DP IB Maths: AA HL

Your notes

3.6 Trigonometric Equations & Identities

Contents

3.6.1 Simple Identities

Your notes

Simple Identities

What is a trigonometric identity?

- Trigonometric identities are statements that are true for all values of x or θ
- They are used to help simplify trigonometric equations before solving them
- Sometimes you may see identities written with the symbol ≡
  - This means 'identical to'

What trigonometric identities do I need to know?

- The two trigonometric identities you must know are
  - $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
    - This is the identity for tan θ
  - $\sin^2\theta + \cos^2\theta = 1$
    - This is the Pythagorean identity
    - Note that the notation $\sin^2\theta$ is the same as $(\sin\theta)^2$
- Both identities can be found **in the formula booklet**
- Rearranging the second identity often makes it easier to work with
  - $\sin^2\theta = 1 - \cos^2\theta$
  - $\cos^2\theta = 1 - \sin^2\theta$

Where do the trigonometric identities come from?

- You do not need to know the proof for these identities but it is a good idea to know where they come from
- From SOHCAHTOA we know that
  - $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{O}{H}$
  - $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}} = \dfrac{A}{H}$
  - $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{O}{A}$
- The identity for $\tan\theta$ can be seen by diving $\sin\theta$ by $\cos\theta$
  - $\dfrac{\sin\theta}{\cos\theta} = \dfrac{\frac{O}{H}}{\frac{A}{H}} = \dfrac{O}{A} = \tan\theta$
- This can also be seen from the unit circle by considering a right-triangle with a hypotenuse of 1

Your notes

- $\tan\theta = \dfrac{O}{A} = \dfrac{\sin\theta}{\cos\theta}$
- The Pythagorean identity can be seen by considering a right-triangle with a hypotenuse of 1
  - Then $(\text{opposite})^2 + (\text{adjacent})^2 = 1$
  - Therefore $\sin^2\theta + \cos^2\theta = 1$
- Considering the equation of the unit circle also shows the Pythagorean identity
  - The equation of the unit circle is $x^2 + y^2 = 1$
  - The coordinates on the unit circle are $(\cos\theta,\ \sin\theta)$
  - Therefore the equation of the unit circle could be written $\cos^2\theta + \sin^2\theta = 1$
- A third very useful identity is $\sin\theta = \cos(90° - \theta)$ or $\sin\theta = \cos\left(\dfrac{\pi}{2} - \theta\right)$
  - This is not included in the formula booklet but is useful to remember

How are the trigonometric identities used?

- Most commonly trigonometric identities are used to change an equation into a form that allows it to be solved
- They can also be used to prove further identities such as the **double angle formulae**

Examiner Tip

- If you are asked to show that one thing is identical (≡) to another, look at what parts are missing – for example, if tan x has gone it must have been substituted

Your notes

Worked example

Show that the equation $2\sin^2 x - \cos x = 0$ can be written in the form $a\cos^2 x + b\cos x + c = 0$, where a, b and c are integers to be found.

$2\sin^2 x - \cos x = 0$

Equation has both $\sin x$ and $\cos x$ so will need changing before it can be solved.

Use the identity $\sin^2 x = 1 - \cos^2 x$

Substitute : $2(1 - \cos^2 x) - \cos x = 0$

Expand : $2 - 2\cos^2 x - \cos x = 0$

Rearrange : $2\cos^2 x + \cos x - 2 = 0$

$a = 2, \quad b = 1, \quad c = -2$

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3.6.2 Compound Angle Formulae

Compound Angle Formulae

What are the compound angle formulae?

- There are six **compound angle formulae** (also known as **addition formulae**), two each for **sin**, **cos** and **tan**:
- For **sin** the +/- sign on the left-hand side **matches** the one on the right-hand side
  - sin(A+B)≡sinAcosB + cosAsinB
  - sin(A-B)≡sinAcosB - cosAsinB
- For **cos** the +/- sign on the left-hand side is **opposite to** the one on the right-hand side
  - cos(A+B)≡cosAcosB - sinAsinB
  - cos(A-B)≡cosAcosB + sinAsinB
- For **tan** the +/- sign on the left-hand side **matches** the one in the **numerator** on the right-hand side, and is **opposite to** the one in the **denominator**
  - $\tan(A+B) \equiv \frac{\tan A + \tan B}{1 - \tan A \tan B}$
  - $\tan(A-B) \equiv \frac{\tan A - \tan B}{1 + \tan A \tan B}$
- The compound angle formulae can all the found in the formula booklet, you do not need to remember them

When are the compound angle formulae used?

- The compound angle formulae are particularly useful when finding the values of trigonometric ratios without the use of a calculator
  - For example to find the value of sin15° rewrite it as sin (45 – 30)° and then
    - apply the compound formula for sin(A – B)
    - use your knowledge of exact values to calculate the answer
- The compound angle formulae are also used…
  - … to derive further multiple angle trig identities such as the double angle formulae
  - … in trigonometric proof
  - … to simplify complicated trigonometric equations before solving

How are the compound angle formulae for cosine proved?

- The proof for the compound angle identity cos (*A* – *B*) = cos *A* cos *B* + sin *A* sin *B* can be seen by considering two coordinates on a unit circle, *P* (cos *A*, sin *A*) and Q (cos *B*, sin *B*)
  - The angle between the positive *x*- axis and the point *P* is *A*
  - The angle between the positive *x*- axis and the point *Q* is *B*
  - The angle between *P* and Q is *B* – *A*

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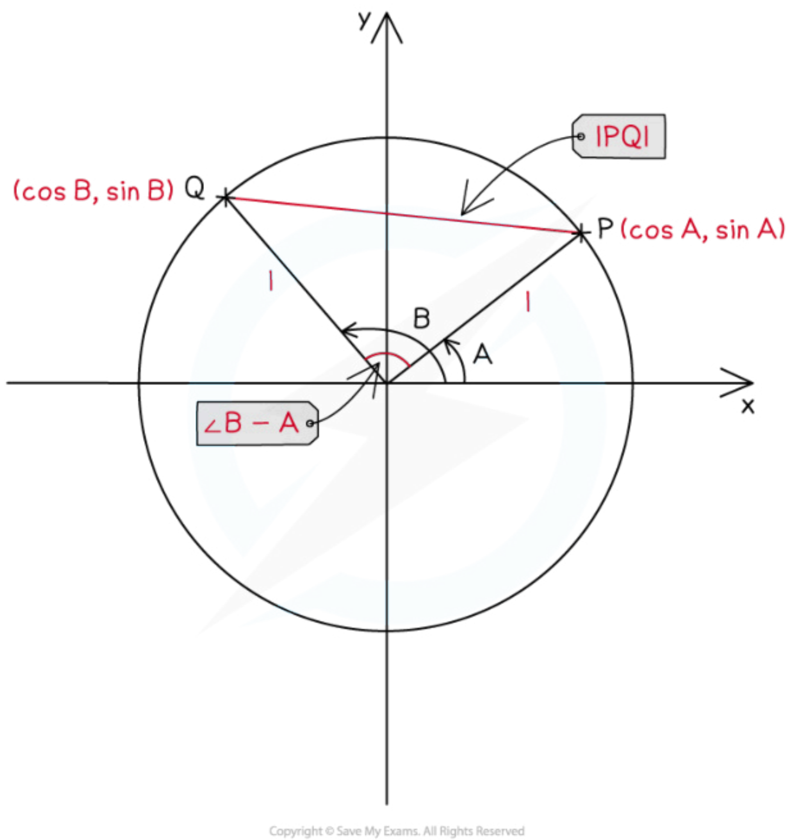
- Using the distance formula (Pythagoras) the distance PQ can be given as
  - $|PQ|^2 = (\cos A - \cos B)^2 + (\sin A - \sin B)^2$
- Using the cosine rule the distance PQ can be given as
  - $|PQ|^2 = 1^2 + 1^2 - 2(1)(1)\cos(B - A) = 2 - 2\cos(B - A)$
- Equating these two formulae, expanding and rearranging gives
  - $2 - 2\cos(B - A) = \cos^2 A + \sin^2 A + \cos^2 B + \sin^2 B - 2\cos A\cos B - 2\sin A\sin B$
  - $2 - 2\cos(B - A) = 2 - 2(\cos A\cos B + \sin A\sin B)$
- Therefore $\cos(B - A) = \cos A\cos B + \sin A\sin B$
- Changing $-A$ for A in this identity and rearranging proves the identity for $\cos(A + B)$
  - $\cos(B - (-A)) = \cos(-A)\cos B + \sin(-A)\sin B = \cos A\cos B - \sin A\sin B$



How are the compound angle formulae for sine proved?

- The proof for the compound angle identity $\sin(A + B)$ can be seen by using the above proof for $\cos(B - A)$ and
  - Considering $\cos(\pi/2 - (A + B)) = \cos(\pi/2)\cos(A + B) + \sin(\pi/2)\sin(A + B)$
  - Therefore $\cos(\pi/2 - (A + B)) = \sin(A + B)$
  - Rewriting $\cos(\pi/2 - (A + B))$ as $\cos((\pi/2 - A) + B)$ gives
    - $\cos(\pi/2 - (A + B)) = \cos(\pi/2 - A)\cos B + \sin(\pi/2 - A)\sin B$

- Using cos (π/2 – A) = sin A and sin (π/2 – A) = cos A and equating gives
  - sin (A + B) = sin A cos B + cos A cos B
- Substituting B for -B proves the result for sin (A – B)

How are the compound angle formulae for tan proved?

- The proof for the compound angle identities tan (A ± B) can be seen by
  - Rewriting tan (A ± B) as $\frac{\sin(A \pm B)}{\cos(A \pm B)}$
  - Substituting the compound angle formulae in
  - Dividing the numerator and denominator by cos A cos B

Examiner Tip

- All these formulae are in the **Topic 3: Geometry and Trigonometry** section of the formula booklet – make sure that you use them correctly paying particular attention to any negative/positive signs

Your notes

Worked example

a) Show that $\tan\left(x+\frac{\pi}{4}\right)-\tan\left(x-\frac{\pi}{4}\right)=\frac{2(\tan^2 x+1)}{1-\tan^2 x}$

Use the compound angle formula for tan:

$$\tan\left(x+\frac{\pi}{4}\right)=\frac{\tan x+\tan\frac{\pi}{4}}{1-\tan x\tan\frac{\pi}{4}}=\frac{\tan x+1}{1-\tan x}$$

$$\tan\left(x-\frac{\pi}{4}\right)=\frac{\tan x-\tan\frac{\pi}{4}}{1+\tan x\tan\frac{\pi}{4}}=\frac{\tan x-1}{1+\tan x}$$

Put together and simplify:

$$\frac{\tan x+1}{1-\tan x}-\frac{\tan x-1}{1+\tan x}=\frac{(\tan x+1)(1+\tan x)-(\tan x-1)(1-\tan x)}{(1-\tan x)(1+\tan x)}$$

$$=\frac{\tan^2 x+2\tan x+1-(-\tan^2 x+2\tan x-1)}{(1-\tan x)(1+\tan x)}$$

$$=\frac{2\tan^2 x+2}{1-\tan^2 x}$$

$$\boxed{\frac{2(\tan^2 x+1)}{1-\tan^2 x}}$$

b) Hence, solve $\tan\left(x+\frac{\pi}{4}\right)-\tan\left(x-\frac{\pi}{4}\right)=-4$ for $0\le x\le\frac{\pi}{2}$

Your notes

Use the answer found in (a) to write a new equation:

$$\frac{2(\tan^2 x + 1)}{1 - \tan^2 x} = -4$$

Rearrange and bring all terms in $\tan x$ to one side:

$$2(\tan^2 x + 1) = -4(1 - \tan^2 x)$$

$$2\tan^2 x + 2 = -4 + 4\tan^2 x$$

$$2\tan^2 x - 6 = 0$$

$$\tan^2 x = 3$$

$$\tan x = \pm\sqrt{3}$$

$$x = \frac{\pi}{3},\ -\frac{\pi}{3}$$ ← outside of given range

$$x = \frac{\pi}{3}$$

Your notes

3.6.3 Double Angle Formulae

Double Angle Formulae

What are the double angle formulae?

- The **double angle formulae** for **sine** and **cosine** are:
  - $\sin 2\theta = 2\sin\theta\cos\theta$
  - $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
  - $\tan 2\theta \equiv \dfrac{2\tan\theta}{1-\tan^2\theta}$
- These can be found in the formula booklet
  - The formulae for sin and cos can be found in the SL section
  - The formula for tan can be found in the HL section

How are the double angle formulae derived?

- The double angle formulae can be derived from the compound angle formulae
- Simply replace *B* for *A* in each of the formulae and simplify
- For example
  - Sin 2*A* = sin (*A* + *A*) = sin*A*cos*A* + sin*A*cos*A* = 2sin*A*cos*A*

How are the double angle formulae used?

- Double angle formulae will often be used with...
  - ... trigonometry exact values
  - ... graphs of trigonometric functions
  - ... relationships between trigonometric ratios
- To help solve trigonometric equations which contain $\sin\theta\cos\theta$:
  - Substitute $\dfrac{1}{2}\sin 2\theta$ for $\sin\theta\cos\theta$
  - Solve for 2θ, finding all values in the range for 2θ
    - The range will need adapting for 2θ
  - Find the solutions for θ
- To help solve trigonometric equations which contain $\sin 2\theta$ and $\sin\theta$ or $\cos\theta$
  - Substitute $2\sin\theta\cos\theta$ for $\sin 2\theta$
  - Isolate all terms in θ
  - Factorise or use another identity to write the equation in a form which can be solved
- To help solve trigonometric equations which contain $\cos 2\theta$ and $\sin\theta$ or $\cos\theta$
  - Substitute either $2\cos^2\theta - 1$ or $1 - 2\sin^2\theta$ for $\cos 2\theta$
    - Choose the trigonometric ratio that is already in the equation
  - Isolate all terms in θ

Your notes

    - Solve
        - The equation will most likely be in the form of a quadratic
- To help solve trigonometric equations which contain tan 2θ
    - Substitute the double angle identity for tan 2θ
    - Rearrange, often this will lead to a quadratic equation in terms of tan θ
    - Solve
- Double angle formulae can be used in proving other trigonometric identities

Examiner Tip

- All these formulae are in the **Topic 3: Geometry and Trigonometry** section of the formula booklet
- If you are asked to show that one thing is identical (≡) to another, look at what parts are missing – for example, if sinθ has disappeared you may want to choose the equivalent expression for cos2θ that does not include sinθ

Your notes

Worked example

Without using a calculator, solve the equation $\sin 2\theta = \sin\theta$ for $0° \leq \theta \leq 360°$. Show all working clearly.

Double angle identity: $\sin 2\theta = 2\sin\theta\cos\theta$

$$2\sin\theta\cos\theta = \sin\theta$$

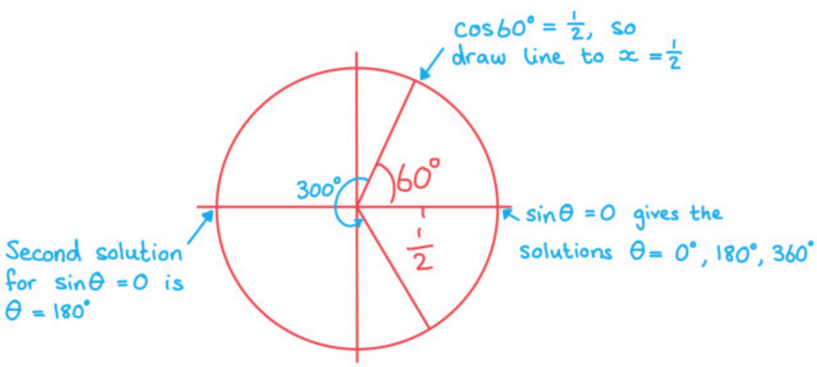
Bring both identities to one side:

$$2\sin\theta\cos\theta - \sin\theta = 0$$

Factorise: $\sin\theta(2\cos\theta - 1) = 0$

Find solutions: $\sin\theta = 0$, $\theta = 0$ $2\cos\theta - 1 = 0$, $\cos\theta = \frac{1}{2}$, $\theta = 60°$

Find secondary values within range:



$$\theta = 0°, 60°, 180°, 300°, 360°$$

3.6.4 Relationship Between Trigonometric Ratios

Your notes

Relationship Between Trigonometric Ratios

What relationships between trigonometric ratios should I know?

- If you know a value for one trig ratio you can often use this to work out the value for the others without needing to find θ
- If you know that $\sin\theta = \frac{a}{b}$, where $a, b \in \mathbb{Z}^+$, you can:
  - Sketch a right-triangle with *a* opposite θ and *b* on the hypotenuse
  - Use Pythagoras' theorem to find the value of the adjacent side
  - Use SOHCAHTOA to find the values of cos θ and tan θ
- If you know a value for sin θ or cos θ you can use the Pythagorean relationship
  - $\sin^2\theta + \cos^2\theta = 1$
  - to find the value of the other
- If you know a value for sin θ or cos θ you can use the double angle formulae to find the value of sin 2θ or cos 2θ
- If you know a value for tan θ you can use the double angle formulae to find the value of tan 2θ
- If you know two out of the three values for sin θ, cos θ or tan θ you can use the identity in tan
  - $\tan\theta = \frac{\sin\theta}{\cos\theta}$
  - to find the value of the third ratio

How do we determine whether a trigonometric ratio will be positive or negative?

- It is possible to determine whether a trigonometric ratio will be positive or negative by looking at the size of the angle and considering the **unit circle**
  - Angles in the range $0° < \theta° < 90°$ will be positive for all three ratios
  - Angles in the range $90° < \theta° < 180°$ will be positive for sin and negative for cos and tan
  - Angles in the range $180° < \theta° < 270°$ will be positive for tan and negative for sin and cos
  - Angles in the range $270° < \theta° < 360°$ will be positive for cos and negative for sin and tan
- The ratios for angles of 0°, 90°, 180°, 270° and 360° are either 0, 1, -1 or undefined
  - You should know these ratios or know how to derive them without a calculator

Examiner Tip

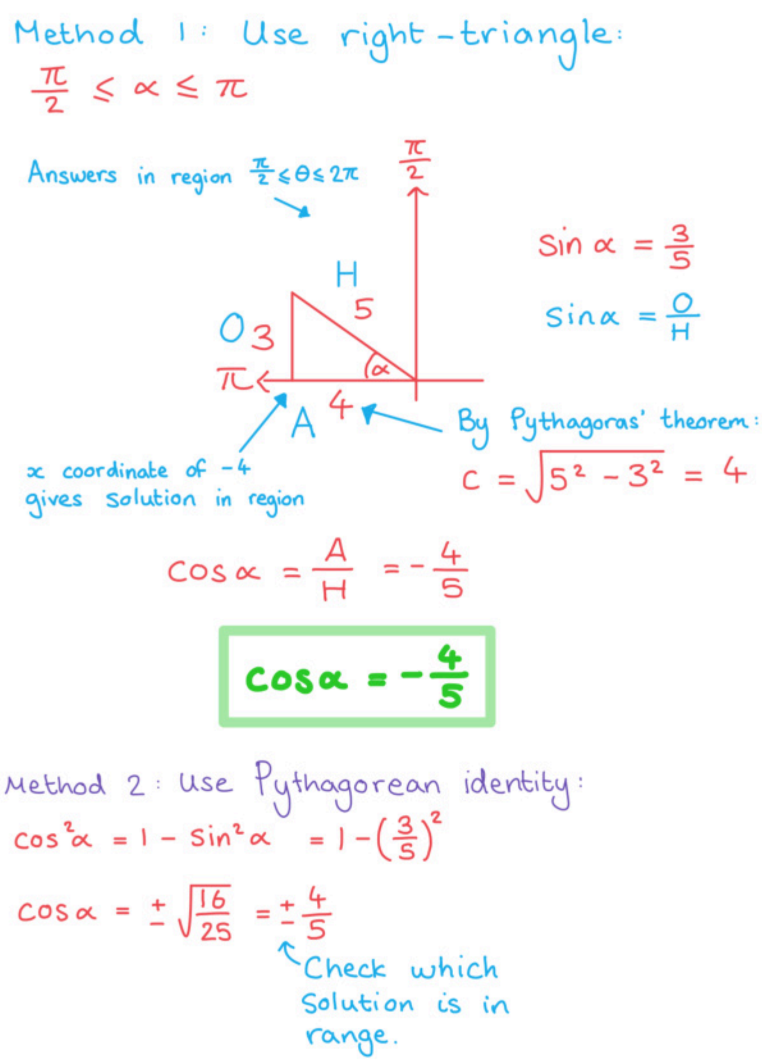
- Being able to sketch out the unit circle and remembering CAST can help you to find all solutions to a problem in an exam question

Your notes

Worked example

The value of $\sin\alpha = \dfrac{3}{5}$ for $\dfrac{\pi}{2} \le \alpha \le \pi$. Find:

i) $\cos\alpha$



ii) $\tan\alpha$

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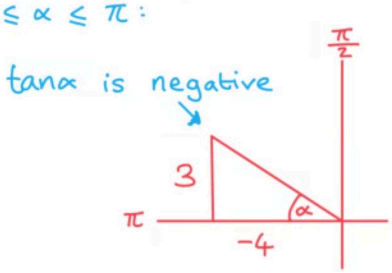
Use $\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\frac{3}{5}}{-\frac{4}{5}} = -\frac{3}{4}$

Check if $\tan\alpha = -\frac{3}{4}$ is in the correct range for $\frac{\pi}{2} \leq \alpha \leq \pi$:



$\tan\alpha = -\frac{3}{4}$

iii) $\sin 2\alpha$

Double angle identity: $\sin 2\theta = 2\sin\theta\cos\theta$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$= 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right)$

$= -\frac{24}{25}$

$\sin\alpha = \frac{3}{5}$

$\cos\alpha = -\frac{4}{5}$

$\sin 2\alpha = -\frac{24}{25}$

iv) $\cos 2\alpha$

Your notes

Double angle identity : $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$$\cos 2\alpha = \left(-\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{7}{25}$$

$$\cos 2\alpha = \frac{7}{25}$$

v) $\tan 2\alpha$

Using identity $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$$\tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha} = \frac{-\frac{24}{25}}{\frac{7}{25}} = -\frac{24}{7}$$

$$\tan 2\alpha = -\frac{24}{7}$$

3.6.5 Linear Trigonometric Equations

Your notes

Trigonometric Equations: sinx = k

How are trigonometric equations solved?

- Trigonometric equations can have an infinite number of solutions
  - For an equation in sin or cos you can add 360° or 2π to each solution to find more solutions
  - For an equation in tan you can add 180° or π to each solution
- When solving a trigonometric equation you will be given a range of values within which you should find all the values
- Solving the equation normally and using the inverse function on your calculator or your knowledge of **exact values** will give you the **primary value**
- The **secondary values** can be found with the help of:
  - The **unit circle**
  - The **graphs of trigonometric functions**

How are trigonometric equations of the form sin x = k solved?

- It is a good idea to sketch the graph of the trigonometric function first
  - Use the given range of values as the domain for your graph
  - The intersections of the graph of the function and the line y = k will show you
    - The location of the solutions
    - The number of solutions
  - You will be able to use the symmetry properties of the graph to find all secondary values within the given range of values
- The method for finding secondary values are:
  - For the equation $\sin x = k$ the primary value is $x_1 = \sin^{-1} k$
    - A secondary value is $x_2 = 180° - \sin^{-1} k$
    - Then all values within the range can be found using $x_1 \pm 360n$ and $x_2 \pm 360n$ where $n \in \mathbb{N}$
  - For the equation $\cos x = k$ the primary value is $x_1 = \cos^{-1} k$
    - A secondary value is $x_2 = -\cos^{-1} k$
    - Then all values within the range can be found using $x_1 \pm 360n$ and $x_2 \pm 360n$ where $n \in \mathbb{N}$
  - For the equation $\tan x = k$ the primary value is $x = \tan^{-1} k$
    - All secondary values within the range can be found using $x \pm 180n$ where $n \in \mathbb{N}$

Your notes

Examiner Tip

- If you are using your GDC it will only give you the principal value and you need to find all other solutions for the given interval
- Sketch out the CAST diagram and the trig graphs on your exam paper to refer back to as many times as you need to

Worked example

Solve the equation $2\cos x = -1$, finding all solutions in the range $-\pi \leq x \leq \pi$.

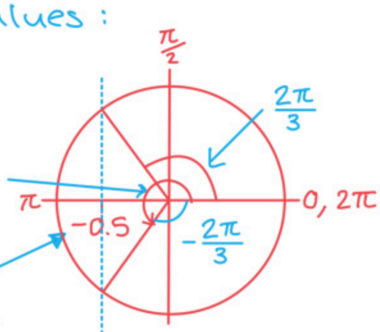
Isolate $\cos x$: $\cos x = \frac{-1}{2}$

$x = \cos^{-1}\left(-\frac{1}{2}\right)$

Use GDC or knowledge of exact values $= \frac{2\pi}{3}$ ← Primary value

Find secondary values:

$2\pi - \frac{2\pi}{3} = \frac{4\pi}{3}$



Working with $\cos x$ so find x-coordinate

$\frac{2\pi}{3} \pm 2\pi n$ and $\frac{4\pi}{3} \pm 2\pi n$

Find all answers in range $-\pi \leq x \leq 3\pi$

$-\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}$

Your notes

Trigonometric Equations: sin(ax + b) = k

How can I solve equations with transformations of trig functions?

- Trigonometric equations in the form $\sin(ax+b)$ can be solved in more than one way
- The easiest method is to consider the transformation of the angle as a substitution
  - For example let $u = ax + b$
- Transform the given interval for the solutions in the same way as the angle
  - For example if the given interval is $0° \le x \le 360°$ the new interval will be
  - $(a(0°)+b) \le u \le (a(360°)+b)$
- Solve the function to find the primary value for u
- Use either the unit circle or sketch the graph to find all the other solutions in the range for u
- Undo the substitution to convert all of the solutions back into the corresponding solutions for x
- Another method would be to sketch the transformation of the function
  - If you use this method then you will not need to use a substitution for the range of values

Examiner Tip

- If you transform the interval, remember to convert the found angles back to the final values at the end!
- If you are using your GDC it will only give you the principal value and you need to find all other solutions for the given interval
- Sketch out the CAST diagram and the trig graphs on your exam paper to refer back to as many times as you need to

Your notes

Worked example

Solve the equation $2\cos(2x - 30°) = -1$, finding all solutions in the range $-360° \le x \le 360°$.

$2\cos(2x - 30°) = -1 \qquad -360° \le x \le 360°$

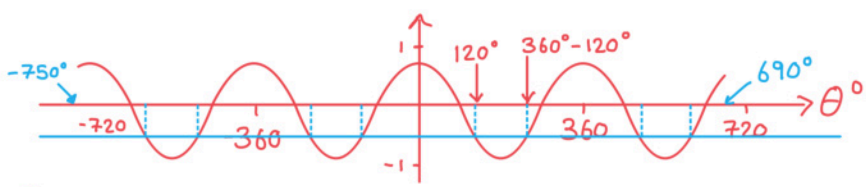
Start by changing the range: $-750° \le 2x - 30 \le 690°$

Substitute $\theta = 2x - 30$:

$2\cos\theta = -1 \qquad -750° \le \theta \le 690°$

$\cos\theta = -\frac{1}{2}$

$\theta = \cos^{-1}\left(-\frac{1}{2}\right) = 120°$ ← primary value



From the sketch you can see there are 8 solutions:

$\theta = 120° \pm 360°$ and $\theta = 240° \pm 360°$

$\theta = -600°, -480°, -240°, -120°, 120°, 240°, 480°, 600°$

Solve for x: $x = \frac{\theta° + 30}{2}$

$x = -285°, -225°, -105°, -45°, 75°, 135°, 255°, 315°$

3.6.6 Quadratic Trigonometric Equations

Your notes

Quadratic Trigonometric Equations

How are quadratic trigonometric equations solved?

- A quadratic trigonometric equation is one that includes either $\sin^2\theta$, $\cos^2\theta$ or $\tan^2\theta$
- Often the **identity** $\sin^2\theta+\cos^2\theta=1$ can be used to rearrange the equation into a form that is possible to solve
  - If the equation involves both sine and cosine then the **Pythagorean identity** should be used to write the equation in terms of just one of these functions
- Solve the **quadratic equation** using your GDC, the quadratic equation or factorisation
  - This can be made easier by changing the function to a single letter
    - Such as changing $2\cos^2\theta-3\cos\theta-1=0$ to $2c^2-3c-1=0$
- A quadratic can give up to two solutions
  - You must consider both solutions to see whether a real value exists
  - Remember that solutions for sin θ = k and cos θ = k only exist for $-1 \le k \le 1$
  - Solutions for tan θ = k exist for all values of k
- Find all solutions within the given interval
  - There will often be more than two solutions for one quadratic equation
  - The best way to check the number of solutions is to sketch the graph of the function

Examiner Tip

- Sketch the trig graphs on your exam paper to refer back to as many times as you need to!
- Be careful to make sure you have found **all** of the solutions in the given interval, being super-careful if you get a negative solution but have a positive interval

Your notes

Worked example

Solve the equation $11\sin x - 7 = 5\cos^2 x$, finding all solutions in the range $0 \leq x \leq 2\pi$.

Use the identity $\cos^2 x = 1 - \sin^2 x$ to write equation in terms of $\sin x$:

$$11\sin x - 7 = 5(1 - \sin^2 x)$$ ← in formula booklet.

$$= 5 - 5\sin^2 x$$

Move all terms to one side:

$$11\sin x - 7 - (5 - 5\sin^2 x) = 0$$

Spot the hidden quadratic:

$$11\sin x - 7 - 5 + 5\sin^2 x = 0$$

$$5\sin^2 x + 11\sin x - 12 = 0$$

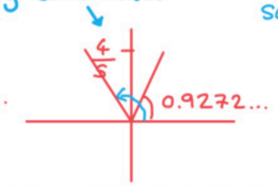
$$\sin x = \frac{4}{5} \quad \text{or} \quad \sin x = -3$$

$\sin x = -3$: no solution so disregard.

$\sin x$ so use y coordinate.

$$\sin x = \frac{4}{5}$$

$$x = 0.9272\ldots$$ ← primary solution



$$x = \pi - 0.9272\ldots$$

$$= 2.214\ldots$$ ← secondary solution

$x = 0.927,\ 2.21$ (3 s.f.)