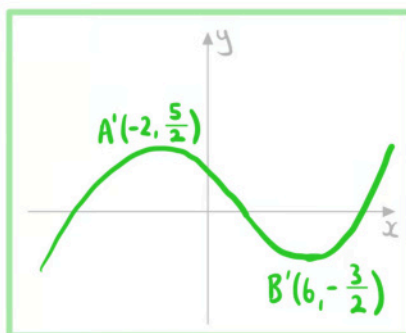
A vertical and horizontal transformation can be done in any order

$y = \frac{1}{2}f(x)$  : vertical stretch scale factor  $\frac{1}{2}$

$y = f\left(\frac{x}{2}\right)$  : horizontal stretch scale factor 2

A becomes  $\left(-2, \frac{5}{2}\right)$

B becomes  $\left(6, -\frac{3}{2}\right)$





Your notes

## Composite Vertical Transformations $af(x)+b$

### How do I deal with multiple vertical transformations?

- **Order matters** when you have **more than one vertical transformations**
- If you are asked to find the equation then **build up the equation** by looking at the transformations in order

- A **vertical stretch** by scale factor  $a$  followed by a **translation** of  $\begin{pmatrix} 0 \\ b \end{pmatrix}$

- Stretch:  $y = af(x)$
- Then translation:  $y = [af(x)] + b$
- Final equation:  $y = af(x) + b$

- A **translation** of  $\begin{pmatrix} 0 \\ b \end{pmatrix}$  followed by a **vertical stretch** by scale factor  $a$

- Translation:  $y = f(x) + b$
- Then stretch:  $y = a[f(x) + b]$
- Final equation:  $y = af(x) + ab$

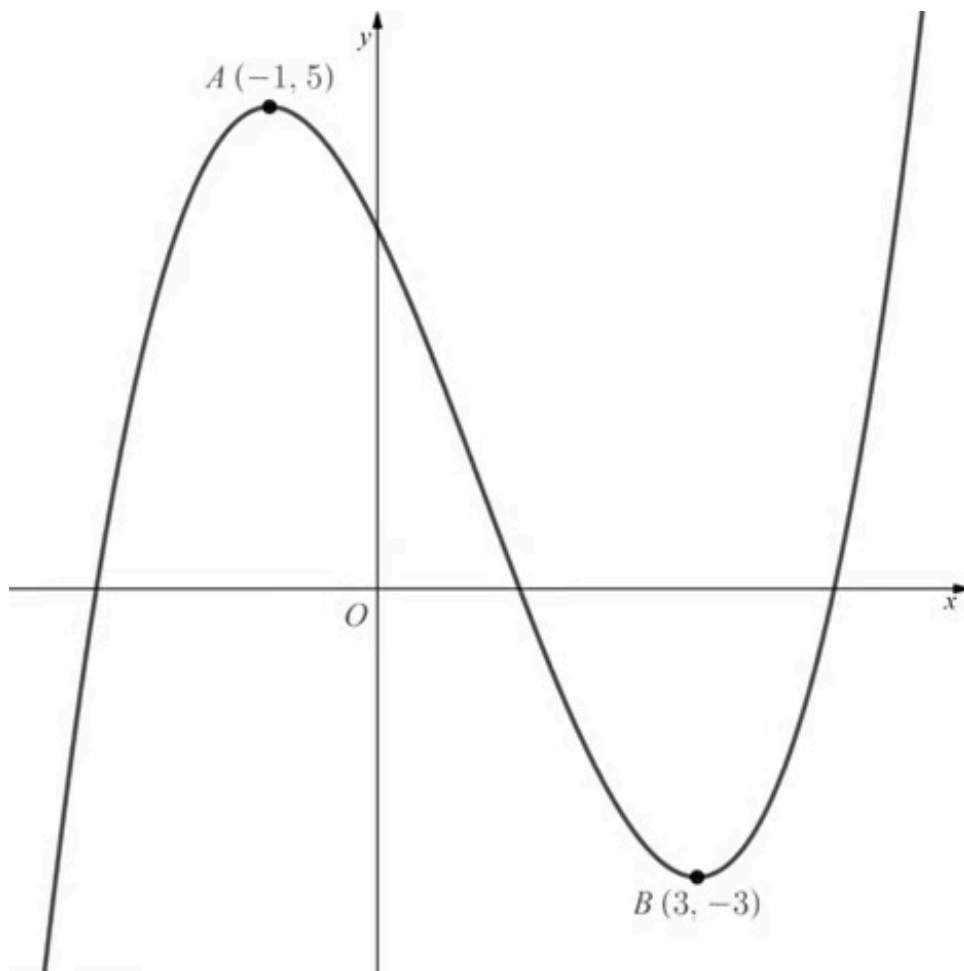
- If you are asked to determine the **order**
  - The order of vertical transformations **follows the order of operations**
  - First write the equation in the form  $y = af(x) + b$ 
    - **First stretch vertically** by scale factor  $a$
    - If  $a$  is negative then the **reflection and stretch** can be **done in any order**
  - **Then translate** by  $\begin{pmatrix} 0 \\ b \end{pmatrix}$



Your notes

 **Worked example**

The diagram below shows the graph of  $y = f(x)$ .



Sketch the graph of  $y = 3f(x) - 2$ .



Your notes

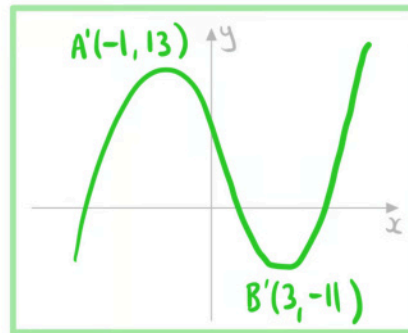
The order vertical transformations follows the order of operations

$y = 3f(x)$ : Vertical stretch scale factor 3

$y = f(x) - 2$ : Translate  $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$

A becomes  $(-1, 13)$

B becomes  $(3, -11)$





Your notes

## Composite Horizontal Transformations $f(ax+b)$

### How do I deal with multiple horizontal transformations?

- **Order matters** when you have **more than one horizontal transformations**
- If you are asked to find the equation then **build up the equation** by looking at the transformations in order

- A **horizontal stretch** by scale factor  $\frac{1}{a}$  followed by a **translation** of  $\begin{pmatrix} -b \\ 0 \end{pmatrix}$

- Stretch:  $y = f(ax)$
- Then translation:  $y = f(a(x + b))$
- Final equation:  $y = f(ax + ab)$

- A **translation** of  $\begin{pmatrix} -b \\ 0 \end{pmatrix}$  followed by a **horizontal stretch** by scale factor  $\frac{1}{a}$

- Translation:  $y = f(x + b)$
- Then stretch:  $y = f((ax) + b)$
- Final equation:  $y = f(ax + b)$

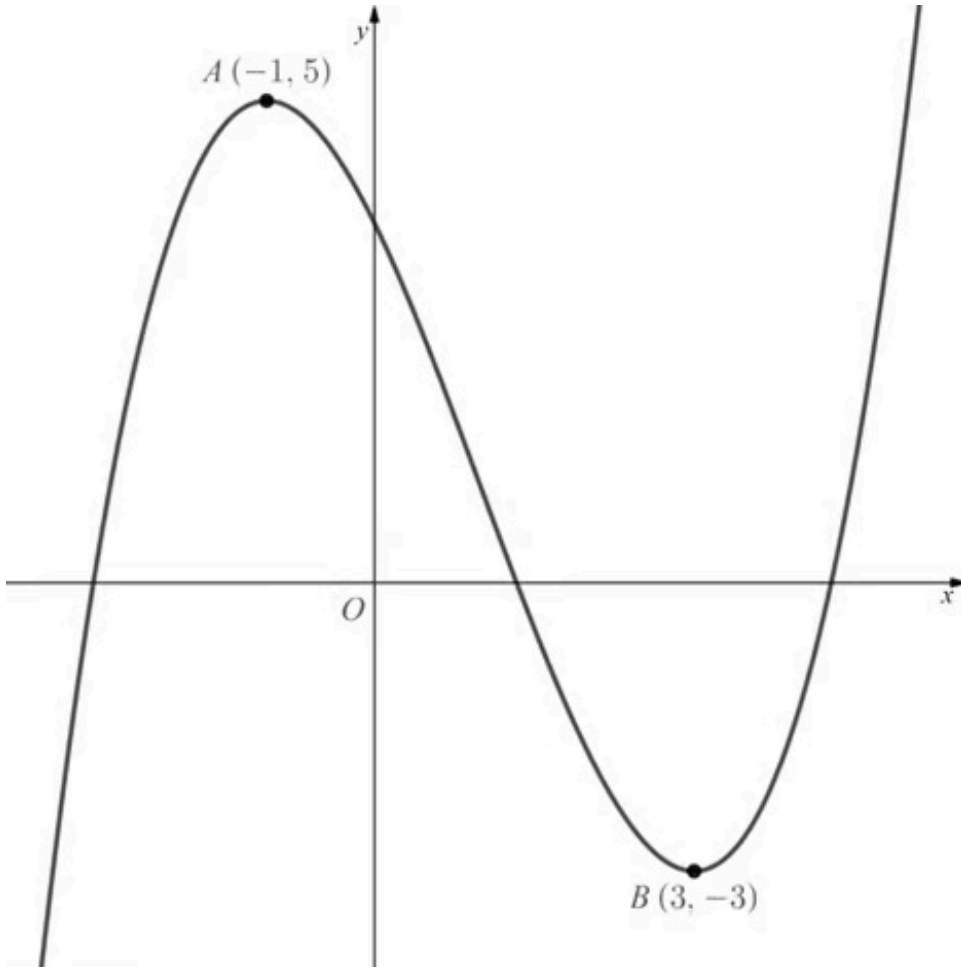
- If you are asked to determine the **order**
  - First write the equation in the form  $y = f(ax + b)$
  - The order of horizontal transformations **is the reverse of the order of operations**
    - **First translate** by  $\begin{pmatrix} -b \\ 0 \end{pmatrix}$
    - **Then stretch** by scale factor  $\frac{1}{a}$
    - If  $a$  is negative then the **reflection and stretch** can be **done in any order**



Your notes

 **Worked example**

The diagram below shows the graph of  $y = f(x)$ .



Sketch the graph of  $y = f(2x - 1)$ .





Your notes

The order of horizontal transformations is the reverse of the order of operations

$y = f(x-1)$ : Translate  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$y = f(2x)$ : Horizontal stretch scale factor  $\frac{1}{2}$

A becomes  $(0, 5)$

B becomes  $(2, -3)$

