

DP IB Maths: AA SL



Your notes

3.6 Trigonometric Equations & Identities

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3.6.1 Simple Identities

Simple Identities

What is a trigonometric identity?

- Trigonometric identities are statements that are true for all values of x or θ
- They are used to help simplify trigonometric equations before solving them
- Sometimes you may see identities written with the symbol \equiv
 - This means 'identical to'

What trigonometric identities do I need to know?

- The two trigonometric identities you must know are
 - $\tan \theta = \frac{\sin \theta}{\cos \theta}$
 - This is the identity for $\tan \theta$
 - $\sin^2 \theta + \cos^2 \theta = 1$
 - This is the Pythagorean identity
 - Note that the notation $\sin^2 \theta$ is the same as $(\sin \theta)^2$
- Both identities can be found **in the formula booklet**
- Rearranging the second identity often makes it easier to work with
 - $\sin^2 \theta = 1 - \cos^2 \theta$
 - $\cos^2 \theta = 1 - \sin^2 \theta$

Where do the trigonometric identities come from?

- You do not need to know the proof for these identities but it is a good idea to know where they come from
- From SOHCAHTOA we know that
 - $\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{O}{H}$
 - $\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{A}{H}$
 - $\tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{O}{A}$
- The identity for $\tan \theta$ can be seen by dividing $\sin \theta$ by $\cos \theta$
 - $\frac{\sin \theta}{\cos \theta} = \frac{\frac{O}{H}}{\frac{A}{H}} = \frac{O}{A} = \tan \theta$
- This can also be seen from the unit circle by considering a right-triangle with a hypotenuse of 1

$$\tan \theta = \frac{O}{A} = \frac{\sin \theta}{\cos \theta}$$

- The Pythagorean identity can be seen by considering a right-triangle with a hypotenuse of 1
 - Then $(\text{opposite})^2 + (\text{adjacent})^2 = 1$
 - Therefore $\sin^2 \theta + \cos^2 \theta = 1$
- Considering the equation of the unit circle also shows the Pythagorean identity
 - The equation of the unit circle is $x^2 + y^2 = 1$
 - The coordinates on the unit circle are $(\cos \theta, \sin \theta)$
 - Therefore the equation of the unit circle could be written $\cos^2 \theta + \sin^2 \theta = 1$
- A third very useful identity is $\sin \theta = \cos (90^\circ - \theta)$ or $\sin \theta = \cos \left(\frac{\pi}{2} - \theta\right)$
 - This is not included in the formula booklet but is useful to remember

How are the trigonometric identities used?

- Most commonly trigonometric identities are used to change an equation into a form that allows it to be solved
- They can also be used to prove further identities such as the **double angle formulae**

Examiner Tip

- If you are asked to show that one thing is identical (\equiv) to another, look at what parts are missing – for example, if $\tan x$ has gone it must have been substituted



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Worked example

Show that the equation $2\sin^2 x - \cos x = 0$ can be written in the form $a\cos^2 x + b\cos x + c = 0$, where a , b and c are integers to be found.

$$2\sin^2 x - \cos x = 0$$

Equation has both $\sin x$ and $\cos x$ so will need changing before it can be solved.

Use the identity $\sin^2 x = 1 - \cos^2 x$

$$\text{Substitute: } 2(1 - \cos^2 x) - \cos x = 0$$

$$\text{Expand: } 2 - 2\cos^2 x - \cos x = 0$$

$$\text{Rearrange: } 2\cos^2 x + \cos x - 2 = 0$$

$$\boxed{a = 2, b = 1, c = -2}$$



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3.6.2 Double Angle Formulae

Double Angle Formulae

What are the double angle formulae?

- The **double angle formulae** for **sine** and **cosine** are:
 - $\sin 2\theta = 2\sin \theta \cos \theta$
 - $\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2\cos^2 \theta - 1 = 1 - 2\sin^2 \theta$
- These can be found in the formula booklet
 - You do not need to know how to derive them

How are the double angle formulae used?

- Double angle formulae will often be used with...
 - ... trigonometry exact values
 - ... graphs of trigonometric functions
 - ... relationships between trigonometric ratios
- To help solve trigonometric equations which contain $\sin \theta \cos \theta$:
 - Substitute $\frac{1}{2} \sin 2\theta$ for $\sin \theta \cos \theta$
 - Solve for 2θ , finding all values in the range for 2θ
 - The range will need adapting for 2θ
 - Find the solutions for θ
- To help solve trigonometric equations which contain $\sin 2\theta$ and $\sin \theta$ or $\cos \theta$
 - Substitute $2\sin \theta \cos \theta$ for $\sin 2\theta$
 - Isolate all terms in θ
 - Factorise or use another identity to write the equation in a form which can be solved
- To help solve trigonometric equations which contain $\cos 2\theta$ and $\sin \theta$ or $\cos \theta$
 - Substitute either $2\cos^2 \theta - 1$ or $1 - 2\sin^2 \theta$ for $\cos 2\theta$
 - Choose the trigonometric ratio that is already in the equation
 - Isolate all terms in θ
 - Solve
 - The equation will most likely be in the form of a quadratic
- Double angle formulae can be used in proving other trigonometric identities

Examiner Tip

- All these formulae are in the **Topic 3: Geometry and Trigonometry** section of the formula booklet
- If you are asked to show that one thing is identical (\equiv) to another, look at what parts are missing – for example, if $\sin\theta$ has disappeared you may want to choose the equivalent expression for $\cos 2\theta$ that does not include $\sin\theta$

Worked example

Without using a calculator, solve the equation $\sin 2\theta = \sin \theta$ for $0^\circ \leq \theta \leq 360^\circ$. Show all working clearly.

$$\text{Double angle identity: } \sin 2\theta = 2\sin\theta\cos\theta$$

$$2\sin\theta\cos\theta = \sin\theta$$

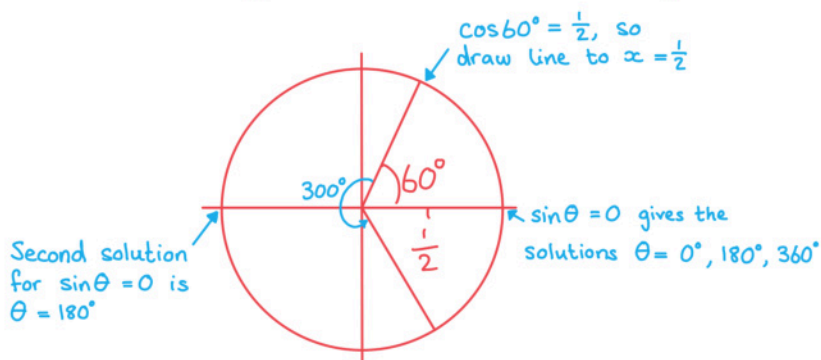
Bring both identities to one side:

$$2\sin\theta\cos\theta - \sin\theta = 0$$

$$\text{Factorise: } \sin\theta(2\cos\theta - 1) = 0$$

$$\text{Find solutions: } \begin{array}{l} \sin\theta = 0 \\ \theta = 0 \end{array} \quad \begin{array}{l} 2\cos\theta - 1 = 0 \\ \cos\theta = \frac{1}{2} \\ \theta = 60^\circ \end{array}$$

Find secondary values within range:



$$\theta = 0^\circ, 60^\circ, 180^\circ, 300^\circ, 360^\circ$$



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3.6.3 Relationship Between Trigonometric Ratios

Relationship Between Trigonometric Ratios

What relationships between trigonometric ratios should I know?

- If you know a value for one trig ratio you can often use this to work out the value for the others without needing to find θ
- If you know that $\sin \theta = \frac{a}{b}$, where $a, b \in \mathbb{Z}^+$, you can:
 - Sketch a right-triangle with a opposite θ and b on the hypotenuse
 - Use Pythagoras' theorem to find the value of the adjacent side
 - Use SOHCAHTOA to find the values of $\cos \theta$ and $\tan \theta$
- If you know a value for $\sin \theta$ or $\cos \theta$ you can use the Pythagorean relationship
 - $\sin^2 \theta + \cos^2 \theta = 1$
 - to find the value of the other
- If you know a value for $\sin \theta$ or $\cos \theta$ you can use the double angle formulae to find the value of $\sin 2\theta$ or $\cos 2\theta$
- If you know two out of the three values for $\sin \theta$, $\cos \theta$ or $\tan \theta$ you can use the identity in \tan
 - $\tan \theta = \frac{\sin \theta}{\cos \theta}$
 - to find the value of the third ratio

How do we determine whether a trigonometric ratio will be positive or negative?

- It is possible to determine whether a trigonometric ratio will be positive or negative by looking at the size of the angle and considering the **unit circle**
 - Angles in the range $0^\circ < \theta < 90^\circ$ will be positive for all three ratios
 - Angles in the range $90^\circ < \theta < 180^\circ$ will be positive for \sin and negative for \cos and \tan
 - Angles in the range $180^\circ < \theta < 270^\circ$ will be positive for \tan and negative for \sin and \cos
 - Angles in the range $270^\circ < \theta < 360^\circ$ will be positive for \cos and negative for \sin and \tan
- The ratios for angles of 0° , 90° , 180° , 270° and 360° are either 0, 1, -1 or undefined
 - You should know these ratios or know how to derive them without a calculator

Examiner Tip

- Being able to sketch out the unit circle and remembering CAST can help you to find all solutions to a problem in an exam question



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Worked example

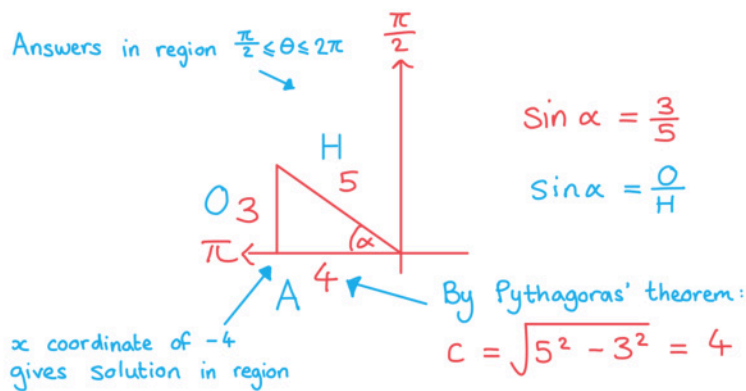
The value of $\sin \alpha = \frac{3}{5}$ for $\frac{\pi}{2} \leq \alpha \leq \pi$. Find:

i) $\cos \alpha$

Method 1: Use right-triangle:

$$\frac{\pi}{2} \leq \alpha \leq \pi$$

Answers in region $\frac{\pi}{2} \leq \theta \leq 2\pi$



$$\sin \alpha = \frac{3}{5}$$

$$\sin \alpha = \frac{O}{H}$$

$$c = \sqrt{5^2 - 3^2} = 4$$

$$\cos \alpha = \frac{A}{H} = -\frac{4}{5}$$

$$\boxed{\cos \alpha = -\frac{4}{5}}$$

Method 2: Use Pythagorean identity:

$$\cos^2 \alpha = 1 - \sin^2 \alpha = 1 - \left(\frac{3}{5}\right)^2$$

$$\cos \alpha = \pm \sqrt{\frac{16}{25}} = \pm \frac{4}{5}$$

Check which solution is in range.

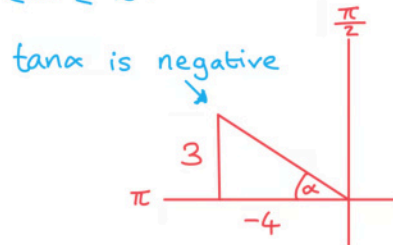
ii) $\tan \alpha$



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$$\text{Use } \tan \alpha = \frac{\sin \alpha}{\cos \alpha} = \frac{\frac{3}{5}}{-\frac{4}{5}} = -\frac{3}{4}$$

Check if $\tan \alpha = -\frac{3}{4}$ is in the correct range for $\frac{\pi}{2} \leq \alpha \leq \pi$:



$$\tan \alpha = -\frac{3}{4}$$

 iii) $\sin 2\alpha$

Double angle identity: $\sin 2\theta = 2\sin \theta \cos \theta$

$$\begin{aligned} \sin 2\alpha &= 2\sin \alpha \cos \alpha \\ &= 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right) \\ &= -\frac{24}{25} \end{aligned}$$

$$\begin{aligned} \sin \alpha &= \frac{3}{5} \\ \cos \alpha &= -\frac{4}{5} \end{aligned}$$

$$\sin 2\alpha = -\frac{24}{25}$$

 iv) $\cos 2\alpha$



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Double angle identity: $\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$

$$\cos 2\alpha = \left(-\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{7}{25}$$

$$\cos 2\alpha = \frac{7}{25}$$

v) $\tan 2\alpha$ Using identity $\tan \theta = \frac{\sin \theta}{\cos \theta}$

$$\tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha} = \frac{-\frac{24}{25}}{\frac{7}{25}} = -\frac{24}{7}$$

$$\tan 2\alpha = -\frac{24}{7}$$



Your notes

3.6.4 Linear Trigonometric Equations

Trigonometric Equations: $\sin x = k$

How are trigonometric equations solved?

- Trigonometric equations can have an infinite number of solutions
 - For an equation in \sin or \cos you can add 360° or 2π to each solution to find more solutions
 - For an equation in \tan you can add 180° or π to each solution
- When solving a trigonometric equation you will be given a range of values within which you should find all the values
- Solving the equation normally and using the inverse function on your calculator or your knowledge of **exact values** will give you the **primary value**
- The **secondary values** can be found with the help of:
 - The **unit circle**
 - The **graphs of trigonometric functions**

How are trigonometric equations of the form $\sin x = k$ solved?

- It is a good idea to sketch the graph of the trigonometric function first
 - Use the given range of values as the domain for your graph
 - The intersections of the graph of the function and the line $y = k$ will show you
 - The location of the solutions
 - The number of solutions
 - You will be able to use the symmetry properties of the graph to find all secondary values within the given range of values
- The method for finding secondary values are:
 - For the equation $\sin x = k$ the primary value is $x_1 = \sin^{-1} k$
 - A secondary value is $x_2 = 180^\circ - \sin^{-1} k$
 - Then all values within the range can be found using $x_1 \pm 360n$ and $x_2 \pm 360n$ where $n \in \mathbb{N}$
 - For the equation $\cos x = k$ the primary value is $x_1 = \cos^{-1} k$
 - A secondary value is $x_2 = -\cos^{-1} k$
 - Then all values within the range can be found using $x_1 \pm 360n$ and $x_2 \pm 360n$ where $n \in \mathbb{N}$
 - For the equation $\tan x = k$ the primary value is $x = \tan^{-1} k$
 - All secondary values within the range can be found using $x \pm 180n$ where $n \in \mathbb{N}$



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Examiner Tip

- If you are using your GDC it will only give you the principal value and you need to find all other solutions for the given interval
- Sketch out the CAST diagram and the trig graphs on your exam paper to refer back to as many times as you need to

Worked example

Solve the equation $2\cos x = -1$, finding all solutions in the range $-\pi \leq x \leq \pi$.

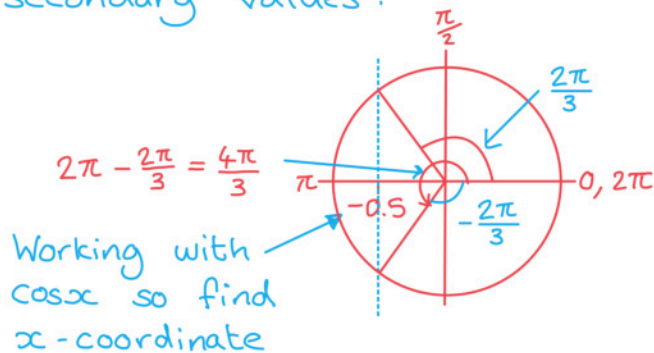
Isolate $\cos x$: $\cos x = \frac{-1}{2}$

use GDC or
knowledge of
exact values

$$x = \cos^{-1}\left(-\frac{1}{2}\right)$$

$$= \frac{2\pi}{3} \leftarrow \text{Primary value}$$

Find secondary values:



$$\frac{2\pi}{3} \pm 2\pi n \quad \text{and} \quad \frac{4\pi}{3} \pm 2\pi n$$

Find all answers in range $-\pi \leq x \leq \pi$

$$\boxed{-\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}}$$

Trigonometric Equations: $\sin(ax + b) = k$

How can I solve equations with transformations of trig functions?

- Trigonometric equations in the form $\sin(ax + b)$ can be solved in more than one way
- The easiest method is to consider the transformation of the angle as a substitution
 - For example let $u = ax + b$
- Transform the given interval for the solutions in the same way as the angle
 - For example if the given interval is $0^\circ \leq x \leq 360^\circ$ the new interval will be $(a(0^\circ) + b) \leq u \leq (a(360^\circ) + b)$
- Solve the function to find the primary value for u
- Use either the unit circle or sketch the graph to find all the other solutions in the range for u
- Undo the substitution to convert all of the solutions back into the corresponding solutions for x
- Another method would be to sketch the transformation of the function
 - If you use this method then you will not need to use a substitution for the range of values

Examiner Tip

- If you transform the interval, remember to convert the found angles back to the final values at the end!
- If you are using your GDC it will only give you the principal value and you need to find all other solutions for the given interval
- Sketch out the CAST diagram and the trig graphs on your exam paper to refer back to as many times as you need to



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Worked example

Solve the equation $2\cos(2x - 30^\circ) = -1$, finding all solutions in the range $-360^\circ \leq x \leq 360^\circ$.

$$2\cos(2x - 30^\circ) = -1 \quad -360^\circ \leq x \leq 360^\circ$$

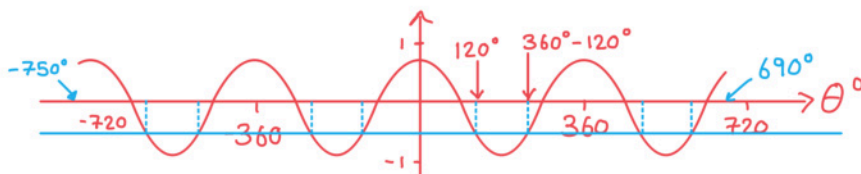
Start by changing the range: $-750^\circ \leq 2x - 30 \leq 690^\circ$

Substitute $\theta = 2x - 30$:

$$2\cos\theta = -1 \quad -750^\circ \leq \theta \leq 690^\circ$$

$$\cos\theta = -\frac{1}{2}$$

$$\theta = \cos^{-1}\left(-\frac{1}{2}\right) = 120^\circ \leftarrow \text{Primary value}$$



From the sketch you can see there are 8 solutions:

$$\theta = 120^\circ \pm 360^\circ \text{ and } \theta = 240^\circ \pm 360^\circ$$

$$\theta = -600^\circ, -480^\circ, -240^\circ, -120^\circ, 120^\circ, 240^\circ, 480^\circ, 600^\circ$$

Solve for x : $x = \frac{\theta + 30}{2}$

$$x = -285^\circ, -225^\circ, -105^\circ, -45^\circ, 75^\circ, 135^\circ, 255^\circ, 315^\circ$$



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3.6.5 Quadratic Trigonometric Equations

Quadratic Trigonometric Equations

How are quadratic trigonometric equations solved?

- A quadratic trigonometric equation is one that includes either $\sin^2 \theta$, $\cos^2 \theta$ or $\tan^2 \theta$
- Often the **identity** $\sin^2 \theta + \cos^2 \theta = 1$ can be used to rearrange the equation into a form that is possible to solve
 - If the equation involves both sine and cosine then the **Pythagorean identity** should be used to write the equation in terms of just one of these functions
- Solve the **quadratic equation** using your GDC, the quadratic equation or factorisation
 - This can be made easier by changing the function to a single letter
 - Such as changing $2\cos^2 \theta - 3\cos \theta - 1 = 0$ to $2c^2 - 3c - 1 = 0$
- A quadratic can give up to two solutions
 - You must consider both solutions to see whether a real value exists
 - Remember that solutions for $\sin \theta = k$ and $\cos \theta = k$ only exist for $-1 \leq k \leq 1$
 - Solutions for $\tan \theta = k$ exist for all values of k
- Find all solutions within the given interval
 - There will often be more than two solutions for one quadratic equation
 - The best way to check the number of solutions is to sketch the graph of the function

Examiner Tip

- Sketch the trig graphs on your exam paper to refer back to as many times as you need to!
- Be careful to make sure you have found **all** of the solutions in the given interval, being super-careful if you get a negative solution but have a positive interval



Your notes

Worked example

Solve the equation $11\sin x - 7 = 5\cos^2 x$, finding all solutions in the range $0 \leq x \leq 2\pi$.

Use the identity $\cos^2 x = 1 - \sin^2 x$ to write equation in terms of $\sin x$:

$$11\sin x - 7 = 5(1 - \sin^2 x) \text{ in formula booklet.}$$

$$= 5 - 5\sin^2 x$$

Move all terms to one side:

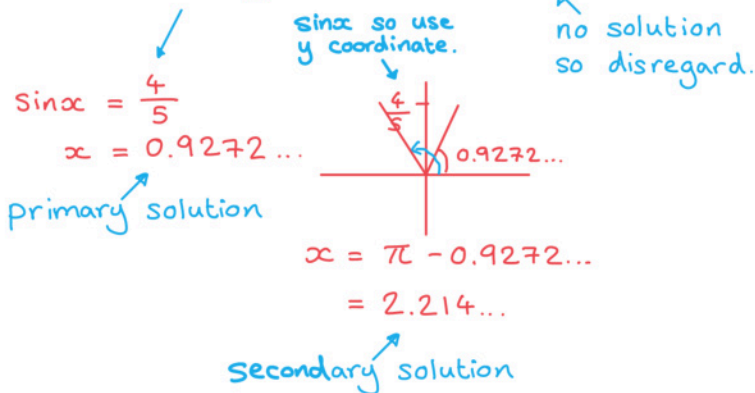
$$11\sin x - 7 - (5 - 5\sin^2 x) = 0$$

Spot the hidden quadratic:

$$11\sin x - 7 - 5 + 5\sin^2 x = 0$$

$$5\sin^2 x + 11\sin x - 12 = 0$$

$$\sin x = \frac{4}{5} \text{ or } \sin x = -3$$



$$x = 0.927, 2.21 \text{ (3 s.f.)}$$