

 **DP IB Maths: AA SL**
Your notes

2.4 Further Functions & Graphs

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Your notes

2.4.1 Reciprocal & Rational Functions

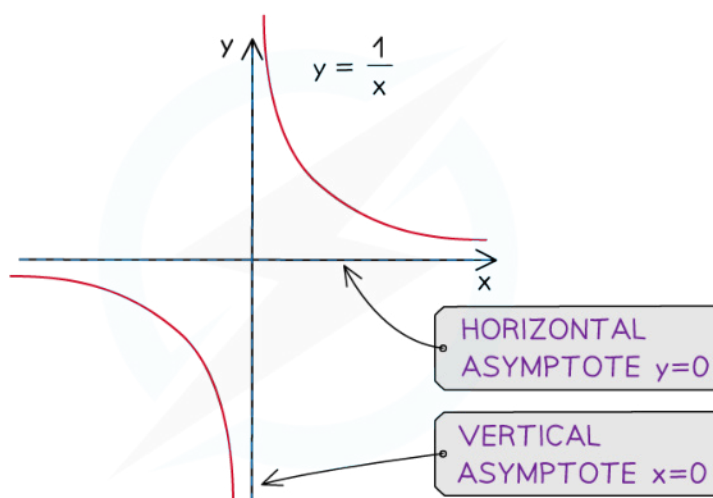
Reciprocal Functions & Graphs

What is the reciprocal function?

- The **reciprocal function** is defined by $f(x) = \frac{1}{x}$, $x \neq 0$
- Its **domain** is the set of **all real values except 0**
- Its **range** is the set of **all real values except 0**
- The reciprocal function has a **self-inverse** nature
 - $f^{-1}(x) = f(x)$
 - $(f \circ f)(x) = x$

What are the key features of the reciprocal graph?

- The graph **does not have a y-intercept**
- The graph **does not have any roots**
- The graph has **two asymptotes**
 - A **horizontal** asymptote at the x-axis: $y = 0$
 - This is the **limiting value** when the absolute value of x gets very large
 - A **vertical** asymptote at the y-axis: $x = 0$
 - This is the value that causes the **denominator to be zero**
- The graph has **two axes of symmetry**
 - $y = x$
 - $y = -x$
- The graph **does not have any minimum or maximum points**



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Linear Rational Functions & Graphs

What is a rational function?

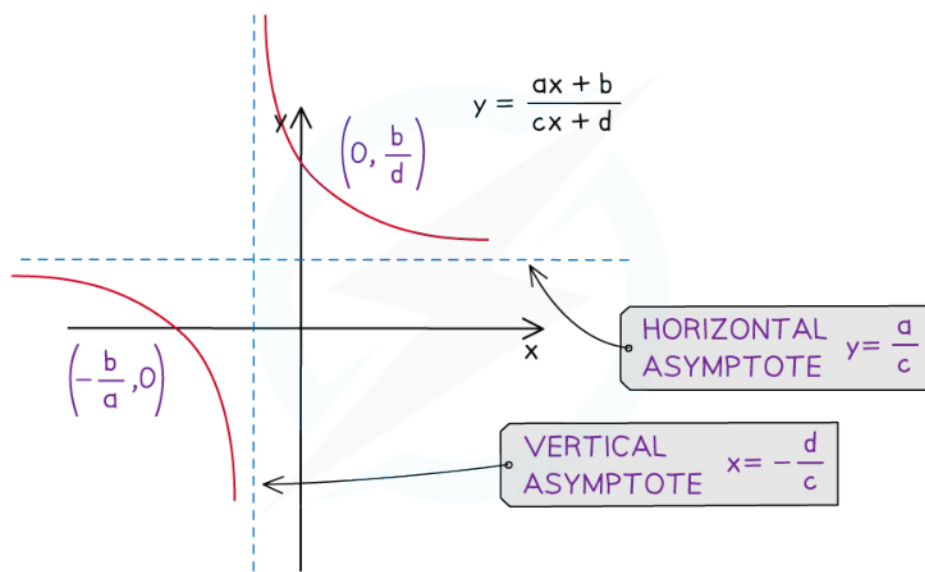
- A **rational function** is of the form $f(x) = \frac{ax + b}{cx + d}$, $x \neq -\frac{d}{c}$
- Its **domain** is the set of **all real values except** $-\frac{d}{c}$
- Its **range** is the set of **all real values except** $\frac{a}{c}$
- The **reciprocal function** is a **special case** of a rational function

What are the key features of rational graphs?

- The graph has a **y-intercept** at $\left(0, \frac{b}{d}\right)$ provided $d \neq 0$
- The graph has **one root** at $\left(-\frac{b}{a}, 0\right)$ provided $a \neq 0$
- The graph has **two asymptotes**
 - A **horizontal asymptote**: $y = \frac{a}{c}$
 - This is the **limiting value** when the absolute value of x gets very large
 - A **vertical asymptote**: $x = -\frac{d}{c}$
 - This is the value that causes the **denominator to be zero**
- The graph **does not have any minimum or maximum points**
- If you are asked to **sketch or draw** a rational graph:
 - Give the **coordinates** of any **intercepts** with the axes
 - Give the **equations** of the **asymptotes**



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 **Examiner Tip**

- If you draw a horizontal line anywhere it should only intersect this type of graph once at most
- The only horizontal line that should not intersect the graph is the horizontal asymptote
 - This can be used to check your sketch in an exam



Your notes

Worked example

The function f is defined by $f(x) = \frac{10 - 5x}{x + 2}$ for $x \neq -2$.

- a) Write down the equation of
- the vertical asymptote of the graph of f ,
 - the horizontal asymptote of the graph of f .

(i) Vertical asymptote is when denominator equals zero

$$x + 2 = 0 \quad \boxed{x = -2}$$

(ii) Horizontal asymptote is limiting value as x gets large

$$\lim_{x \rightarrow \infty} \frac{10 - 5x}{x + 2} = \lim_{x \rightarrow \infty} \frac{-5x}{x} \quad \boxed{y = -5}$$

- b) Find the coordinates of the intercepts of the graph of f with the axes.

y -intercept occurs when $x = 0$

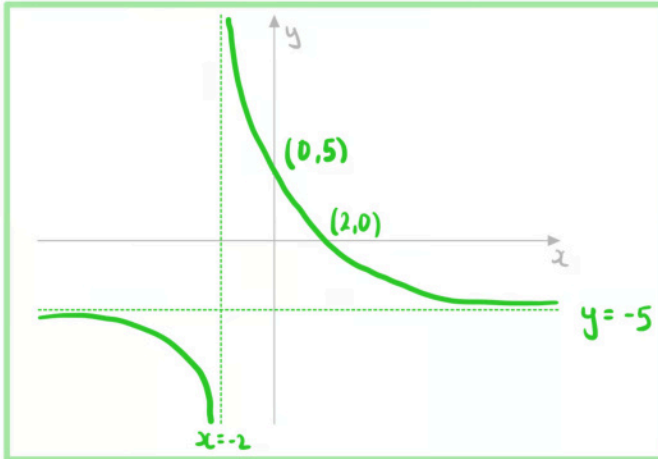
$$y = \frac{10 - 5(0)}{0 + 2} = 5 \quad \boxed{(0, 5)}$$

x -intercept occurs when $y = 0$

$$\frac{10 - 5x}{x + 2} = 0 \Rightarrow 10 - 5x = 0 \Rightarrow x = 2 \quad \boxed{(2, 0)}$$

- c) Sketch the graph of f .

Include asymptotes and intercepts



Your notes

2.4.2 Exponential & Logarithmic Functions



Your notes

Exponential Functions & Graphs

What is an exponential function?

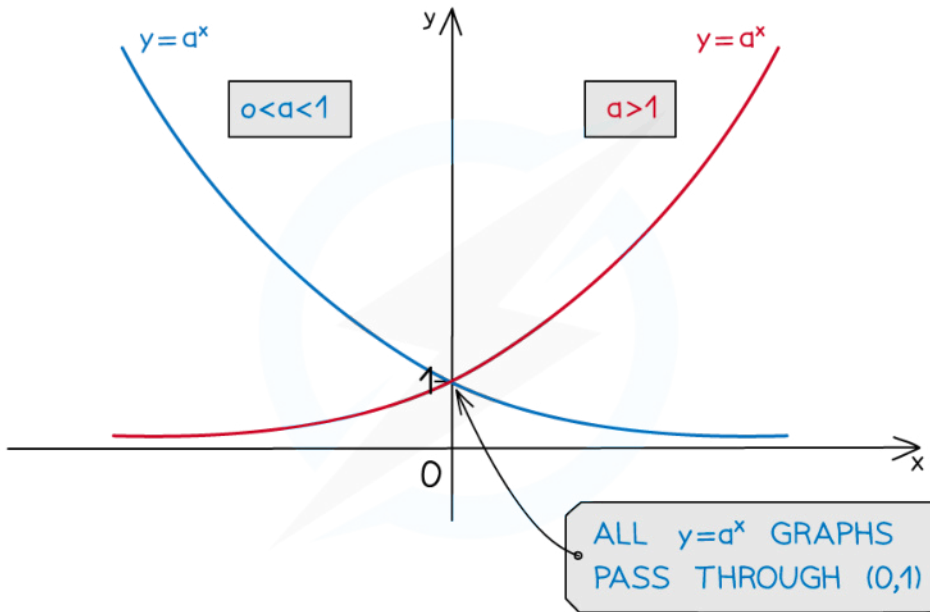
- An **exponential function** is defined by $f(x) = a^x$, $a > 0$
- Its **domain** is the set of **all real values**
- Its **range** is the set of **all positive real values**
- An important exponential function is $f(x) = e^x$
 - Where e is the mathematical constant 2.718...
- Any exponential function can be written using e
 - $a^x = e^{x \ln a}$
 - This is given in the **formula booklet**

What are the key features of exponential graphs?

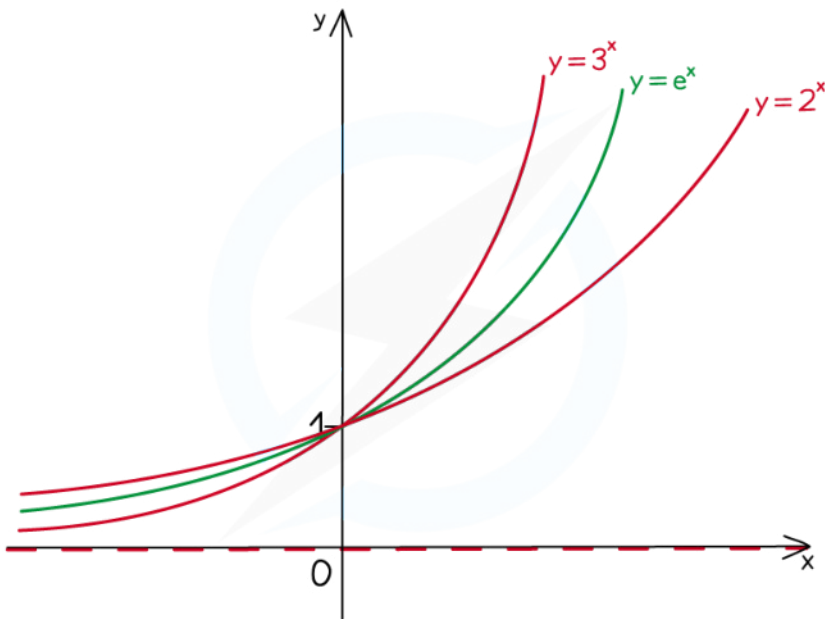
- The graphs have a **y-intercept** at $(0, 1)$
- The graph **will always** pass through the **point** $(1, a)$
- The graphs **do not have any roots**
- The graphs have a **horizontal asymptote** at the x -axis: $y = 0$
 - For $a > 1$ this is the **limiting value** when x tends to **negative infinity**
 - For $0 < a < 1$ this is the **limiting value** when x tends to **positive infinity**
- The graphs **do not have any minimum or maximum points**



Your notes



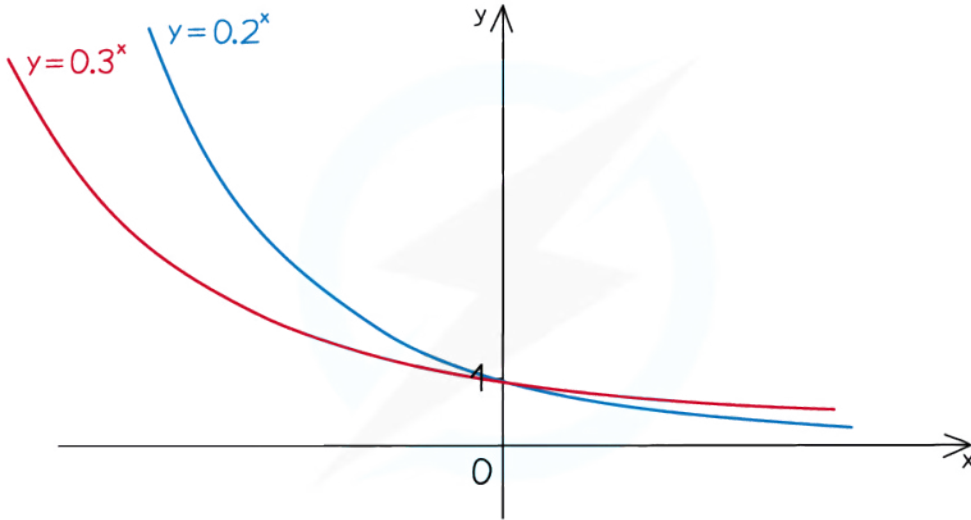
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Logarithmic Functions & Graphs

What is a logarithmic function?

- A **logarithmic function** is of the form $f(x) = \log_a x, x > 0$
- Its **domain** is the set of all **positive real values**
 - You can't take a log of zero or a negative number
- Its **range** is set of **all real values**
- $\log_a x$ and a^x are **inverse** functions
- An important logarithmic function is $f(x) = \ln x$
 - This is the natural logarithmic function $\ln x \equiv \log_e x$
 - This is the inverse of e^x
 - $\ln e^x = x$ and $e^{\ln x} = x$
- Any logarithmic function can be written using \ln
 - $\log_a x = \frac{\ln x}{\ln a}$ using the change of base formula

What are the key features of logarithmic graphs?

- The graphs **do not have a y-intercept**
- The graphs have **one root** at (1, 0)
- The graphs **will always** pass through the point (a, 1)
- The graphs have a **vertical asymptote** at the y-axis: $x = 0$
- The graphs **do not have any minimum or maximum points**



Your notes

Worked example

The function f is defined by $f(x) = \log_5 x$ for $x > 0$.

- a) Write down the inverse of f . Give your answer in the form $e^{g(x)}$.

Formula booklet

Exponents & logarithms	$a^x = b \Leftrightarrow x = \log_a b$	$a > 0, b > 0, a \neq 1$
------------------------	--	--------------------------

$$x = \log_5 y \Leftrightarrow y = 5^x$$

Formula booklet

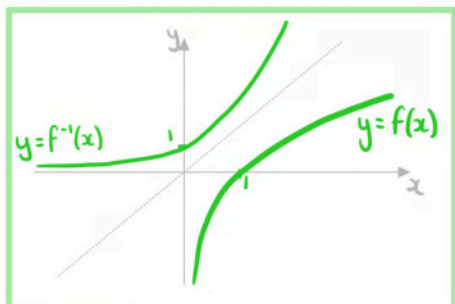
Exponential & logarithmic functions	$a^x = e^{x \ln a}$
-------------------------------------	---------------------

$$5^x = e^{x \ln 5}$$

$$f^{-1}(x) = e^{x \ln 5}$$

- b) Sketch the graphs of f and its inverse on the same set of axes.

f and f^{-1} are reflections in line $y=x$





Your notes

2.4.3 Solving Equations

Solving Equations Analytically

How can I solve equations analytically where the unknown appears only once?

- These equations can be **solved by rearranging**
- For **one-to-one functions** you can just apply the **inverse**
 - Addition and subtraction are inverses
 - $y = x + k \Leftrightarrow x = y - k$
 - Multiplication and division are inverses
 - $y = kx \Leftrightarrow x = \frac{y}{k}$
 - Taking the reciprocal is a self-inverse
 - $y = \frac{1}{x} \Leftrightarrow x = \frac{1}{y}$
 - Odd powers and roots are inverses
 - $y = x^n \Leftrightarrow x = \sqrt[n]{y}$
 - $y = x^n \Leftrightarrow x = y^{\frac{1}{n}}$
 - Exponentials and logarithms are inverses
 - $y = a^x \Leftrightarrow x = \log_a y$
 - $y = e^x \Leftrightarrow x = \ln y$
- For **many-to-one functions** you will need to use your knowledge of the functions to find the **other solutions**
 - Even powers lead to positive and negative solutions
 - $y = x^n \Leftrightarrow x = \pm \sqrt[n]{y}$
 - Modulus functions lead to positive and negative solutions
 - $y = |x| \Leftrightarrow x = \pm y$
 - Trigonometric functions lead to infinite solutions using their symmetries
 - $y = \sin x \Leftrightarrow x = 2k\pi + \sin^{-1}y$ or $x = (1 + 2k)\pi - \sin^{-1}y$
 - $y = \cos x \Leftrightarrow x = 2k\pi \pm \cos^{-1}y$
 - $y = \tan x \Leftrightarrow x = k\pi + \tan^{-1}y$
- Take care when you apply **many-to-one functions** to **both sides** of an equation as this can create **additional solutions** which are incorrect
 - For example: squaring both sides
 - $x + 1 = 3$ has one solution $x = 2$
 - $(x + 1)^2 = 3^2$ has two solutions $x = 2$ and $x = -4$
- Always **check your solutions** by substituting back into the **original equation**



Your notes

How can I solve equations analytically where the unknown appears more than once?

- Sometimes it is possible to **simplify expressions** to make the **unknown appear only once**
- **Collect all terms** involving x on **one side** and try to simplify into one term
 - For **exponents** use
 - $a^{f(x)} \times a^{g(x)} = a^{f(x) + g(x)}$
 - $\frac{a^{f(x)}}{a^{g(x)}} = a^{f(x) - g(x)}$
 - $(a^{f(x)})^{g(x)} = a^{f(x) \times g(x)}$
 - $a^{f(x)} = e^{f(x) \ln a}$
 - For **logarithms** use
 - $\log_a f(x) + \log_a g(x) = \log_a (f(x) \times g(x))$
 - $\log_a f(x) - \log_a g(x) = \log_a \left(\frac{f(x)}{g(x)} \right)$
 - $n \log_a f(x) = \log_a (f(x))^n$

How can I solve equations analytically when the equation can't be simplified?

- Sometimes it is **not possible to simplify** equations
- Most of these equations **cannot be solved analytically**
- A **special case** that can be solved is where the equation can be **transformed into a quadratic** using a substitution
 - These will have **three terms** and involve the same type of function
- **Identify the suitable substitution** by considering which **function is a square of another**
 - For example: the following can be transformed into $2y^2 + 3y - 4 = 0$
 - $2x^4 + 3x^2 - 4 = 0$ using $y = x^2$
 - $2x + 3\sqrt{x} - 4 = 0$ using $y = \sqrt{x}$
 - $\frac{2}{x^6} + \frac{3}{x^3} - 4 = 0$ using $y = \frac{1}{x^3}$
 - $2e^{2x} + 3e^x - 4 = 0$ using $y = e^x$
 - $2 \times 25^x + 3 \times 5^x - 4 = 0$ using $y = 5^x$
 - $2^{2x+1} + 3 \times 2^x - 4 = 0$ using $y = 2^x$
 - $2(x^3 - 1)^2 + 3(x^3 - 1) - 4 = 0$ using $y = x^3 - 1$
- **To solve:**
 - Make the **substitution** $y = f(x)$
 - **Solve the quadratic equation** $ay^2 + by + c = 0$ to get y_1 & y_2
 - **Solve** $f(x) = y_1$ and $f(x) = y_2$
 - Note that some equations might have **zero or several solutions**



Your notes

Can I divide both sides of an equation by an expression?

- When dividing by an expression you must consider whether the **expression could be zero**
- Dividing by an expression that could be zero could result in you **losing solutions to the original equation**
 - For example: $(x + 1)(2x - 1) = 3(x + 1)$
 - If you divide both sides by $(x + 1)$ you get $2x - 1 = 3$ which gives $x = 2$
 - However $x = -1$ is also a solution to the original equation
- To ensure you **do not lose solutions** you can:
 - **Split the equation into two equations**
 - One where the dividing expression equals zero: $x + 1 = 0$
 - One where the equation has been divided by the expression: $2x - 1 = 3$
 - **Make the equation equal zero and factorise**
 - $(x + 1)(2x - 1) - 3(x + 1) = 0$
 - $(x + 1)(2x - 1 - 3) = 0$ which gives $(x + 1)(2x - 4) = 0$
 - Set each factor equal to zero and solve: $x + 1 = 0$ and $2x - 4 = 0$

Examiner Tip

- A common mistake that students make in exams is applying functions to each term rather than to each side
 - For example: Starting with the equation $\ln x + \ln(x - 1) = 5$ it would be incorrect to write $e^{\ln x} + e^{\ln(x - 1)} = e^5$ or $x + (x - 1) = e^5$
 - Instead it would be correct to write $e^{\ln x + \ln(x - 1)} = e^5$ and then simplify from there



Your notes

Worked example

Find the exact solutions for the following equations:

a) $5 - 2\log_4 x = 0$.

Rearrange using inverse functions

$$\begin{aligned}
 5 - 2\log_4 x &= 0 \\
 2\log_4 x &= 5 \\
 \log_4 x &= \frac{5}{2} \\
 x &= 4^{\frac{5}{2}} \\
 x &= (\sqrt{4})^5
 \end{aligned}$$

$$y = x - k \Leftrightarrow x = y + k$$

$$y = kx \Leftrightarrow x = \frac{y}{k}$$

$$y = \log_a x \Leftrightarrow x = a^y$$

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m$$

$$x = 32$$

b) $x = \sqrt{x+2}$.

Square both sides (Many-to-one function)

$$x^2 = x + 2 \Rightarrow x^2 - x - 2 = 0$$

$$(x - 2)(x + 1) = 0 \Rightarrow x = 2 \text{ or } x = -1$$

Check whether each solution is valid

$$x = 2: \text{ LHS} = 2 \quad \text{RHS} = \sqrt{2+2} = 2 \quad \checkmark$$

$$x = -1: \text{ LHS} = -1 \quad \text{RHS} = \sqrt{-1+2} = 1 \quad \times$$

$$x = 2$$

c) $e^{2x} - 4e^x - 5 = 0$.



Your notes

Notice $e^{2x} = (e^x)^2$, let $y = e^x$

$$y^2 - 4y - 5 = 0 \Rightarrow (y+1)(y-5) = 0$$

$$y = -1 \text{ or } y = 5$$

Solve using $y = e^x$

$e^x = -1$ has no solutions as $e^x > 0$

$$e^x = 5 \quad \therefore x = \ln 5$$

$$x = \ln 5$$

Solving Equations Graphically

How can I solve equations graphically?

- To solve $f(x) = g(x)$
 - One method is to **draw the graphs** $y = f(x)$ and $y = g(x)$
 - The **solutions** are the **x-coordinates** of the points of **intersection**
 - Another method is to **draw the graph** $y = f(x) - g(x)$ or $y = g(x) - f(x)$
 - The **solutions** are the **roots (zeros)** of this graph
 - This method is sometimes quicker as it involves **drawing only one graph**

Why do I need to solve equations graphically?

- Some equations **cannot be solved analytically**
 - **Polynomials** of degree higher than 4
 - $x^5 - x + 1 = 0$
 - Equations involving **different types of functions**
 - $e^x = x^2$

Examiner Tip

- On a calculator paper you are allowed to solve equations using your GDC unless the question asks for an algebraic method
- If your answer needs to be an exact value then you might need to solve analytically to get the exact value



Your notes

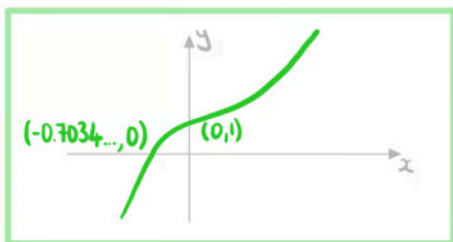


Your notes

 **Worked example**

- a) Sketch the graph $y = e^x - x^2$.

Sketch using GDC



- b) Hence find the solution to $e^x = x^2$.

$$e^x = x^2 \text{ when } e^x - x^2 = 0$$

Solution is the x-intercept of $y = e^x - x^2$

$$x = -0.703 \text{ (3sf)}$$



Your notes

2.4.4 Modelling with Functions

Modelling with Functions

What is a mathematical model?

- A **mathematical model** simplifies a real-world situation so it can be described using mathematics
 - The model can then be used to make predictions
- **Assumptions** about the situation are made in order to simplify the mathematics
- Models can be **refined** (improved) if further information is available or if the model is compared to real-world data

How do I set up the model?

- The question could:
 - give you the equation of the model
 - tell you about the relationship
 - It might say the relationship is linear, quadratic, etc
 - ask you to suggest a **suitable model**
 - Use your knowledge of each model
 - E.g. if it is compound interest then an exponential model is the most appropriate
- You may have to determine a **reasonable domain**
 - Consider real-life context
 - E.g. if dealing with hours in a day then $0 \leq t < 24$
 - E.g. if dealing with physical quantities (such as length) then, $x > 0$
 - Consider the **possible ranges**
 - If the outcome cannot be negative then you want to choose a domain which corresponds to a range with no negative values
 - **Sketching the graph** is helpful to determine a suitable domain

Which models might I need to use?

- You could be given any model and be expected to use it
- Common models include:
 - **Linear**
 - Arithmetic sequences
 - Linear regression
 - **Quadratic**
 - Projectile motion
 - The height of a cable supporting a bridge
 - Profit
 - **Exponential**
 - Geometric sequences
 - Exponential growth and decay
 - Compound interest

- **Logarithmic**
 - Richter scale for the magnitude of earthquakes
- **Rational**
 - Temperature of a cup of coffee
- **Trigonometric**
 - The depth of a tide



Your notes

How do I use a model?

- You can use a model by substituting in values for the variable to **estimate outputs**
 - For example: Let $h(t)$ be the height of a football t seconds after being kicked
 - $h(3)$ will be an estimate for the height of the ball 3 seconds after being kicked
- Given an **output** you can **form an equation** with the model to **estimate the input**
 - For example: Let $P(n)$ be the profit made by selling n items
 - Solving $P(n) = 100$ will give you an estimate for the number of items needing to be sold to make a profit of 100
- If your variable is **time** then substituting $t = 0$ will give you the **initial value** according to the model
- Fully understand the **units for the variables**
 - If the units of P are measured in **thousand dollars** then $P = 3$ represents \$3000
- Look out for **key words** such as:
 - Initially
 - Minimum/maximum
 - Limiting value

What do I do if some of the parameters are unknown?

- A general method is to **form equations** by substituting in given values
 - You can form **multiple equations** and **solve them simultaneously** using your GDC
 - This method **works for all models**
- The **initial value** is the value of the function when the variable is 0
 - This is **normally one of the parameters** in the equation of the model



Your notes

Worked example

The temperature, $T^{\circ}\text{C}$, of a cup of coffee is monitored. Initially the temperature is 80°C and 5 minutes later it is 40°C . It is suggested that the temperature follows the model:

$$T(t) = Ae^{kt} + 16, t \geq 0.$$

where t is the time, in minutes, after the coffee has been made.

- a) State the value of A .

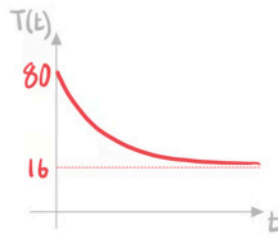
Initially temperature is 80°C

$$T(0) = 80$$

$$Ae^{-k(0)} + 16 = 80$$

$$A + 16 = 80$$

$$A = 64$$



- b) Find the exact value of k .

$$t = 5, T = 40$$

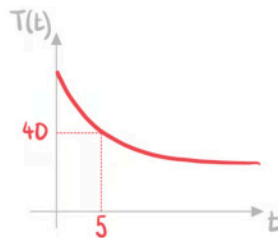
$$40 = 64e^{5k} + 16$$

$$64e^{5k} = 24$$

$$e^{5k} = \frac{3}{8}$$

$$5k = \ln \frac{3}{8}$$

$$k = \frac{1}{5} \ln \frac{3}{8}$$



- c) Find the time taken for the temperature of the coffee to reach 30°C .



Your notes

Find t such that $T(t) = 30$

$$30 = 64e^{kt} + 16$$

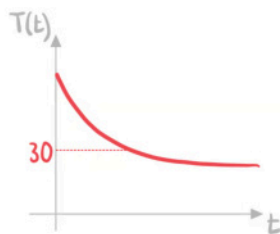
Leave as k until the end to save writing $\frac{1}{5} \ln \frac{3}{8}$ each time

$$64e^{kt} = 14$$

$$e^{kt} = \frac{7}{32}$$

$$kt = \ln \frac{7}{32}$$

$$t = \frac{\ln \frac{7}{32}}{k} = \frac{\ln \frac{7}{32}}{\frac{1}{5} \ln \frac{3}{8}} = 7.7476..$$



7.75 minutes (3sf)