

DP IB Maths: AA SL



Your notes

2.3 Functions Toolkit

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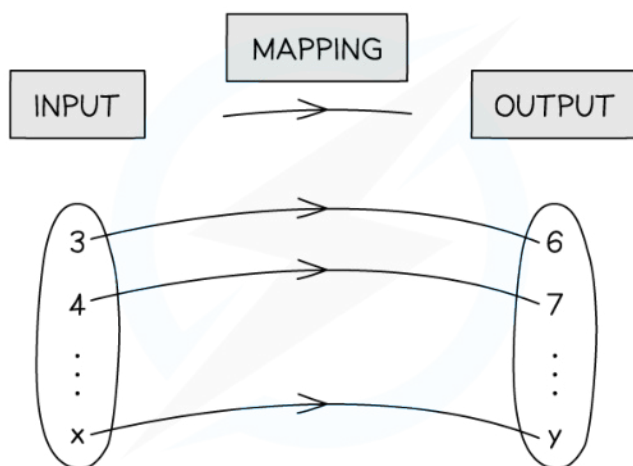
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2.3.1 Language of Functions

Language of Functions

What is a mapping?

- A **mapping transforms** one set of values (**inputs**) into another set of values (**outputs**)
- Mappings can be:
 - **One-to-one**
 - Each input gets mapped to **exactly one unique** output
 - No two inputs are mapped to the same output
 - For example: A mapping that cubes the input
 - **Many-to-one**
 - Each input gets mapped to **exactly one** output
 - Multiple inputs can be mapped to the same output
 - For example: A mapping that squares the input
 - **One-to-many**
 - An input can be mapped to **more than one** output
 - No two inputs are mapped to the same output
 - For example: A mapping that gives the numbers which when squared equal the input
 - **Many-to-many**
 - An input can be mapped to **more than one** output
 - Multiple inputs can be mapped to the same output
 - For example: A mapping that gives the factors of the input



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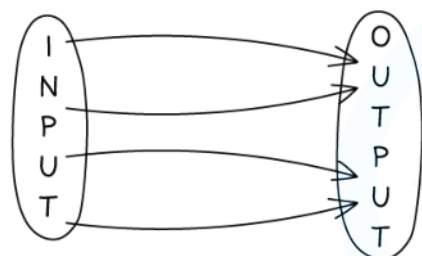


What is a function?

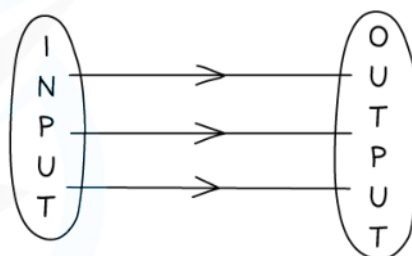


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- A **function** is a mapping between two sets of numbers where **each input** gets mapped to **exactly one output**
 - The output does not need to be unique
- **One-to-one** and **many-to-one** mappings are functions
- A mapping is a function if its graph passes the **vertical line test**
 - Any **vertical line** will intersect with the graph **at most once**



MANY-TO-ONE MAPPINGS ARE FUNCTIONS



ONE-TO-ONE MAPPINGS ARE FUNCTIONS

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What notation is used for functions?

- Functions are denoted using letters (such as f , v , g , etc)
 - A function is followed by a variable in a bracket
 - This shows the input for the function
 - The letter f is used most commonly for functions and will be used for the remainder of this revision note
- $f(x)$ represents an expression for the value of the function f when evaluated for the variable x
- Function notation gets rid of the need for words which makes it **universal**
 - $f = 5$ when $x = 2$ can simply be written as $f(2) = 5$

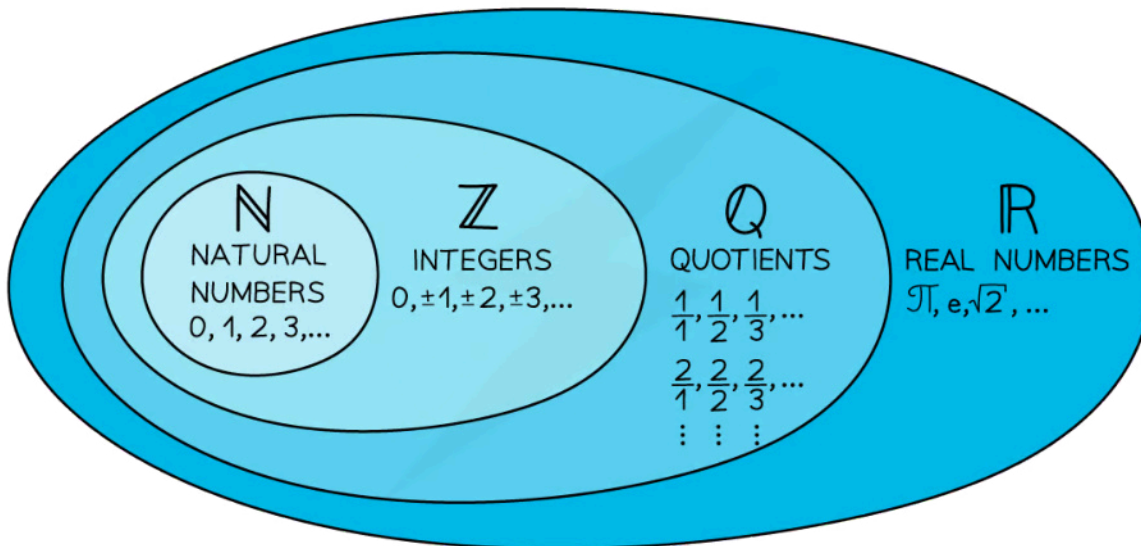
What are the domain and range of a function?

- The **domain** of a function is the set of values that are used as **inputs**
- A domain should be stated with a function
 - If a domain is not stated then it is assumed the domain is all the real values which would work as inputs for the function
 - Domains are expressed in terms of the input
 - $x \leq 2$
- The **range** of a function is the set of values that are given as **outputs**
 - The range depends on the domain
 - Ranges are expressed in terms of the output



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- $f(x) \geq 0$
- To graph a function we use the **inputs as the x-coordinates** and the **outputs as the y-coordinates**
 - $f(2) = 5$ corresponds to the coordinates (2, 5)
- Graphing the function can help you visualise the range
- Common sets of numbers have special symbols:
 - \mathbb{R} represents all the real numbers that can be placed on a number line
 - $x \in \mathbb{R}$ means x is a real number
 - \mathbb{Q} represents all the rational numbers $\frac{a}{b}$ where a and b are integers and $b \neq 0$
 - \mathbb{Z} represents all the integers (positive, negative and zero)
 - \mathbb{Z}^+ represents positive integers
 - \mathbb{N} represents the natural numbers (0,1,2,3...)



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What are piecewise functions?

- **Piecewise functions** are defined by different functions depending on which interval the input is in
 - E.g. $f(x) = \begin{cases} x + 1 & x \leq 5 \\ 2x - 4 & 5 < x < 10 \\ x^2 & 10 \leq x \leq 20 \end{cases}$
- The region for the individual functions **cannot overlap**
- To evaluate a piecewise function for a particular value $x = k$
 - Find which interval includes k
 - Substitute $x = k$ into the corresponding function



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- The function **may or may not be continuous** at the ends of the intervals
 - In the example above the function is
 - continuous at $x = 5$ as $5 + 1 = 2(5) - 4$
 - not continuous at $x = 10$ as $2(10) - 4 \neq 10^2$

Examiner Tip

- Questions may refer to "the largest possible domain"
 - This would usually be $x \in \mathbb{R}$ unless \mathbb{N} , \mathbb{Z} or \mathbb{Q} has already been stated
 - There are usually some exceptions
 - e.g. square roots; $x \geq 0$ for a function involving \sqrt{x}
 - e.g. reciprocal functions; $x \neq 2$ for a function with denominator $(x - 2)$

Worked example

For the function $f(x) = x^3 + 1$, $2 \leq x \leq 10$:

- a) write down the value of $f(7)$.

Substitute $x = 7$

$$f(7) = 7^3 + 1$$

$$f(7) = 344$$

- b) find the range of $f(x)$.

Find the values of $x^3 + 1$ when $2 \leq x \leq 10$

$$2 \leq x \leq 10$$

$$8 \leq x^3 \leq 1000$$

$$9 \leq x^3 + 1 \leq 1001$$

$$9 \leq f(x) \leq 1001$$



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2.3.2 Composite & Inverse Functions

Composite Functions

What is a composite function?

- A **composite function** is where a function is applied to another function
- A composite function can be denoted
 - $(f \circ g)(x)$
 - $fg(x)$
 - $f(g(x))$
- The order matters
 - $(f \circ g)(x)$ means:
 - First apply g to x to get $g(x)$
 - Then apply f to the previous output to get $f(g(x))$
 - Always start with the function **closest to the variable**
 - $(f \circ g)(x)$ is not usually equal to $(g \circ f)(x)$

How do I find the domain and range of a composite function?

- The domain of $f \circ g$ is the set of values of X ...
 - which are a **subset** of the **domain of g**
 - which maps g to a value that is in the **domain of f**
- The range of $f \circ g$ is the set of values of X ...
 - which are a **subset** of the **range of f**
 - found by **applying f** to the **range of g**
- To find the **domain** and **range** of $f \circ g$
 - First find the **range of g**
 - **Restrict** these values to the values that are **within the domain** of f
 - The **domain** is the set of values that **produce the restricted range** of g
 - The **range** is the set of values that are **produced using the restricted range** of g as the domain for f
- For example: let $f(x) = 2x + 1$, $-5 \leq x \leq 5$ and $g(x) = \sqrt{x}$, $1 \leq x \leq 49$
 - The **range of g** is $1 \leq g(x) \leq 7$
 - **Restricting** this to fit the **domain of f** results in $1 \leq g(x) \leq 5$
 - The **domain** of $f \circ g$ is therefore $1 \leq x \leq 25$
 - These are the values of x which map to $1 \leq g(x) \leq 5$
 - The **range** of $f \circ g$ is therefore $3 \leq (f \circ g)(x) \leq 11$
 - These are the values which f maps $1 \leq g(x) \leq 5$ to

Examiner Tip

- Make sure you know what your GDC is capable of with regard to functions
 - You may be able to store individual functions and find composite functions and their values for particular inputs
 - You may be able to graph composite functions directly and so deduce their domain and range from the graph
- $ff(x)$ is not the same as $[f(x)]^2$



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Worked example

Given $f(x) = \sqrt{x+4}$ and $g(x) = 3 + 2x$:

a) Write down the value of $(g \circ f)(12)$.

First apply function closest to input

$$(g \circ f)(12) = g(f(12))$$

$$f(12) = \sqrt{12+4} = \sqrt{16} = 4$$

$$g(4) = 3 + 2(4) = 11$$

$$(g \circ f)(12) = 11$$

b) Write down an expression for $(f \circ g)(x)$.

First apply function closest to input

$$(f \circ g)(x) = f(g(x))$$

$$= f(3+2x)$$

$$= \sqrt{3+2x+4}$$

$$(f \circ g)(x) = \sqrt{7+2x}$$

c) Write down an expression for $(g \circ g)(x)$.

$$(g \circ g)(x) = g(g(x))$$

$$= g(3+2x)$$

$$= 3 + 2(3+2x)$$

$$= 3 + 6 + 4x$$

$$(g \circ g)(x) = 9 + 4x$$



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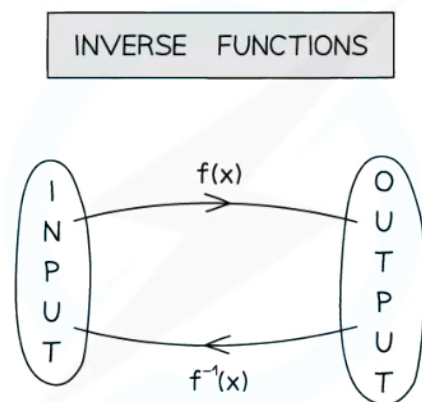


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Inverse Functions

What is an inverse function?

- Only **one-to-one** functions have inverses
- A function has an inverse if its graph passes the **horizontal line test**
 - Any **horizontal line** will intersect with the graph **at most once**
- The **identity function** id maps each value to itself
 - $\text{id}(x) = x$
- If $f \circ g$ and $g \circ f$ have the **same effect as the identity function** then f and g are **inverses**
- Given a function $f(x)$ we denote the **inverse function** as $f^{-1}(x)$
- An inverse function **reverses the effect** of a function
 - $f(2) = 5$ means $f^{-1}(5) = 2$
- Inverse functions are used to solve equations
 - The solution of $f(x) = 5$ is $x = f^{-1}(5)$
- A composite function made of f and f^{-1} has the **same effect as the identity function**
 - $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$



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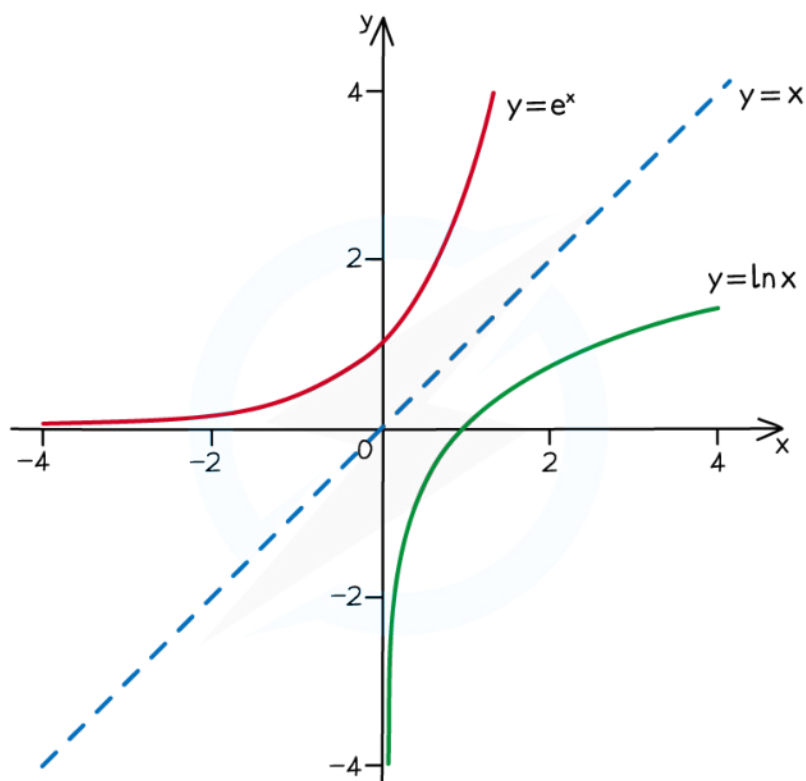


What are the connections between a function and its inverse function?

- The **domain of a function** becomes the **range of its inverse**
- The **range of a function** becomes the **domain of its inverse**
- The graph of $y = f^{-1}(x)$ is a **reflection** of the graph $y = f(x)$ in the line $y = x$
 - Therefore solutions to $f(x) = x$ or $f^{-1}(x) = x$ will also be solutions to $f(x) = f^{-1}(x)$
 - There could be other solutions to $f(x) = f^{-1}(x)$ that don't lie on the line $y = x$



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How do I find the inverse of a function?

- STEP 1: **Swap** the x and y in $y = f(x)$
 - If $y = f^{-1}(x)$ then $x = f(y)$
- STEP 2: **Rearrange** $x = f(y)$ to make y the subject
- Note this can be done in any order
 - Rearrange $y = f(x)$ to make x the subject
 - Swap x and y

Examiner Tip

- Remember that an inverse function is a reflection of the original function in the line $y = x$
 - Use your GDC to plot the function and its inverse on the same graph to visually check this
- $f^{-1}(x)$ is not the same as $\frac{1}{f(x)}$



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Worked example

For the function $f(x) = \frac{2x}{x-1}$, $x > 1$:

a) Find the inverse of $f(x)$.

Let $y = f^{-1}(x)$ and rearrange $x = f(y)$

$$x = \frac{2y}{y-1}$$

$$x(y-1) = 2y$$

$$xy - x = 2y$$

$$xy - 2y = x$$

$$y(x-2) = x$$

$$y = \frac{x}{x-2}$$

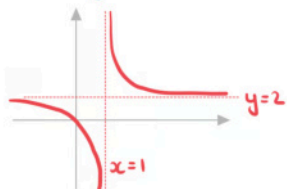
$$f^{-1}(x) = \frac{x}{x-2}$$

b) Find the domain of $f^{-1}(x)$.

Domain of f^{-1} is the range of f

Sketch $y = f(x)$ to see range

For $x > 1$, $f(x) > 2$



$$\text{Domain of } f^{-1} : x > 2$$

c) Find the value of k such that $f(k) = 6$.

Use inverse $f(a) = b \Leftrightarrow a = f^{-1}(b)$

$$k = f^{-1}(6) = \frac{6}{6-2}$$

$$k = \frac{3}{2}$$



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2.3.3 Graphing Functions

Graphing Functions

How do I graph the function $y = f(x)$?

- A point (a, b) lies on the graph $y = f(x)$ if $f(a) = b$
- The **horizontal axis** is used for the **domain**
- The **vertical axis** is used for the **range**
- You will be able to graph some functions by hand
- For some functions you will need to use your GDC
- You might be asked to graph the **sum** or **difference** of two functions
 - Use your GDC to graph $y = f(x) + g(x)$ or $y = f(x) - g(x)$
 - Just type the functions into the graphing mode

What is the difference between “draw” and “sketch”?

- If asked to sketch you should:
 - Show the general shape
 - Label any key points such as the intersections with the axes
 - Label the axes
- If asked to draw you should:
 - Use a pencil and ruler
 - Draw to scale
 - Plot any points **accurately**
 - Join points with a straight line or smooth curve
 - Label any key points such as the intersections with the axes
 - Label the axes

How can my GDC help me sketch/draw a graph?

- You use your GDC to plot the graph
 - Check the scales on the graph to make sure you see the full shape
- Use your GDC to find any key points
- Use your GDC to check specific points to help you plot the graph

Key Features of Graphs

What are the key features of graphs?

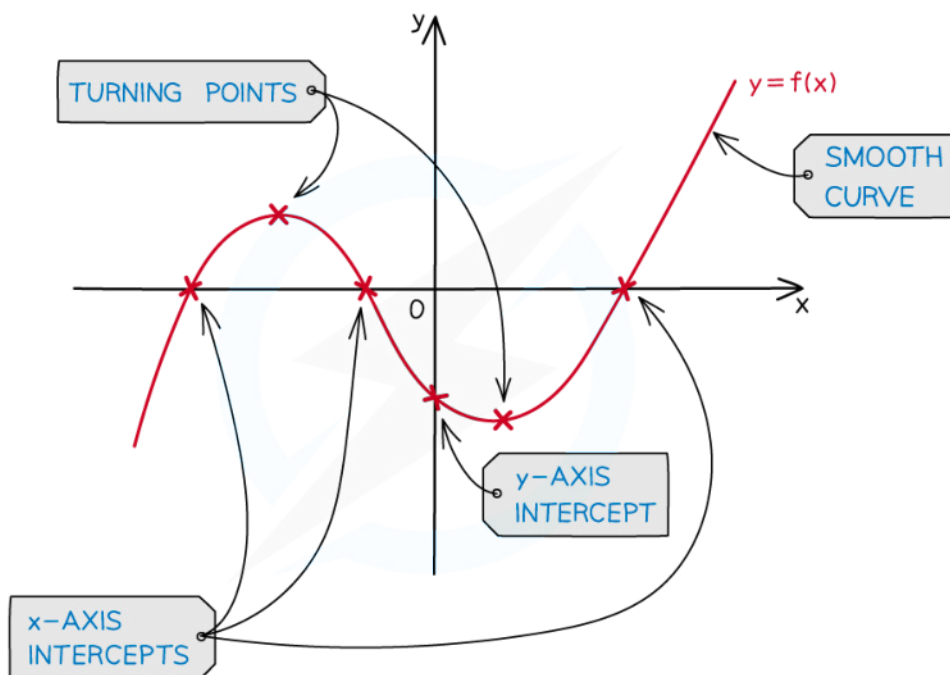
- You should be familiar with the following key features and know how to use your GDC to find them
- Local minimums/maximums
 - These are points where the graph has a minimum/maximum for a small region
 - They are also called **turning points**
 - This is where the graph changes its direction between upwards and downwards directions
 - A graph can have multiple local minimums/maximums
 - A local minimum/maximum is not necessarily the minimum/maximum of the whole graph
 - This would be called the **global** minimum/maximum
 - For quadratic graphs the minimum/maximum is called the **vertex**
- Intercepts
 - y - intercepts are where the graph crosses the y -axis
 - At these points $x = 0$
 - x - intercepts are where the graph crosses the x -axis
 - At these points $y = 0$
 - These points are also called the **zeros of the function** or **roots of the equation**
- Symmetry
 - Some graphs have lines of symmetry
 - A quadratic will have a vertical line of symmetry
- Asymptotes
 - These are lines which the graph will get closer to but not cross
 - These can be horizontal or vertical
 - Exponential graphs have horizontal asymptotes
 - Graphs of variables which vary inversely can have vertical and horizontal asymptotes



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Examiner Tip

- Most GDC makes/models will not plot/show asymptotes just from inputting a function
 - Add the asymptotes as additional graphs for your GDC to plot
 - You can then check the equations of your asymptotes visually
 - You may have to zoom in or change the viewing window options to confirm an asymptote
- Even if using your GDC to plot graphs and solve problems sketching them as part of your working is good exam technique
 - Label the key features of the graph and anything else relevant to the question on your sketch



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Worked example

Two functions are defined by

$$f(x) = x^2 - 4x - 5 \text{ and } g(x) = 2 + \frac{1}{x+1}.$$

a) Draw the graph $y = f(x)$.

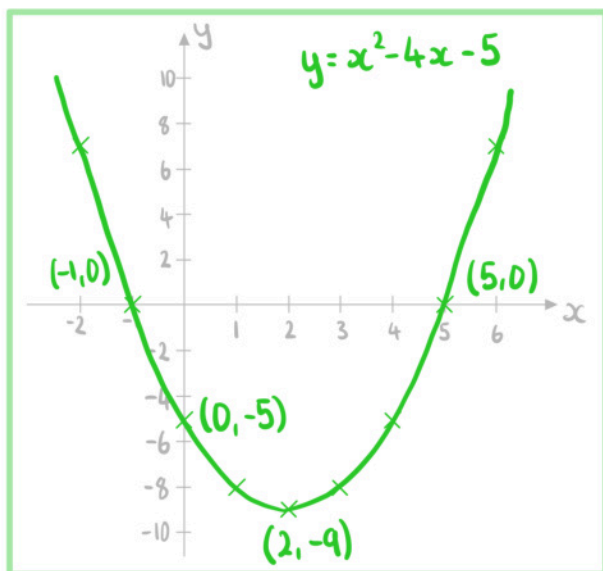
Draw means accurately

Use GDC to find vertex, roots and y-intercepts

Vertex = $(2, -9)$

Roots = $(-1, 0)$ and $(5, 0)$

y-intercept = $(0, -5)$



b) Sketch the graph $y = g(x)$.



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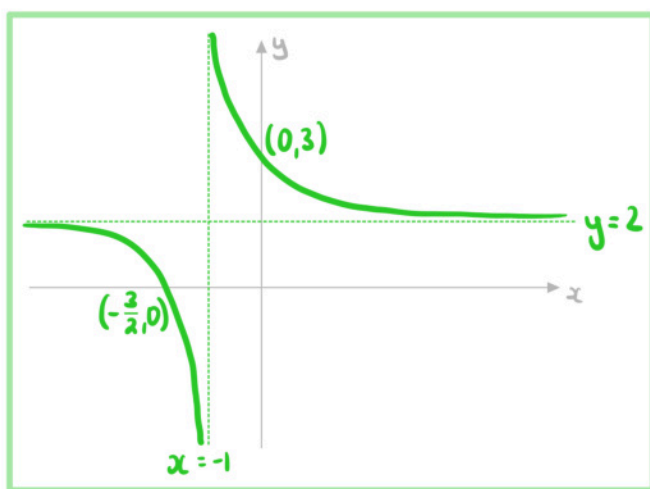
Sketch means rough but showing key points

Use GDC to find x and y -intercepts and asymptotes

$$x\text{-intercept} = \left(-\frac{3}{2}, 0\right)$$

$$y\text{-intercept} = (0, 3)$$

Asymptotes : $x = -1$ and $y = 2$



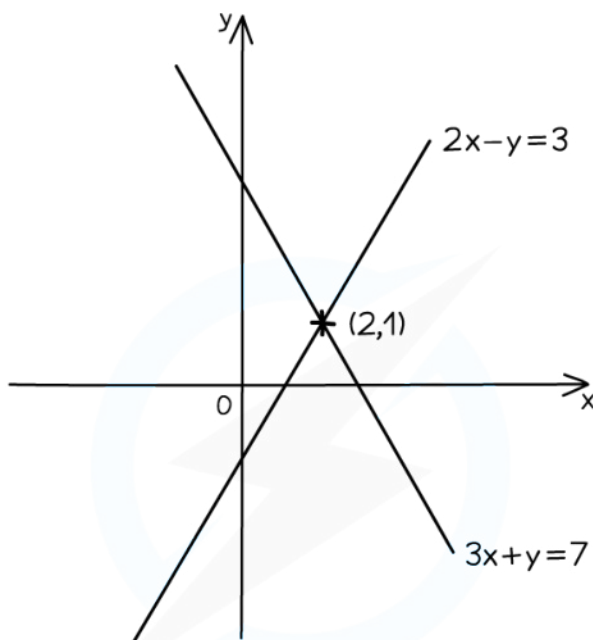


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Intersecting Graphs

How do I find where two graphs intersect?

- Plot both graphs on your GDC
- Use the intersect function to find the intersections
- Check if there is more than one point of intersection



- LINES INTERSECT AT (2,1)
- SOLVING $2x - y = 3$ AND $3x + y = 7$ SIMULTANEOUSLY IS $x = 2, y = 1$

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How can I use graphs to solve equations?

- One method to solve equations is to use graphs
- To solve $f(x) = a$
 - Plot the two graphs $y = f(x)$ and $y = a$ on your GDC
 - Find the points of intersections
 - The **x-coordinates** are the **solutions** of the equation
- To solve $f(x) = g(x)$

- Plot the two graphs $y = f(x)$ and $y = g(x)$ on your GDC
- Find the points of intersections
- The **x-coordinates** are the **solutions** of the equation
- Using graphs makes it easier to see **how many solutions** an equation will have

Examiner Tip

- You can use graphs to solve equations
 - Questions will not necessarily ask for a drawing/sketch or make reference to graphs
 - Use your GDC to plot the equations and find the intersections between the graphs



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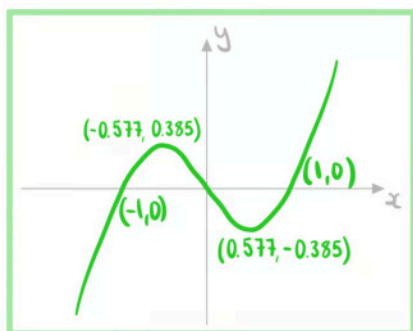
Worked example

Two functions are defined by

$$f(x) = x^3 - x \text{ and } g(x) = \frac{4}{x}$$

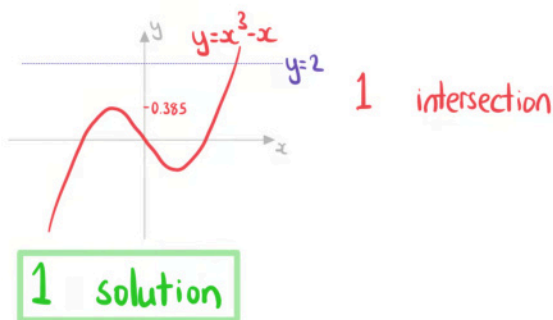
a) Sketch the graph $y = f(x)$.

Use GDC to find max, min, intercepts



b) Write down the number of real solutions to the equation $x^3 - x = 2$.

Identify the number of intersections between $y = x^3 - x$ and $y = 2$

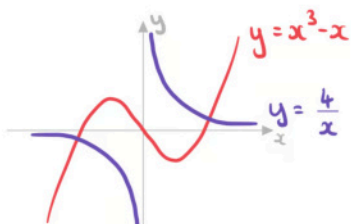


c) Find the coordinates of the points where $y = f(x)$ and $y = g(x)$ intersect.



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Use GDC to sketch both graphs



$(-1.60, -2.50)$ and $(1.60, 2.50)$

d)

Write down the solutions to the equation $x^3 - x = \frac{4}{x}$.

Solutions to $x^3 - x = \frac{4}{x}$ are the x coordinates of the points of intersection.

$x = -1.60$ and $x = 1.60$