DP IB Maths: AA HL

Your notes

1.6 Binomial Theorem

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1.6.1 Binomial Theorem

Binomial Theorem

What is the Binomial Theorem?

- The **binomial theorem (**sometimes known as the binomial expansion) gives a method for expanding a **two-term** expression in a bracket raised to a power
  - A **binomial expression** is in fact any two terms inside the bracket, however in IB the expression will usually be linear
- To expand a bracket with a two-term expression in:
  - First choose the most appropriate parts of the expression to assign to *a* and *b*
  - Then use the formula for the binomial theorem:

$$(a+b)^n = a^n + {}^nC_1\, a^{n-1} b + \ldots + {}^nC_r\, a^{n-r} b^r + \ldots + b^n$$

  - where ${}^nC_r = \dfrac{n!}{r!(n-r)!}$
    - See below for more information on ${}^nC_r$
    - You may also see ${}^nC_r$ written as $\begin{pmatrix} n \\ r \end{pmatrix}$ or ${}_nC_r$
- You will usually be asked to find the first three or four terms of an expansion
- Look out for whether you should give your answer in **ascending** or **descending** powers of *x*
  - For **ascending** powers start with the constant term, a^n
  - For **descending** powers start with the term with *x* in
    - You may wish to swap *a* and *b* over so that you can follow the general formula given in the formula book
- If you are not writing the full expansion you can either
  - show that the sequence continues by putting an ellipsis (…) after your final term
  - or show that the terms you have found are an approximation of the full sequence by using the sign for approximately equals to (≈)

How do I find the coefficient of a single term?

- Most of the time you will be asked to find the coefficient of a term, rather than carry out the whole expansion
- Use the formula for the general term

$${}^nC_r\, a^{n-r} b^r$$

- The question will give you the power of *x* of the term you are looking for

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- Use this to choose which value of r you will need to use in the formula
- This will depend on where the x is in the bracket
- The laws of indices can help you decide which value of r to use:
    - For $(a + bx)^n$ to find the coefficient of x^r use $a^{n-r}(bx)^r$
    - For $(a + bx^2)^n$ to find the coefficient of x^r use $a^{n-\frac{r}{2}}(bx^2)^{\frac{r}{2}}$
    - For $\left(a + \frac{b}{x}\right)^n$ look at how the powers will cancel out to decide which value of r to use
    - So for $\left(3x + \frac{2}{x}\right)^8$ to find the coefficient of x^2 use the term with $r = 3$ and to find the constant term use the term with $r = 4$
    - There are a lot of variations of this so it is usually easier to see this by inspection of the exponents
- You may also be given the coefficient of a particular term and asked to find an unknown in the brackets
    - Use the laws of indices to choose the correct term and then use the binomial theorem formula to form and solve and equation

Examiner Tip

- Binomial expansion questions can get messy, use separate lines to keep your working clear and always put terms in brackets

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Worked example

Find the first three terms, in ascending powers of x, in the expansion of $(3-2x)^5$.

$a = 3 \quad b = -2x \quad n = 5$

Substitute values into the formula for $(a+b)^n$

$(a+b)^n = a^n + {}^nC_1 a^{n-1} b + \ldots + {}^nC_r a^{n-r} b^r + \ldots + b^n$

Question asks for ascending powers of x, so start with the constant term, a^n.

$(3-2x)^5 = 3^5 + {}^5C_1 (3)^{5-1}(-2x) + {}^5C_2 (3)^{5-2}(-2x)^2 + \ldots$

Watch out for the negative

$\approx 243 + 5 \times 81 \times -2x + 10 \times 27 \times 4x^2$

$\approx 243 - 810x + 1080x^2$

$(3-2x)^5 \approx 243 - 810x + 1080x^2$

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The Binomial Coefficient nCr

What is ${}^{n}C_{r}$?

- If we want to find the number of ways to **choose** *r* items out of *n* different objects we can use the formula for ${}^{n}C_{r}$
  - The formula for *r* **combinations** of *n* items is ${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$
  - This formula is given in the formula booklet along with the formula for the binomial theorem
  - The function ${}^{n}C_{r}$ can be written $\binom{n}{r}$ or ${}_{n}C_{r}$ and is often read as '*n choose r*'
    - Make sure you can find and use the button on your GDC

How does ${}^{n}C_{r}$ relate to the binomial theorem?

- The formula ${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$ is also known as a **binomial** coefficient
- For a binomial expansion $(a + b)^{n}$ the coefficients of each term will be ${}^{n}C_{0}, {}^{n}C_{1}$ and so on up to ${}^{n}C_{n}$
  - The coefficient of the r^{th} term will be ${}^{n}C_{r}$
- ${}^{n}C_{n} = {}^{n}C_{0} = 1$
- The binomial coefficients are symmetrical, so ${}^{n}C_{r} = {}^{n}C_{n-r}$
  - This can be seen by considering the formula for ${}^{n}C_{r}$
  - ${}^{n}C_{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{r!(n-r)!} = nC_{r}$

Examiner Tip

- You will most likely need to use the formula for nCr at some point in your exam
  - Practice using it and don't always rely on your GDC
  - Make sure you can find it easily in the formula booklet

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Worked example

Without using a calculator, find the coefficient of the term in x^3 in the expansion of $(1 + x)^9$.

$n = 9, \quad a = 1, \quad b = x$

Substitute values into the formula for the binomial theorem:

$$(a+b)^n = a^n + \ldots + {}^nC_r\, a^{n-r} b^r + \ldots + b^n$$

where $\quad {}^nC_r = \dfrac{n!}{r!(n-r)!}$

$$(1+x)^9 = \sum_{r=0}^{9} {}^9C_r\, (1)^{9-r}\, (x)^r$$

← Coefficient of x^3 occurs when $r = 3$.

$r = 3$ gives $\quad {}^9C_3 \times (1)^{9-3} (x)^3$

Non-calculator, so work out ${}^nC_r$ separately:

$$ {}^9C_3 = \frac{9!}{3!(9-3)!} = \frac{9 \times 8 \times 7 \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2}}{(3 \times 2)(\cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2})}$$

$$= \frac{9 \times 8 \times 7}{6} = 84$$

so the term when $r = 3$ is $\quad 84 \times (1)^6 \times x^3$

$$= 84x^3$$

Coefficient of x^3 = 84

Pascal's Triangle

What is Pascal's Triangle?

- Pascal's triangle is a way of arranging the binomial coefficients and neatly shows how they are formed
  - Each term is formed by adding the two terms above it
  - The first row has just the number 1
  - Each row begins and ends with a number 1
  - From the third row the terms in between the 1s are the sum of the two terms above it

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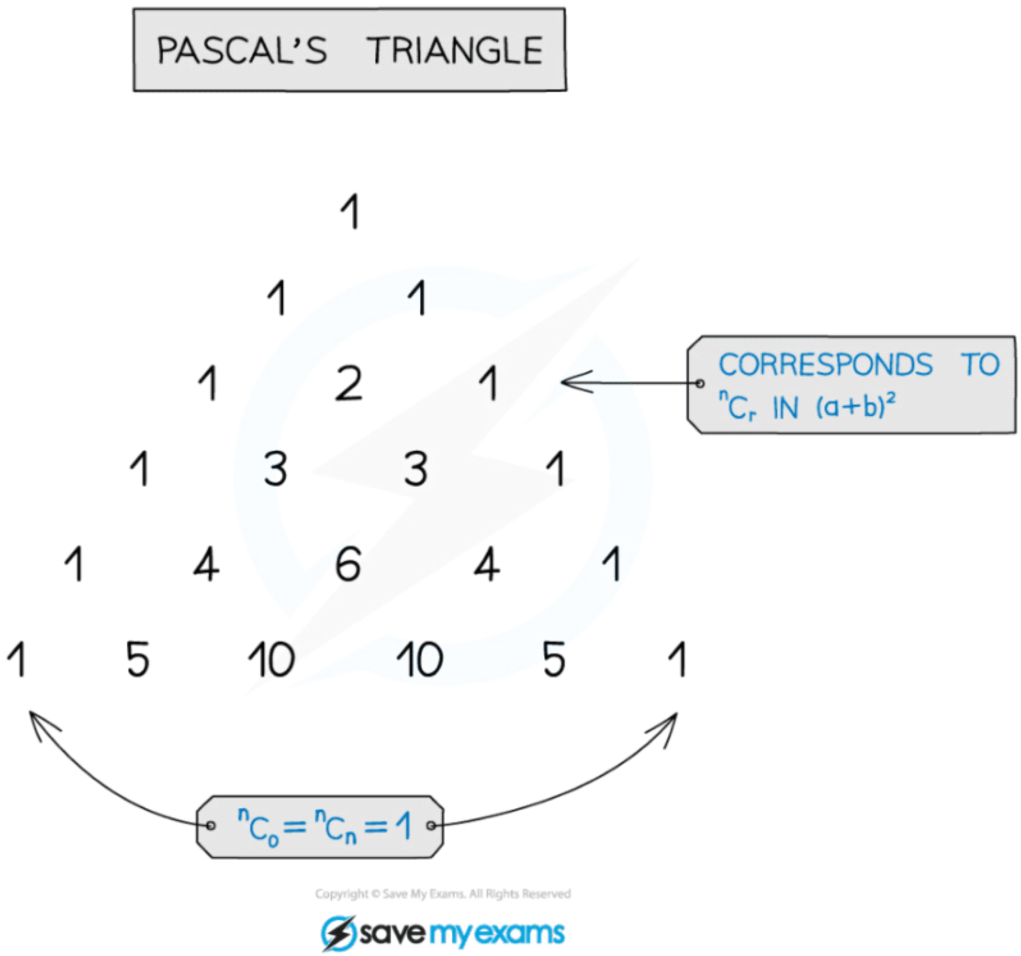

How does Pascal's Triangle relate to the binomial theorem?

- Pascal's triangle is an alternative way of finding the binomial coefficients, ^nC_r
  - It can be useful for finding for smaller values of n without a calculator

- However for larger values of n it is slow and prone to arithmetic errors
- Taking the first row as zero, $\left({}^{0}\mathrm{C}_{0} = 1\right)$, each row corresponds to the n^{th} row and the term within that row corresponds to the r^{th} term

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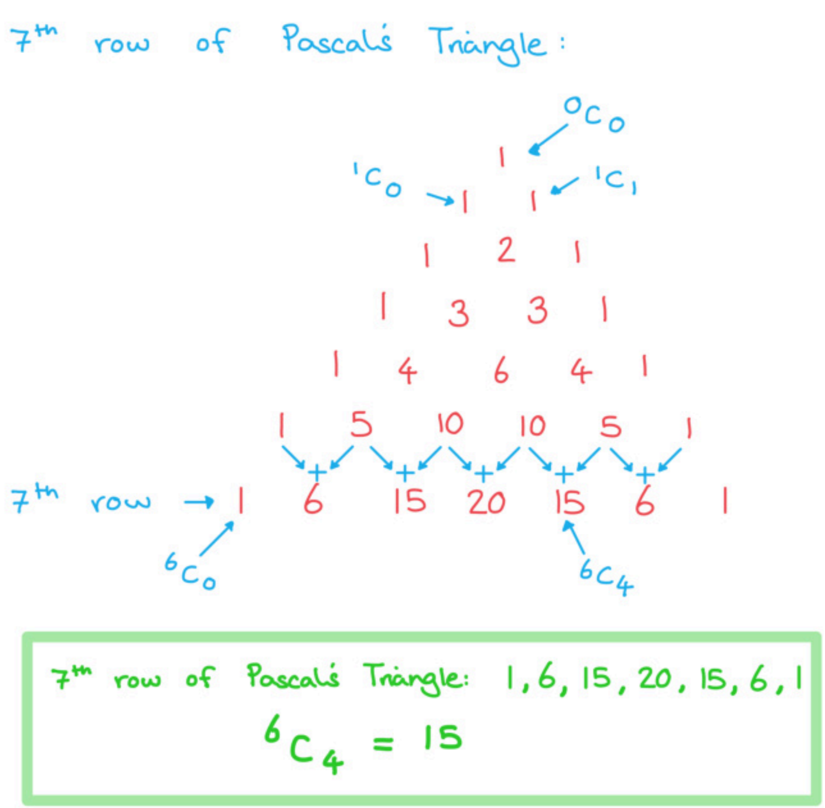
Examiner Tip

- In the non-calculator exam Pascal's triangle can be helpful if you need to get the coefficients of an expansion quickly, provided the value of n is not too big

Worked example

Write out the 7th row of Pascal's triangle and use it to find the value of ${}^{6}\mathrm{C}_{4}$.



1.6.2 Extension of The Binomial Theorem

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Binomial Theorem: Fractional & Negative Indices

How do I use the binomial theorem for fractional and negative indices?

- The formula given in the formula booklet for the binomial theorem applies to positive integers only
  - $(a+b)^n = a^n + {}^nC_1 a^{n-1}b + \ldots + {}^nC_r a^{n-r}b^r + \ldots + b^n$
  - where ${}^nC_r = \dfrac{n!}{r!(n-r)!}$
- For **negative** or **fractional powers** the expression in the brackets must first be changed such that the value for *a* is 1
  - $(a+b)^n = a^n\left(1+\dfrac{b}{a}\right)^n$
  - $(a+b)^n = a^n\left(1+n\left(\dfrac{b}{a}\right)+\dfrac{n(n-1)}{2!}\left(\dfrac{b}{a}\right)^2+\ \ldots\ \right),\ n\in\mathbb{Q}$
  - This **is given in the formula booklet**
- If *a* = 1 and *b* = *x* the binomial theorem is simplified to
  - $(1+x)^n = 1+nx+\dfrac{n(n-1)}{2!}x^2+\dfrac{n(n-1)(n-2)}{3!}x^3+\ldots,\ \ n\in\mathbb{Q}\ ,\quad |x|<1$
  - This is **not** in the formula booklet, you must remember it or be able to derive it from the formula given
- You need to be able to recognise a negative or fractional power
  - The expression may be on the denominator of a fraction
    - $\dfrac{1}{(a+b)^n} = (a+b)^{-n}$
  - Or written as a surd
    - $\sqrt[n]{(a+b)^m} = (a+b)^{\frac{m}{n}}$
- For $n \notin \mathbb{N}$ the expansion is infinitely long
  - You will usually be asked to find the first three terms
- The expansion is only valid for $|x|<1$
  - This means $-1<x<1$
  - This is known as the **interval of convergence**
  - For an expansion $(a\ +\ bx)^n$ the interval of convergence would be $-\dfrac{a}{b}<x<\dfrac{a}{b}$

How do we use the binomial theorem to estimate a value?

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- The binomial expansion can be used to form an approximation for a value raised to a power
- Since $|x| < 1$ higher powers of *x* will be very small
  - Usually only the first three or four terms are needed to form an approximation
  - The more terms used the closer the approximation is to the true value
- The following steps may help you use the binomial expansion to approximate a value
  - STEP 1: Compare the value you are approximating to the expression being expanded
    - e.g. $(1 - x)^{\frac{1}{2}} = 0.96^{\frac{1}{2}}$
  - STEP 2: Find the value of *x* by solving the appropriate equation
    - e.g. $1 - x = 0.96$
      $x = 0.04$
  - STEP 3: Substitute this value of *x* into the expansion to find the approximation
    - e.g. $1 - \frac{1}{2}(0.04) - \frac{1}{8}(0.04)^2 = 0.9798$
- Check that the value of *x* is within the **interval of convergence** for the expression
  - If *x* is outside the interval of convergence then the approximation may not be valid

Examiner Tip

- Students often struggle with the extension of the binomial theorem questions in the exam, however the formula is given in the formula booklet
  - Make sure you can locate the formula easily and practice substituting values in
  - Mistakes are often made with negative numbers or by forgetting to use brackets properly
    - Writing one term per line can help with both of these

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Worked example

Consider the binomial expansion of $\dfrac{1}{\sqrt{9-3x}}$.

a) Write down the first three terms.

Rewrite $\dfrac{1}{\sqrt{9-3x}}$ in the form $k(1+\frac{x}{a})^n$

$$\frac{1}{\sqrt{9-3x}} = (9-3x)^{-\frac{1}{2}} = 9^{-\frac{1}{2}}\left(1-\frac{3x}{9}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{3}\left(1-\frac{x}{3}\right)^{-\frac{1}{2}}$$

Substitute values into the formula for $(1+x)^n$

$$\frac{1}{3}\left(1-\frac{x}{3}\right)^{-\frac{1}{2}} = \frac{1}{3}\left[1+\left(-\frac{1}{2}\right)\left(-\frac{x}{3}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{x}{3}\right)^2+\dots\right]$$

$$= \frac{1}{3}\left[1+\frac{x}{6}+\frac{x^2}{24}+\dots\right]$$

$$= \frac{1}{3}+\frac{x}{18}+\frac{x^2}{72}+\dots$$

$$\frac{1}{\sqrt{9-3x}} \approx \frac{1}{3}+\frac{x}{18}+\frac{x^2}{72}$$

b) State the interval of convergence for the complete expansion.

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$n \geq 0$ and $n \notin \mathbb{N}$, so the series converges when $|x| < 1$

$\frac{1}{3}\left(1 - \frac{x}{3}\right)^{-\frac{1}{2}}$ ← x-term

$\left|-\frac{x}{3}\right| < 1$

$|x| < 3 \Rightarrow -3 < x < 3$

Converges for $-3 < x < 3$

c) Use the terms found in part (a) to estimate $\frac{1}{\sqrt{10}}$. Give your answer as a fraction.

Find the value of x for which $\frac{1}{\sqrt{9-3x}} = \frac{1}{\sqrt{10}}$

$9 - 3x = 10$

$x = -\frac{1}{3}$ ← $-3 < x < 3$ so can use the expansion

Substitute $x = -\frac{1}{3}$ into the expansion for $\frac{1}{\sqrt{9-3x}}$

$\frac{1}{\sqrt{9-3\left(-\frac{1}{3}\right)}} \approx \frac{1}{3} + \frac{\left(-\frac{1}{3}\right)}{18} + \frac{\left(-\frac{1}{3}\right)^2}{72}$

$\frac{1}{\sqrt{10}} \approx \frac{205}{648}$