

# DP IB Maths: AI HL



Your notes

## 1.6 Further Complex Numbers

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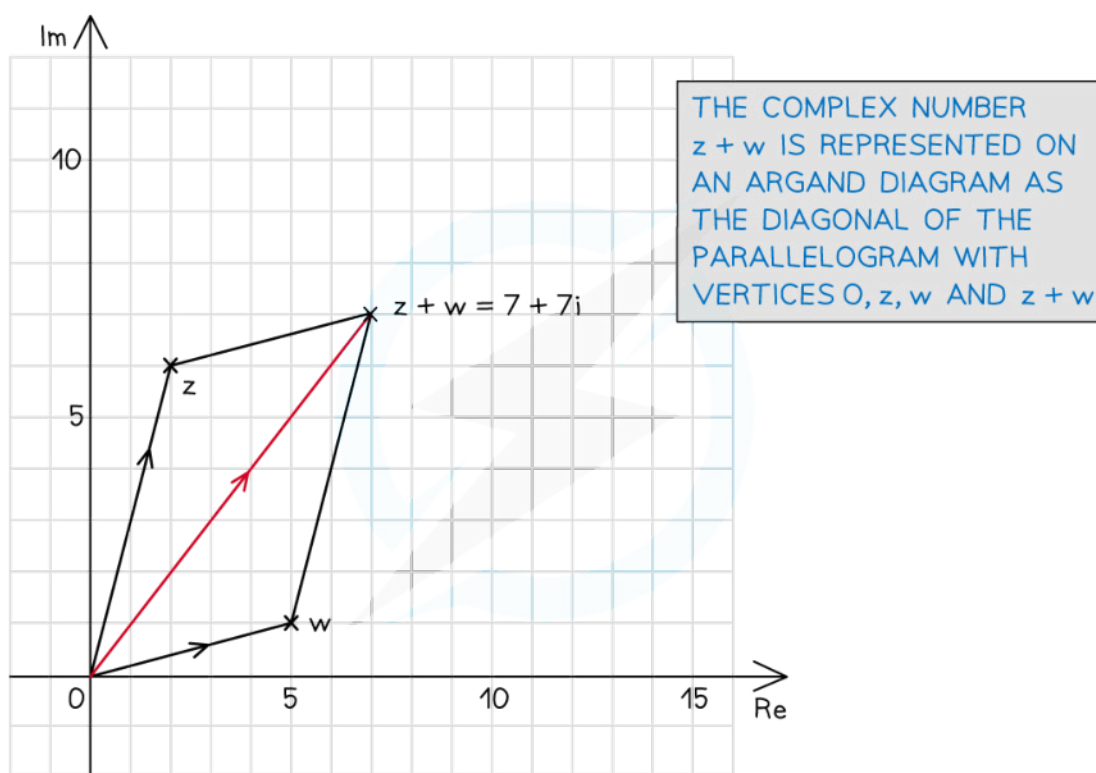
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## 1.6.1 Geometry of Complex Numbers

### Geometry of Complex Addition & Subtraction

#### What does addition look like on an Argand diagram?

- In Cartesian form two complex numbers are added by adding the real and imaginary parts
- When plotted on an Argand diagram the complex number  $z_1 + z_2$  is the longer diagonal of the parallelogram with vertices at the origin,  $z_1$ ,  $z_2$  and  $z_1 + z_2$



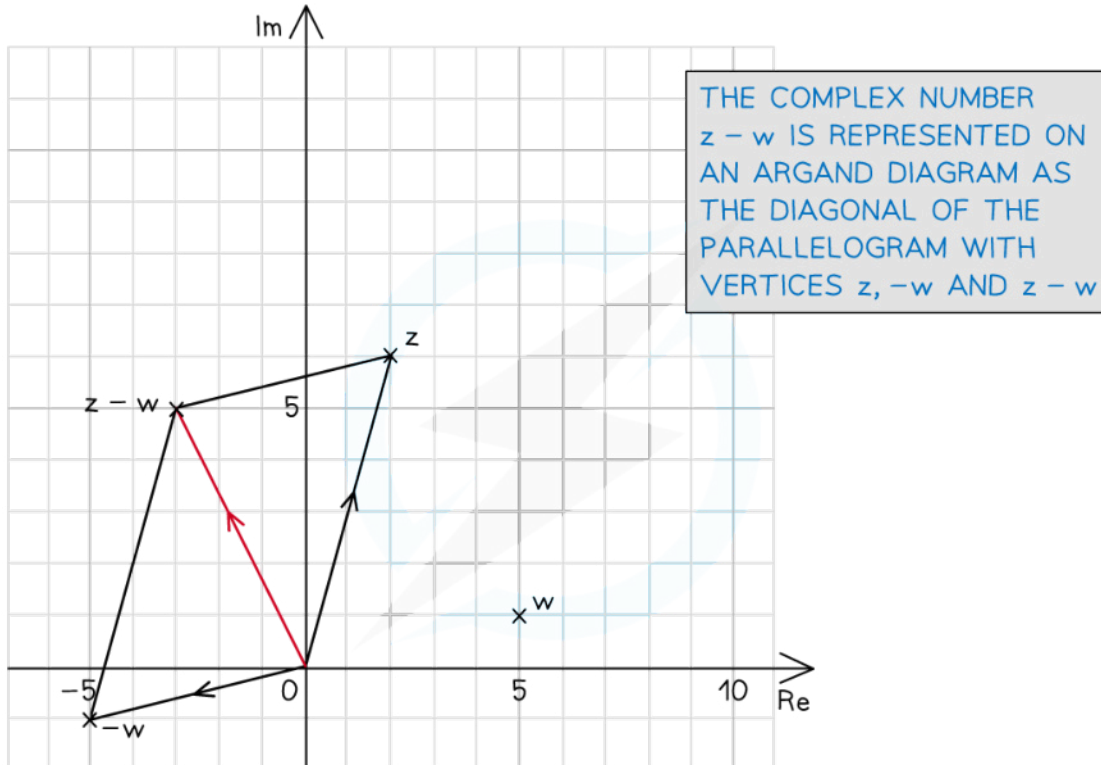
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#### What does subtraction look like on an Argand diagram?

- In Cartesian form the difference of two complex numbers is found by subtracting the real and imaginary parts
- When plotted on an Argand diagram the complex number  $z_1 - z_2$  is the shorter diagonal of the parallelogram with vertices at the origin,  $z_1$ ,  $-z_2$  and  $z_1 - z_2$



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REMEMBER TO PLOT THE POINT  $-w$  BEFORE DRAWING THE PARALLELOGRAM

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### What are the geometrical representations of complex addition and subtraction?

- Let  $w$  be a given complex number with **real part  $a$**  and **imaginary part  $b$** 
  - $w = a + bi$
- Let  $z$  be any complex number represented on an Argand diagram
- **Adding  $w$  to  $z$**  results in  $z$  being:
  - Translated by vector  $\begin{pmatrix} a \\ b \end{pmatrix}$
- **Subtracting  $w$  from  $z$**  results in  $z$  being:
  - Translated by vector  $\begin{pmatrix} -a \\ -b \end{pmatrix}$



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### Examiner Tip

- Take extra care when representing a subtraction of a complex number geometrically
  - Remember that your answer will be a translation of the shorter diagonal of the parallelogram made up by the two complex numbers

### Worked example

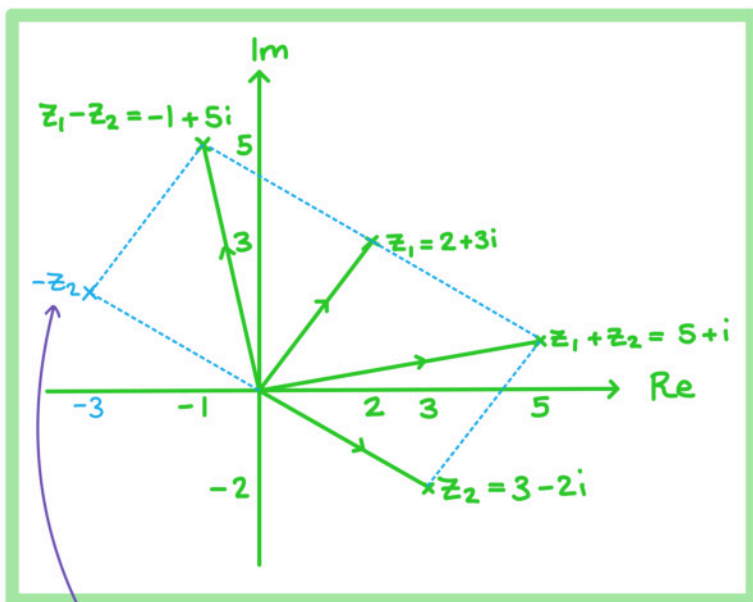
Consider the complex numbers  $z_1 = 2 + 3i$  and  $z_2 = 3 - 2i$ .

On an Argand diagram represent the complex numbers  $z_1, z_2, z_1 + z_2$  and  $z_1 - z_2$ .

First find  $z_1 + z_2$  and  $z_1 - z_2$ :

$$z_1 + z_2 = (2 + 3i) + (3 - 2i) = 5 + i$$

$$z_1 - z_2 = (2 + 3i) - (3 - 2i) = -1 + 5i$$



The geometrical properties can be seen by adding in  $-z_2 = -3 + 2i$



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## Geometry of Complex Multiplication & Division

### What do multiplication and division look like on an Argand diagram?

- The geometrical effect of multiplying a complex number by a real number,  $a$ , will be an enlargement of the vector by scale factor  $a$ 
  - For positive values of  $a$  the direction of the vector will not change but the distance of the point from the origin will increase by scale factor  $a$
  - For negative values of  $a$  the direction of the vector will change and the distance of the point from the origin will increase by scale factor  $a$
- The geometrical effect of dividing a complex number by a real number,  $a$ , will be an enlargement of the vector by scale factor  $1/a$ 
  - For positive values of  $a$  the direction of the vector will not change but the distance of the point from the origin will increase by scale factor  $1/a$
  - For negative values of  $a$  the direction of the vector will change and the distance of the point from the origin will increase by scale factor  $1/a$
- The geometrical effect of multiplying a complex number by  $i$  will be a rotation of the vector  $90^\circ$  counter-clockwise
  - $i(x + yi) = -y + xi$
- The geometrical effect of multiplying a complex number by an imaginary number,  $ai$ , will be a rotation  $90^\circ$  counter-clockwise and an enlargement by scale factor  $a$ 
  - $ai(x + yi) = -ay + axi$
- The geometrical effect of multiplying or dividing a complex number by a complex number will be an enlargement and a rotation
  - The direction of the vector will change
    - The angle of rotation is the **argument**
  - The distance of the point from the origin will change
    - The scale factor is the **modulus**

### What does complex conjugation look like on an Argand diagram?

- The geometrical effect of plotting a **complex conjugate** on an Argand diagram is a reflection in the real axis
  - The **real** part of the complex number will stay the same and the **imaginary** part will change sign

#### Examiner Tip

- Make sure you remember the transformations that different operations have on complex numbers, this could help you check your calculations in an exam



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### Worked example

Consider the complex number  $z = 2 - i$ .

On an Argand diagram represent the complex numbers  $z$ ,  $3z$ ,  $iz$ ,  $z^*$  and  $zz^*$ .

First find  $3z$ ,  $iz$  and  $z^*$

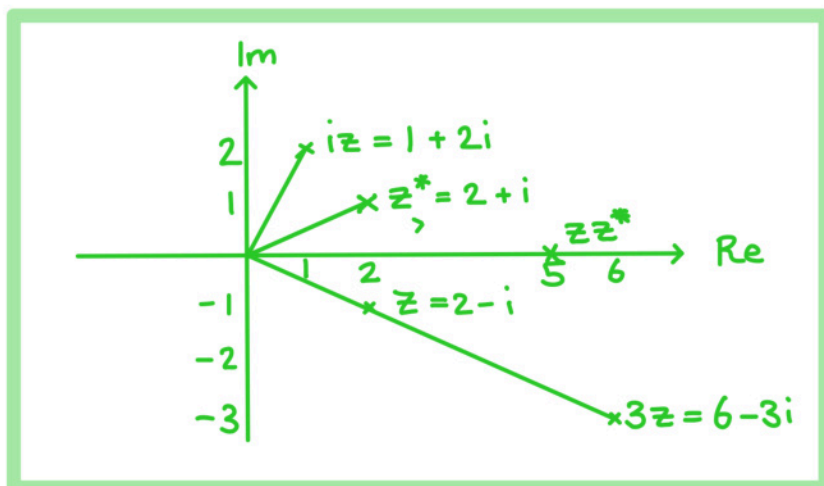
$$z = 2 - i$$

$$3z = 3(2 - i) = 6 - 3i$$

$$iz = i(2 - i) = 2i - i^2 = 2i - (-1) = 1 + 2i$$

$$z^* = 2 + i$$

$$zz^* = (2 - i)(2 + i) = 4 - i^2 = 4 - (-1) = 5$$





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## 1.6.2 Forms of Complex Numbers

### Modulus-Argument (Polar) Form

#### How do I write a complex number in modulus-argument (polar) form?

- The **Cartesian form** of a complex number,  $z = x + iy$ , is written in terms of its real part,  $x$ , and its imaginary part,  $y$
- If we let  $r = |z|$  and  $\theta = \arg z$ , then it is possible to write a complex number in terms of its modulus,  $r$ , and its argument,  $\theta$ , called the **modulus-argument (polar) form**, given by...
  - $z = r(\cos \theta + i \sin \theta)$ 
    - This is often written as  $z = r \operatorname{cis} \theta$
    - This is given in the formula book under Modulus-argument (polar) form and exponential (Euler) form
- It is usual to give arguments in the range  $-\pi < \theta \leq \pi$  or  $0 \leq \theta < 2\pi$ 
  - Negative arguments should be shown clearly
  - e.g.  $z = 2\left(\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right)\right) = 2 \operatorname{cis}\left(-\frac{\pi}{3}\right)$ 
    - without simplifying  $\cos\left(-\frac{\pi}{3}\right)$  to either  $\cos\left(\frac{\pi}{3}\right)$  or  $\frac{1}{2}$
- The **complex conjugate** of  $r \operatorname{cis} \theta$  is  $r \operatorname{cis}(-\theta)$
- If a complex number is given in the form  $z = r(\cos \theta - i \sin \theta)$ , then it is not in modulus-argument (polar) form due to the minus sign
  - It can be converted by considering transformations of trigonometric functions
    - $-\sin \theta = \sin(-\theta)$  and  $\cos \theta = \cos(-\theta)$
    - So  $z = r(\cos \theta - i \sin \theta) = z = r(\cos(-\theta) + i \sin(-\theta)) = r \operatorname{cis}(-\theta)$
- To convert from modulus-argument (polar) form back to Cartesian form, evaluate the real and imaginary parts
  - E.g.  $z = 2\left(\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right)\right)$  becomes  $z = 2\left(\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right) = 1 - \sqrt{3}i$

#### How do I multiply complex numbers in modulus-argument (polar) form?

- The main benefit of writing complex numbers in modulus-argument (polar) form is that they multiply and divide very easily
- To **multiply** two complex numbers in modulus-argument (polar) form we **multiply their moduli** and **add their arguments**
  - $|z_1 z_2| = |z_1| |z_2|$
  - $\arg(z_1 z_2) = \arg z_1 + \arg z_2$
- So if  $z_1 = r_1 \operatorname{cis}(\theta_1)$  and  $z_2 = r_2 \operatorname{cis}(\theta_2)$ 
  - $z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$



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- Sometimes the new argument,  $\theta_1 + \theta_2$ , does not lie in the range  $-\pi < \theta \leq \pi$  (or  $0 \leq \theta < 2\pi$  if this is being used)
  - An out-of-range argument can be adjusted by either **adding or subtracting  $2\pi$**
  - E.g. If  $\theta_1 = \frac{2\pi}{3}$  and  $\theta_2 = \frac{\pi}{2}$  then  $\theta_1 + \theta_2 = \frac{7\pi}{6}$
  - This is currently not in the range  $-\pi < \theta \leq \pi$
  - Subtracting  $2\pi$  from  $\frac{7\pi}{6}$  to give  $-\frac{5\pi}{6}$ , a new argument is formed
    - This lies in the correct range and represents the same angle on an Argand diagram
- The rules of **multiplying the moduli** and **adding the arguments** can also be applied when...
  - ...multiplying three complex numbers together,  $z_1 z_2 z_3$ , or more
  - ...finding powers of a complex number (e.g.  $z^2$  can be written as  $zz$ )
- The rules for multiplication can be proved algebraically by multiplying  $z_1 = r_1 \text{cis}(\theta_1)$  by  $z_2 = r_2 \text{cis}(\theta_2)$ , expanding the brackets and using compound angle formulae

### How do I divide complex numbers in modulus-argument (polar) form?

- To **divide** two complex numbers in modulus-argument (polar) form, we **divide their moduli** and **subtract their arguments**
  - $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$
  - $\arg \left( \frac{z_1}{z_2} \right) = \arg z_1 - \arg z_2$
- So if  $z_1 = r_1 \text{cis}(\theta_1)$  and  $z_2 = r_2 \text{cis}(\theta_2)$  then
  - $\frac{z_1}{z_2} = \frac{r_1}{r_2} \text{cis}(\theta_1 - \theta_2)$
- Sometimes the new argument,  $\theta_1 - \theta_2$ , can lie out of the range  $-\pi < \theta \leq \pi$  (or the range  $0 < \theta \leq 2\pi$  if this is being used)
  - You can **add or subtract  $2\pi$**  to bring out-of-range arguments back in range
- The rules for division can be proved algebraically by dividing  $z_1 = r_1 \text{cis}(\theta_1)$  by  $z_2 = r_2 \text{cis}(\theta_2)$  using **complex division** and the compound angle formulae



### Examiner Tip

- Remember that  $r \operatorname{cis} \theta$  only refers to  $r(\cos \theta + i \sin \theta)$ 
  - If you see a complex number written in the form  $z = r(\cos \theta - i \sin \theta)$  then you will need to convert it to the correct form first
  - Make sure you are confident with basic trig identities to help you do this



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### Worked example

$$\text{Let } z_1 = 4\sqrt{2} \operatorname{cis} \frac{3\pi}{4} \text{ and } z_2 = \sqrt{8} \left( \cos\left(\frac{\pi}{2}\right) - i \sin\left(\frac{\pi}{2}\right) \right)$$

a) Find  $z_1 z_2$ , giving your answer in the form  $r(\cos\theta + i\sin\theta)$  where  $0 \leq \theta < 2\pi$

$$z_1 = 4\sqrt{2} \operatorname{cis}\left(\frac{3\pi}{4}\right), \quad z_2 = \sqrt{8} \left( \cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right) \right) = 2\sqrt{2} \operatorname{cis}\left(-\frac{\pi}{2}\right)$$

For  $z_1 z_2$ , multiply the moduli and add the arguments.

$$\begin{aligned} z_1 z_2 &= (4\sqrt{2} \operatorname{cis}\left(\frac{3\pi}{4}\right))(2\sqrt{2} \operatorname{cis}\left(-\frac{\pi}{2}\right)) \\ &= (4\sqrt{2})(2\sqrt{2}) \operatorname{cis}\left(\frac{3\pi}{4} + \left(-\frac{\pi}{2}\right)\right) \\ &= 16 \operatorname{cis}\left(\frac{\pi}{4}\right) \end{aligned}$$

$$z_1 z_2 = 16 \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$$

b) Find  $\frac{z_1}{z_2}$ , giving your answer in the form  $r(\cos\theta + i\sin\theta)$  where  $-\pi \leq \theta < \pi$

For  $\frac{z_1}{z_2}$ , divide the moduli and subtract the arguments

$$\frac{z_1}{z_2} = \frac{4\sqrt{2} \operatorname{cis}\left(\frac{3\pi}{4}\right)}{2\sqrt{2} \operatorname{cis}\left(-\frac{\pi}{2}\right)} = \frac{4\sqrt{2}}{2\sqrt{2}} \operatorname{cis}\left(\frac{3\pi}{4} - \left(-\frac{\pi}{2}\right)\right)$$

$$\begin{aligned} &= 2 \operatorname{cis}\left(\frac{5\pi}{4}\right) \\ &= 2 \operatorname{cis}\left(\frac{5\pi}{4} - 2\pi\right) \end{aligned}$$

$\frac{5\pi}{4}$  is not in the range  $-\pi \leq \theta \leq \pi$  so subtract  $2\pi$  to bring it into range

$$\frac{z_1}{z_2} = 2 \left( \cos\left(\frac{-3\pi}{4}\right) + i \sin\left(\frac{-3\pi}{4}\right) \right)$$



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## Exponential (Euler's) Form

### How do we write a complex number in Euler's (exponential) form?

- A complex number can be written in Euler's form as  $z = re^{i\theta}$ 
  - This relates to the modulus-argument (polar) form as  $z = re^{i\theta} = r \operatorname{cis} \theta$
  - This shows a clear link between exponential functions and trigonometric functions
  - This is given in the formula booklet under 'Modulus-argument (polar) form and exponential (Euler) form'
- The argument is normally given in the range  $0 \leq \theta < 2\pi$ 
  - However in exponential form other arguments can be used and the same convention of adding or subtracting  $2\pi$  can be applied

### How do we multiply and divide complex numbers in Euler's form?

- Euler's form allows for quick and easy multiplication and division of complex numbers
- If  $z_1 = r_1 e^{i\theta_1}$  and  $z_2 = r_2 e^{i\theta_2}$  then
  - $z_1 \times z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$ 
    - Multiply the moduli and add the arguments
  - $\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$ 
    - Divide the moduli and subtract the arguments
- Using these rules makes multiplying and dividing more than two complex numbers much easier than in Cartesian form
- When a complex number is written in Euler's form it is easy to raise that complex number to a power
  - If  $z = re^{i\theta}$ ,  $z^2 = r^2 e^{2i\theta}$  and  $z^n = r^n e^{ni\theta}$

### What are some common numbers in exponential form?

- As  $\cos(2\pi) = 1$  and  $\sin(2\pi) = 0$  you can write:
  - $1 = e^{2\pi i}$
- Using the same idea you can write:
  - $1 = e^0 = e^{2\pi i} = e^{4\pi i} = e^{6\pi i} = e^{2k\pi i}$ 
    - where  $k$  is any integer
- As  $\cos(\pi) = -1$  and  $\sin(\pi) = 0$  you can write:
  - $e^{\pi i} = -1$
  - Or more commonly written as  $e^{i\pi} + 1 = 0$ 
    - This is known as Euler's identity and is considered by some mathematicians as the most beautiful equation

- As  $\cos\left(\frac{\pi}{2}\right) = 0$  and  $\sin\left(\frac{\pi}{2}\right) = 1$  you can write:
  - $i = e^{\frac{\pi}{2}i}$



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### Examiner Tip

- Euler's form allows for easy manipulation of complex numbers, in an exam it is often worth the time converting a complex number into Euler's form if further calculations need to be carried out
  - Familiarise yourself with which calculations are easier in which form, for example multiplication and division are easiest in Euler's form but adding and subtracting are easiest in Cartesian form

### Worked example

Consider the complex number  $z = 2e^{\frac{\pi}{3}i}$ . Calculate  $z^2$  giving your answer in the form  $re^{i\theta}$ .

$$z^2 = \left(2e^{\frac{\pi}{3}i}\right)^2 = \left(2e^{\frac{\pi}{3}i}\right)\left(2e^{\frac{\pi}{3}i}\right) = 4e^{2\left(\frac{\pi}{3}i\right)}$$


 multiply the moduli  
add the arguments

$$z^2 = 4e^{\frac{2\pi}{3}i}$$



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## Conversion of Forms

### Converting from Cartesian form to modulus-argument (polar) form or exponential (Euler's) form

- To convert from Cartesian form to modulus-argument (polar) form or exponential (Euler) form use
  - $r = |z| = \sqrt{x^2 + y^2}$
- and
  - $\theta = \arg z$

### Converting from modulus-argument (polar) form or exponential (Euler's) form to Cartesian form

- To convert from modulus-argument (polar) form to Cartesian form
  - You may need to use your knowledge of trig exact values
  - $a = r \cos \theta$  and  $b = r \sin \theta$
  - Write  $z = r(\cos \theta + i \sin \theta)$  as  $z = r \cos \theta + (r \sin \theta)i$
  - Find the values of the trigonometric ratios  $r \sin \theta$  and  $r \cos \theta$
  - Rewrite as  $z = a + bi$  where
- To convert from exponential (Euler's) form to Cartesian form first rewrite  $z = r e^{i\theta}$  in the form  $z = r \cos \theta + (r \sin \theta)i$  and then follow the steps above

### Converting between complex number forms using your GDC

- Your GDC may also be able to convert complex numbers between the various forms
  - TI calculators, for example, have 'Convert to Polar' and 'Convert to Rectangular' (i.e. Cartesian) as options in the 'Complex Number Tools' menu
  - Make sure you are familiar with your GDC and what it can (and cannot) do with complex numbers

#### Examiner Tip

- When converting from Cartesian form into Polar or Euler's form, always leave your modulus and argument as an exact value
  - Rounding values too early may result in inaccuracies later on



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### Worked example

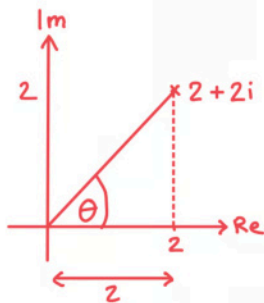
Two complex numbers are given by  $z_1 = 2 + 2i$  and  $z_2 = 3e^{\frac{2\pi}{3}i}$ .

a) Write  $z_1$  in the form  $re^{i\theta}$ .

$$z_1 = 2 + 2i$$

$$\text{Find the modulus: } |z_1| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$$

Draw a sketch to help find the argument:



$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{2}{2}\right) = \tan^{-1}(1) \\ &= \frac{\pi}{4} \end{aligned}$$

$$r = 2\sqrt{2}, \quad \theta = \frac{\pi}{4}$$

$$z_1 = 2\sqrt{2} e^{\frac{\pi}{4}i}$$

b) Write  $z_2$  in the form  $r(\cos\theta + i\sin\theta)$  and then convert it to Cartesian form.

$$\begin{aligned} z_2 &= 3e^{\frac{2\pi}{3}i} = 3\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) \\ &= 3\left(-\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right) \end{aligned}$$

$$z_2 = \frac{3}{2}(-1 + \sqrt{3}i)$$



Your notes



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## 1.6.3 Applications of Complex Numbers

### Frequency & Phase of Trig Functions

#### How are complex numbers and trig functions related?

- A sinusoidal function is of the form  $a \sin(bx + c)$ 
  - $a$  represents the **amplitude**
  - $b$  represents the **period** (also known as frequency)
  - $c$  represents the **phase shift**
    - The function may be written  $a \sin(bx + bc) = a \sin b(x + c)$  where the phase shift is represented by  $bc$
    - This will be made clear in the exam
- When written in **modulus-argument** form the **imaginary part** of a complex number relates only to the **sin** part and the **real** part relates to the **cos** part
  - This means that the complex number can be rewritten in Euler's form and relates to the sinusoidal functions as follows:
    - $a \sin(bx + c) = \text{Im}(ae^{i(bx+c)})$
    - $a \cos(bx + c) = \text{Re}(ae^{i(bx+c)})$
- Complex numbers are particularly useful when working with electrical currents or voltages as these follow sinusoidal wave patterns
  - AC voltages may be given in the form  $V = a \sin(bt + c)$  or  $V = a \cos(bt + c)$

#### How are complex numbers used to add two sinusoidal functions?

- Complex numbers can help to add two **sinusoidal** functions if they have the same **frequency** but different **amplitudes** and **phase shifts**
  - e.g.  $2\sin(3x + 1)$  can be added to  $3\sin(3x - 5)$  but **not**  $2\sin(5x + 1)$
- To add  $a\sin(bx + c)$  to  $d\sin(bx + e)$ 
  - or  $a\cos(bx + c)$  to  $d\cos(bx + e)$
- STEP 1: Consider the complex numbers  $z_1 = ae^{i(bx+c)}$  and  $z_2 = de^{i(bx+e)}$ 
  - Then  $a\sin(bx + c) + d\sin(bx + e) = \text{Im}(z_1 + z_2)$
  - Or  $a\cos(bx + c) + d\cos(bx + e) = \text{Re}(z_1 + z_2)$
- STEP 2: Factorise  $z_1 + z_2 = ae^{i(bx+c)} + de^{i(bx+e)} = e^{ibx}(ae^{ci} + de^{ei})$
- STEP 3: Convert  $ae^{ci} + de^{ei}$  into a single complex number in exponential form
  - You may need to convert it into Cartesian form first, simplify and then convert back into exponential form
  - Your GDC will be able to do this quickly
- STEP 4: Simplify the whole term and use the rules of indices to collect the powers
- STEP 5: Convert into polar form and take...
  - only the imaginary part for sin
  - or only the real part for cos





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### Examiner Tip

- An exam question involving applications of complex numbers will often be made up of various parts which build on each other
  - Remember to look back at your answers from previous question parts to see if they can help you, especially when looking to convert from Euler's form to a sinusoidal graph form

### Worked example

Two AC voltage sources are connected in a circuit. If  $V_1 = 20\sin(30t)$  and  $V_2 = 30\sin(30t + 5)$  find an expression for the total voltage in the form  $V = A\sin(30t + B)$ .

$$20\sin(30t) + 30\sin(30t+5)$$

↑                      ↑  
Frequencies are the same so they can be added

STEP 1: Let  $z_1 = 20e^{i(30t)}$  and  $z_2 = 30e^{i(30t+5)}$

In polar form the imaginary parts are the sinusoidal functions we want to add.

STEP 2: Find  $z_1 + z_2$

$$\begin{aligned} z_1 + z_2 &= 20e^{i(30t)} + 30e^{i(30t+5)} \\ &= 10e^{30ti} (2 + 3e^{5i}) \end{aligned}$$

←  $2 + (3(\cos(5) + i \sin(5)))$   
 $= 2.850... - 2.876...i$

STEP 3: Use GDC to find  $2 + 3e^{5i}$  in Euler's form

$$= 10e^{30ti} (4.05... e^{-0.789...i})$$

STEP 4: Use index laws to simplify.

$$= 40.5e^{i(30t - 0.789...)}$$

STEP 5: Convert to Polar form

$$= 40.5(\cos(30t - 0.789...) + i \sin(30t - 0.789...))$$

The imaginary part is the solution

$$\boxed{V = 40.5 \sin(30t - 0.790)}$$