DP IB Maths: AI HL

Your notes

3.1 Geometry Toolkit

Contents

Your notes

3.1.1 Coordinate Geometry

Basic Coordinate Geometry

What are cartesian coordinates?

- **Cartesian** coordinates are basically the *x*-*y* coordinate system
  - They allow us to label where things are in a two-dimensional plane
- In the 2D cartesian system, the horizontal axis is labelled *x* and the vertical axis is labelled *y*

What can we do with coordinates?

- If we have two points with coordinates (x_1, y_1) and (x_2, y_2) then we should be able to find
  - The **midpoint** of the two points
  - The **distance** between the two points
  - The **gradient** of the line between them

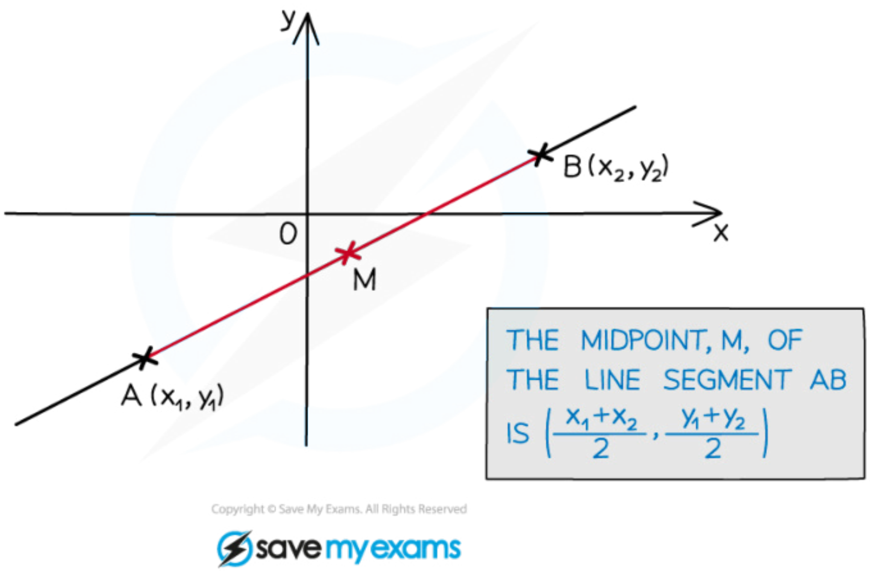
How do I find the midpoint of two points?

- The midpoint is the **average (middle) point**
  - It can be found by finding the middle of the *x*-coordinates and the middle of the *y*-coordinates
- The coordinates of the midpoint will be

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

  - This is given in the formula booklet under the prior learning section at the beginning

Your notes



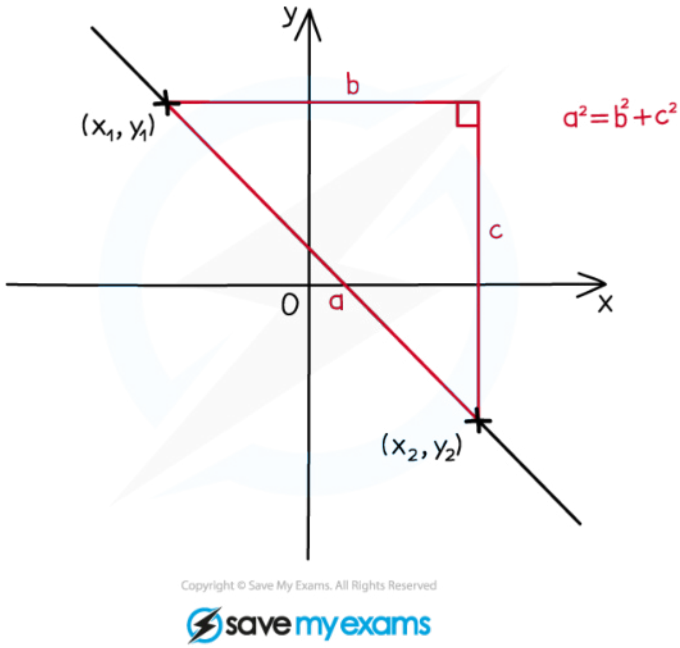
How do I find the distance between two points?

- The distance between two points with coordinates (x_1, y_1) and (x_2, y_2) can be found using the formula

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

  - This is given in the formula booklet in the *prior learning* section at the beginning
- Pythagoras' Theorem $a^2 = b^2 + c^2$ is used to find the length of a line between two coordinates
- If the coordinates are labelled A and B then the line segment between them is written with the notation [AB]

Your notes



How do I find the gradient of the line between two points?

- The gradient of a line between two points with coordinates (x_1, y_1) and (x_2, y_2) can be found using the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

  - This is given in the formula booklet under section 2.1 Gradient formula
  - This is usually known as $m = \frac{\text{rise}}{\text{run}}$

Your notes

Worked example

Point *A* has coordinates (3, -4) and point *B* has coordinates (-5, 2).

i) Calculate the distance of the line segment *AB*.

A: (3, −4) B: (−5, 2)

x_1 ↗ (3), y_1 ↗ (−4), x_2 ↗ (−5), y_2 ↗ (2)

Formula for distance between two points:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Sub coordinates for A and B into the formula:

$$d = \sqrt{(3-(-5))^2 + (-4-2)^2}$$

$$= \sqrt{8^2 + (-6)^2} = \sqrt{100}$$

d = 10 units

ii) Find the gradient of the line connecting points *A* and *B*.

Your notes

A: $(3, -4)$ B: $(-5, 2)$

$x_1 = 3$, $y_1 = -4$, $x_2 = -5$, $y_2 = 2$

Formula for gradient of a line segment:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Sub coordinates for A and B into the formula:

$$m = \frac{2 - -4}{-5 - 3} = \frac{6}{-8} = -\frac{3}{4}$$

$$\boxed{m = -\frac{3}{4}}$$

iii) Find the midpoint of [*AB*] .

A: $(3, -4)$ B: $(-5, 2)$

$x_1 = 3$, $y_1 = -4$, $x_2 = -5$, $y_2 = 2$

Formula for the midpoint of two coordinates:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Sub values in:

$$\text{Midpoint} = \left(\frac{3 + (-5)}{2}, \frac{-4 + 2}{2}\right) = (-1, -1)$$

$$\boxed{\text{Midpoint} = (-1, -1)}$$

Perpendicular Bisectors

Your notes

What is a perpendicular bisector?

- A perpendicular bisector of a line segment cuts the line segment in half at a right angle
  - Perpendicular lines meet at right angles
  - Bisect means to cut in half
- Two lines are perpendicular if the **product of their gradients is -1**

How do I find the equation of the perpendicular bisector of a line segment?

- To find the equation of a straight line you need to find
  - The gradient of the line
  - A coordinate of a point on the line
- To find the equation of the **perpendicular bisector** of a line segment follow these steps:
  - STEP 1: Find the coordinates of the midpoint of the line segment
    - We know that the perpendicular bisector will cut the line segment in half so we can use the midpoint of the line segment as the known coordinate on the bisector
  - STEP 2: Find the gradient of the line segment
  - STEP 3: Find the gradient of the perpendicular bisector
    - This will be -1 divided by the gradient of the line segment
  - STEP 4: Substitute the gradient of the perpendicular bisector and the coordinates of the midpoint into an equation for a straight line
    - The **point-gradient** form $y - y_1 = m(x - x_1)$ is the easiest
  - STEP 5: Rearrange into the required form
    - Either $y = mx + c$ or $ax + by + d = 0$
    - These equations for a straight line are given in the formula booklet

Your notes

Worked example

Point A has coordinates (4, -6) and point B has coordinates (8, 6). Find the equation of the perpendicular bisector to [AB]. Give your answer in the form $ax + by + d = 0$.

Step 1: find the coordinates of the midpoint:

A: (4, −6) B: (8, 6)

x_1, y_1, x_2, y_2

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Sub values in:

$$\text{Midpoint} = \left(\frac{4+8}{2}, \frac{-6+6}{2}\right) = (6, 0)$$

Step 2: Find the gradient of [AB]:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - -6}{8 - 4} = \frac{12}{4} = 3$$

Step 3: Find the gradient of the perpendicular bisector:

$$m_{\perp r} = -\frac{1}{m} = -\frac{1}{3}$$

Step 4: Substitute gradient and coordinate into an equation for a straight line.

insert coordinates of the midpoint.

$$(y - y_1) = m(x - x_1)$$

$$(y - 0) = -\frac{1}{3}(x - 6)$$

Step 5: Rearrange into the form $ax + by + d = 0$

$$(y - 0) = -\frac{1}{3}(x - 6) \quad (\times -3)$$

$$-3y = x - 6 \quad (+3y)$$

$$x + 3y - 6 = 0$$

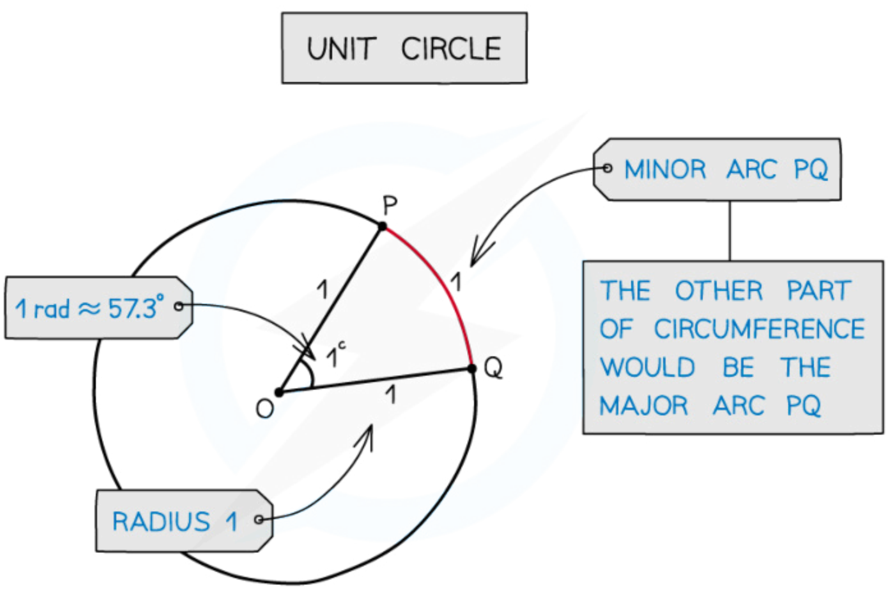
3.1.2 Radian Measure

Your notes

Radian Measure

What are radians?

- Radians are an alternative to degrees for measuring angles
- 1 radian is the angle in a **sector** of radius 1 and arc length 1
  - A circle with radius 1 is called a **unit circle**
- Radians are normally quoted in terms of π
  - 2π radians = 360°
  - π radians = 180°
- The symbol for radians is ^{c} but it is more usual to see **rad**
  - Often, when π is involved, no symbol is given as it is obvious it is in radians
  - Whilst it is okay to omit the symbol for radians, you should never omit the symbol for degrees
- In the exam you should use radians unless otherwise indicated

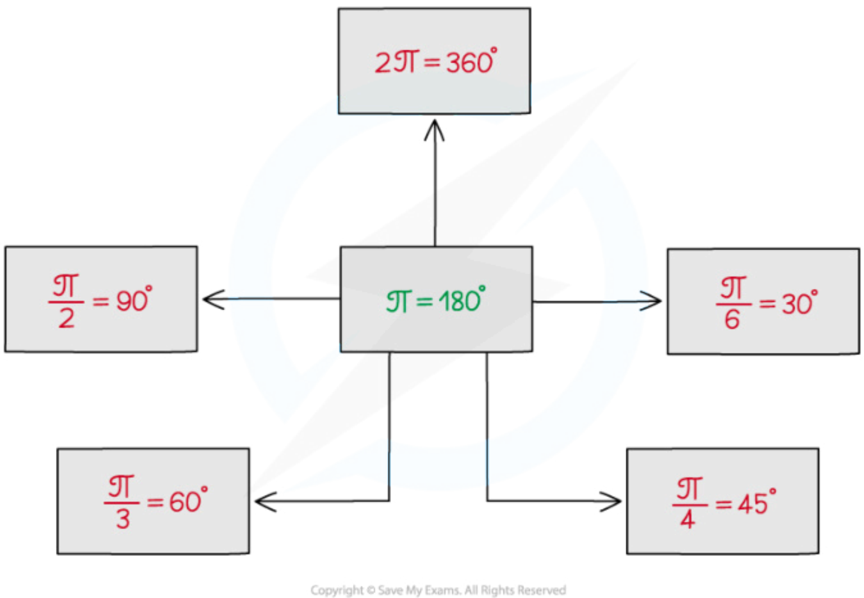


How do I convert between radians and degrees?

- Use $\pi^c = 180°$ to convert between radians and degrees
  - To convert from radians to degrees multiply by $\frac{180}{\pi}$

Your notes

- To convert from degrees to radians multiply by $\frac{\pi}{180}$
- Some of the common conversions are:
  - $2\pi^c = 360°$
  - $\pi^c = 180°$
  - $\frac{\pi}{2}^c = 90°$
  - $\frac{\pi}{3}^c = 60°$
  - $\frac{\pi}{4}^c = 45°$
  - $\frac{\pi}{6}^c = 30°$
- It is a good idea to remember some of these and use them to work out other conversions
- Your GDC will be able to work with both radians and degrees



Examiner Tip

- Sometimes an exam question will specify whether you should be using degrees or radians and sometimes it will not, if it doesn't it is expected that you will work in radians
- If the question involves π then working in radians is useful as there will likely be opportunities where you can cancel out π
- Make sure that your calculator is in the correct mode for the type of angle you are working with

Your notes

Worked example

i) Convert 43.8° to radians.

$$43.8^\circ \xrightarrow{\div 180^\circ} \frac{73}{300} \xrightarrow{\times \pi^c} \frac{73\pi}{300}^c \qquad (\pi^c = 180^\circ)$$

$$43.8^\circ = 0.764^c \ (3 \text{ s.f.})$$

ii) Convert $\frac{5\pi}{4}$ to degrees.

$$\frac{5\pi}{4} \xrightarrow{\div \pi^c} \frac{5}{4} \xrightarrow{\times 180^\circ} 225^\circ \qquad (\pi^c = 180^\circ)$$

$$\frac{5\pi}{4} = 225^\circ$$

3.1.3 Arcs & Sectors

Your notes

Length of an Arc

What is an arc?

- An arc is a part of the **circumference** of a circle
  - It is easiest to think of it as the crust of a single slice of pizza
- The length of an arc depends of the size of the angle at the centre of the circle
- If the angle at the centre is **less than 180°** then the arc is known as a **minor arc**
  - This could be considered as the crust of a single slice of pizza
- If the angle at the centre is **more than 180°** then the arc is known as a **major arc**
  - This could be considered as the crust of the remaining pizza after a slice has been taken away

How do I find the length of an arc?

- The length of an arc is simply a fraction of the circumference of a circle
  - The fraction can be found by dividing the angle at the centre by 360°
- The formula for the length, l, of an arc is

$$l = \frac{\theta}{360} \times 2\pi r$$

  - Where θ is the angle measured in degrees
  - r is the radius
  - This is **in the formula booklet**, you do not need to remember it

Examiner Tip

- Make sure that you read the question carefully to determine if you need to calculate the arc length of a sector, the perimeter or something else that incorporates the arc length!

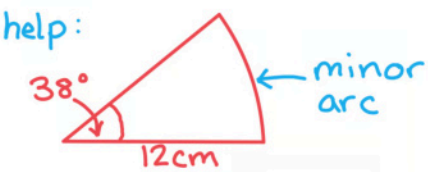
Your notes

Worked example

A circular pizza has had a slice cut from it, the angle of the slice that was cut was 38 °. The radius of the pizza is 12 cm. Find

i) the length of the outside crust of the slice of pizza (the minor arc),

A diagram will help:



Formula for the length of an arc:

$$l = \frac{\theta}{360} \times 2\pi r$$

Substitute:

$$l = \frac{38}{360} \times 2\pi(12)$$

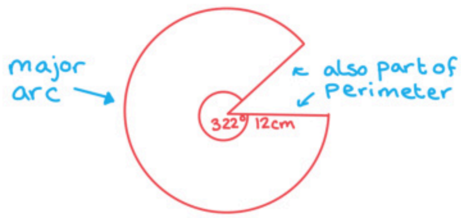
$$= \frac{38\pi}{15} = 7.9587\ldots \text{ cm}$$

Length of crust = 7.96 cm (3s.f)

ii) the perimeter of the remaining pizza.

Your notes

A diagram will help:



Formula for the Length of an arc:

$$l = \frac{\theta}{360} \times 2\pi r$$

Substitute:

$$l = \frac{322}{360} \times 2\pi(12)$$

$$= \frac{322}{15}\pi$$ ← Length of major arc

Find perimeter:

$$P = \text{major arc} + \text{radius} + \text{radius}$$

$$= \frac{322\pi}{15} + 12 + 12 = 91.4395... \text{ cm}$$

Perimeter = 91.4 cm (3s.f)

Area of a Sector

What is a sector?

- A sector is a part of a circle enclosed by two radii (radiuses) and an arc
  - It is easier to think of this as the shape of a single slice of pizza
- The area of a sector depends of the size of the angle at the centre of the sector
- If the angle at the centre is **less than 180°** then the sector is known as a **minor sector**
  - This could be considered as the shape of a single slice of pizza
- If the angle at the centre is **more than 180°** then the sector is known as a **major sector**
  - This could be considered as the shape of the remaining pizza after a slice has been taken away

How do I find the area of a sector?

- The area of a sector is simply a fraction of the area of the whole circle
  - The fraction can be found by dividing the angle at the centre by 360°
- The formula for the area, A, of a sector is

$$A = \frac{\theta}{360} \times \pi r^2$$

  - Where θ is the angle measured in degrees
  - r is the radius
  - This is **in the formula booklet**, you do not need to remember it

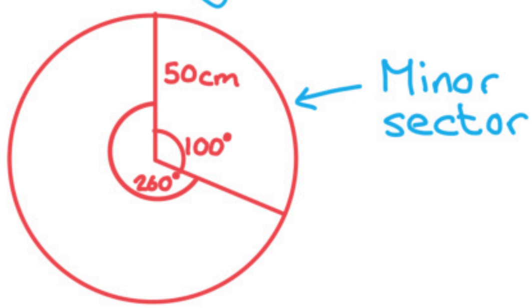
Your notes

Worked example

Jamie has divided a circle of radius 50 cm into two sectors; a minor sector of angle 100° and a major sector of angle 260°. He is going to paint the minor sector blue and the major sector yellow. Find

i) the area Jamie will paint blue,

Start with a diagram:



Formula for the area of a sector:

$$A = \frac{\theta}{360°} \times \pi r^2$$

Substitute: $A = \frac{100}{360} \times \pi \times 50^2$

$$= \frac{6250}{9}\pi$$

$$= 2181.66\ldots \text{ cm}^2$$

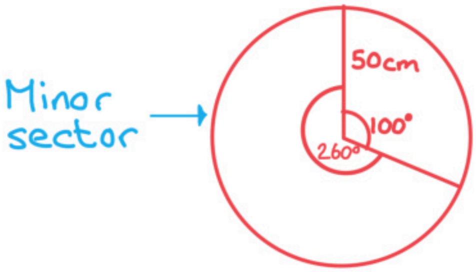
Blue area = 2180 cm² (3sf)

ii) the area Jamie will paint yellow.

Your notes

Your notes

Start with a diagram:



Formula for the area of a sector:

$$A = \frac{\theta}{360°} \times \pi r^2$$

Substitute: $A = \frac{260}{360} \times \pi \times 50^2$

$$= \frac{16250}{9}\pi$$

$$= 5672.32\ldots \text{ cm}^2$$

Yellow area = 5670 cm² (3sf)

Your notes

Arcs & Sectors Using Radians

How do I use radians to find the length of an arc?

- As the radian measure for a **full turn** is 2π, the fraction of the circle becomes $\frac{\theta}{2\pi}$
- Working in radians, the formula for the length of an arc will become

$$l = \frac{\theta}{2\pi} \times 2\pi r$$

- Simplifying, the formula for the length, l, of an arc is

$$l = r\theta$$

  - θ is the angle measured in **radians**
  - r is the radius
  - This is **given in the formula booklet,** you do not need to remember it

How do I use radians to find the area of a sector?

- As the radian measure for a **full turn** is 2π, the fraction of the circle becomes $\frac{\theta}{2\pi}$
- Working in radians, the formula for the area of a sector will become

$$A = \frac{\theta}{2\pi} \times \pi r^2$$

- Simplifying, the formula for the area, A, of a sector is

$$A = \frac{1}{2} r^2 \theta$$

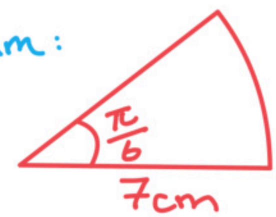
  - θ is the angle measured in **radians**
  - r is the radius
  - This is **given in the formula booklet,** you do not need to remember it

Your notes

Worked example

A slice of cake forms a sector of a circle with an angle of $\frac{\pi}{6}$ radians and radius of 7 cm. Find the area of the surface of the slice of cake and its perimeter.

Draw a diagram:



Area of a sector: $A = \frac{1}{2}r^2\theta$

Substitute: $r = 7, \ \theta = \frac{\pi}{6}$

$$A = \frac{1}{2}(7)^2\left(\frac{\pi}{6}\right) = \frac{49\pi}{12}$$

Area = 12.8 cm² (3 s.f.)

Perimeter = arc length + 2(radius)

Length of an arc: $l = r\theta$

$$P = 7\left(\frac{\pi}{6}\right) + 2(7)$$

Perimeter = 17.7 cm (3 s.f.)