

# DP IB Maths: AA HL



## 2.7 Polynomial Functions

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Your notes

## 2.7.1 Factor & Remainder Theorem

### Factor Theorem

#### What is the factor theorem?

- The **factor theorem** is used to find the linear factors of **polynomial** equations
- This topic is closely tied to finding the **zeros** and **roots** of a **polynomial** function/equation
  - As a rule of thumb a **zero** refers to the polynomial function and a **root** refers to a polynomial equation
- For any **polynomial** function  $P(x)$ 
  - $(x - k)$  is a **factor** of  $P(x)$  if  $P(k) = 0$
  - $P(k) = 0$  if  $(x - k)$  is a **factor** of  $P(x)$

#### How do I use the factor theorem?

- Consider the polynomial function  $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  and  $(x - k)$  is a **factor**
  - Then, due to the factor theorem  $P(k) = a_n k^n + a_{n-1} k^{n-1} + \dots + a_1 k + a_0 = 0$
  - $P(x) = (x - k) \times Q(x)$ , where  $Q(x)$  is a **polynomial** that is a factor of  $P(x)$
  - Hence,  $\frac{P(x)}{x - k} = Q(x)$ , where  $Q(x)$  is another factor of  $P(x)$
- If the linear factor has a **coefficient of x** then you must first factorise out the coefficient
  - If the linear factor is  $(ax - b) = a\left(x - \frac{b}{a}\right) \rightarrow P\left(\frac{b}{a}\right) = 0$

#### Examiner Tip

- A common mistake in exams is using the incorrect sign for either the root or the factor
- If you are asked to find integer solutions to a polynomial then you only need to consider factors of the constant term



Your notes

### Worked example

Determine whether  $(x - 2)$  is a factor of the following polynomials:

a)  $f(x) = x^3 - 2x^2 - x + 2$ .

Step 1: Determine k

Our linear function is  $x - 2$

→ so  $k = 2$

Step 2: Apply factor theorem

For  $x - 2$  to be a factor of  $f(x)$ ,

$f(2)$  has to equal zero

$$\begin{aligned} f(2) &= (2)^3 - 2(2)^2 - (2) + 2 \\ &= 8 - 8 - 2 + 2 \\ &= 0 \end{aligned}$$

$f(2) = 0$ ,  
so  $x - 2$  is a factor of  $f(x)$

b)  $g(x) = 2x^3 + 3x^2 - x + 5$ .



Your notes

### Step 1: Determine k

Our linear function is  $x - 2$

→ so  $k = 2$

### Step 2: Apply factor theorem

For  $x - 2$  to be a factor of  $g(x)$ ,  
 $g(2)$  has to equal zero

$$\begin{aligned}g(2) &= 2(2)^3 + 3(2)^2 - (2) + 5 \\ &= 16 - 12 - 2 + 5 \\ &= 7\end{aligned}$$

$g(2) = 7$ ,  
so  $x - 2$  is not a factor of  $g(x)$

It is given that  $(2x - 3)$  is a factor of  $h(x) = 2x^3 - bx^2 + 7x - 6$ .

- c) Find the value of  $b$ .



Your notes

Step 1: Determine k

Our linear function is  $2x - 3$

$$\rightarrow \text{so } k = \frac{3}{2}$$

Step 2: Apply factor theorem to find b

Since  $2x - 3$  is a factor of  $h(x)$ ,

$$h\left(\frac{3}{2}\right) = 0$$

$$0 = 2\left(\frac{3}{2}\right)^3 - b\left(\frac{3}{2}\right)^2 + 7\left(\frac{3}{2}\right) - 6$$

$$= \frac{54}{8} - \frac{9}{4}b + \frac{21}{2} - 6$$

$$b = 5$$



Your notes

## Remainder Theorem

### What is the remainder theorem?

- The **remainder theorem** is used to find the remainder when we divide a **polynomial** function by a linear function
- When any polynomial  $P(x)$  is divided by any linear function  $(x - k)$  the value of the remainder  $R$  is given by  $P(k) = R$ 
  - Note, when  $P(k) = 0$  then  $(x - k)$  is a factor of  $P(x)$

### How do I use the remainder theorem?

- Consider the polynomial function  $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  and the linear function  $(x - k)$ 
  - Then, due to the remainder theorem  $P(k) = a_n k^n + a_{n-1} k^{n-1} + \dots + a_1 k + a_0 = R$
  - $P(x) = (x - k) \times Q(x) + R$ , where  $Q(x)$  is a **polynomial**
  - Hence,  $\frac{P(x)}{x - k} = Q(x) + \frac{R}{x - k}$ , where  $R$  is the remainder
- If the linear function has a **coefficient of x** then you must first factorise out the coefficient
  - If the linear function is  $(ax - b) = a\left(x - \frac{b}{a}\right) \rightarrow P\left(\frac{b}{a}\right) = R$



Your notes

 **Worked example**

Let  $f(x) = 2x^4 - 2x^3 - x^2 - 3x + 1$ , find the remainder  $R$  when  $f(x)$  is divided by:

a)  $x - 3$ .

Step 1: Determine k

Our linear function is  $x - 3$

→ so  $k = 3$

Step 2: Apply remainder theorem

$$f(3) = R$$

$$f(3) = 2(3)^4 - 2(3)^3 - (3)^2 - 3(3) + 1$$

$$f(3) = 162 - 54 - 9 - 9 + 1$$

$$f(3) = 91$$

$$R = 91$$

b)  $x + 2$ .



Your notes

### Step 1: Determine k

Our linear function is  $x + 2$

→ so  $k = -2$

### Step 2: Apply remainder theorem

$$f(-2) = R$$

$$f(-2) = 2(-2)^4 - 2(-2)^3 - (-2)^2 - 3(-2) + 1$$

$$f(-2) = 32 + 16 - 4 + 6 + 1$$

$$f(-2) = 51$$

$$R = 51$$

The remainder when  $f(x)$  is divided by  $(2x + k)$  is  $\frac{893}{8}$ .

c) Given that  $k > 0$ , find the value of  $k$ .

### Step 1: Apply remainder theorem

$$2x + k = 2\left(x + \frac{k}{2}\right) \quad f\left(-\frac{k}{2}\right) = \frac{893}{8}$$

$$\frac{893}{8} = 2\left(-\frac{k}{2}\right)^4 - 2\left(-\frac{k}{2}\right)^3 - \left(-\frac{k}{2}\right)^2 - 3\left(-\frac{k}{2}\right) + 1$$

### Step 2: Solve for k using your GDC

$$k = 5$$





Your notes



Your notes

## 2.7.2 Polynomial Division

### Polynomial Division

#### What is polynomial division?

- Polynomial division is the process of **dividing two polynomials**
  - This is usually only useful when the **degree of the denominator** is **less than or equal** to the **degree of the numerator**
- To do this we use an algorithm similar to that used for **division of integers**
- To divide the polynomial  $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  by the polynomial

$$D(x) = b_k x^k + b_{k-1} x^{k-1} + \dots + b_1 x + b_0 \text{ where } k \leq n$$

- STEP 1

**Divide** the **leading term of the polynomial**  $P(x)$  by the **leading term of the divisor**  $D(x)$  :

$$\frac{a_n x^n}{b_k x^k} = q_m x^m$$

- STEP 2

**Multiply the divisor** by this term:  $D(x) \times q_m x^m$

- STEP 3

**Subtract this** from the **original polynomial**  $P(x)$  to cancel out the leading term:

$$R(x) = P(x) - D(x) \times q_m x^m$$

- Repeat steps 1 – 3 using the new polynomial  $R(x)$  in place of  $P(x)$  until the subtraction results in an expression for  $R(x)$  with degree less than the divisor
  - The quotient  $Q(x)$  is the **sum of the terms** you multiplied the divisor by:
 
$$Q(x) = q_m x^m + q_{m-1} x^{m-1} + \dots + q_1 x + q_0$$
  - The remainder  $R(x)$  is the polynomial after the final subtraction

#### Division by linear functions

- If  $P(x)$  has degree  $n$  and is divided by a linear function  $(ax + b)$  then
  - $\frac{P(x)}{ax + b} = Q(x) + \frac{R}{ax + b}$  where
    - $ax + b$  is the **divisor** (degree 1)
    - $Q(x)$  is the **quotient** (degree  $n - 1$ )
    - $R$  is the **remainder** (degree 0)
  - Note that  $P(x) = Q(x) \times (ax + b) + R$



Your notes

## Division by quadratic functions

- If  $P(x)$  has degree  $n$  and is divided by a quadratic function  $(ax^2 + bx + c)$  then
  - $\frac{P(x)}{ax^2 + bx + c} = Q(x) + \frac{ex + f}{ax^2 + bx + c}$  where
    - $ax^2 + bx + c$  is the **divisor** (degree 2)
    - $Q(x)$  is the **quotient** (degree  $n - 2$ )
    - $ex + f$  is the **remainder** (degree less than 2)
  - The remainder will be **linear** (degree 1) if  $e \neq 0$ , and **constant** (degree 0) if  $e = 0$
  - Note that  $P(x) = Q(x) \times (ax^2 + bx + c) + ex + f$

## Division by polynomials of degree $k \leq n$

- If  $P(x)$  has degree  $n$  and is divided by a polynomial  $D(x)$  with degree  $k \leq n$ 
  - $\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$  where
    - $D(x)$  is the **divisor** (degree  $k$ )
    - $Q(x)$  is the **quotient** (degree  $n - k$ )
    - $R(x)$  is the **remainder** (degree less than  $k$ )
  - Note that  $P(x) = Q(x) \times D(x) + R(x)$

## Are there other methods for dividing polynomials?

- Synthetic division** is a faster and shorter way of setting out a division when dividing by a linear term of the form
  - To divide  $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  by  $(x - c)$ :
    - Set  $b_n = a_n$
    - Calculate  $b_{n-1} = a_{n-1} + c \times b_n$
    - Continue this iterative process  $b_{i-1} = a_{i-1} + c \times a_i$
    - The quotient is  $Q(x) = b_n x^{n-1} + b_{n-1} x^{n-2} + \dots + b_2 x + b_1$  and the remainder is  $r = b_0$
- You can also find quotients and remainders by **comparing coefficients**
  - Given a polynomial  $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$
  - And a divisor  $D(x) = d_k x^k + d_{k-1} x^{k-1} + \dots + d_1 x + d_0$
  - Write  $Q(x) = q_{n-k} x^{n-k} + \dots + q_1 x + q_0$  and  $R(x) = r_{k-1} x^{k-1} + \dots + r_1 x + r_0$
  - Write  $P(x) = Q(x)D(x) + R(x)$ 
    - Expand the right-hand side
    - Equate the coefficients
    - Solve to find the unknowns  $q$ 's &  $r$ 's

### Examiner Tip

- In an exam you can use whichever method to divide polynomials – just make sure your method is written clearly so that if you make a mistake you can still get a mark for your method!



Your notes



Your notes

 **Worked example**

- a) Perform the division  $\frac{x^4 + 11x^2 - 1}{x + 3}$ . Hence write  $x^4 + 11x^2 - 1$  in the form  $Q(x) \times (x + 3) + R$ .

Step 1: what do we multiply  $x$  by to get  $x^4$ ?

$$x + 3 \overline{) \begin{array}{r} x^4 + 0x^3 + 11x^2 + 0x - 1 \end{array}}$$

Note:  $0x^3$  and  $0x$  are used to keep like terms together.

Step 2: subtract  $x^3(x + 3) = x^4 + 3x^3$  from  $x^4 + 0x^3$

$$x + 3 \overline{) \begin{array}{r} x^4 + 0x^3 + 11x^2 + 0x - 1 \\ - (x^4 + 3x^3) \\ \hline - 3x^3 \end{array}}$$



Your notes

Step 3: bring the  $11x^2$  down and return to step 1.

$$\begin{array}{r}
 x^3 - 3x^2 + 20x - 60 \\
 x + 3 \overline{) x^4 + 0x^3 + 11x^2 + 0x - 1} \\
 \underline{-(x^4 + 3x^3)} \phantom{+ 0x^2 + 0x - 1} \\
 -3x^3 + 11x^2 \phantom{+ 0x - 1} \\
 \underline{-(-3x^3 - 9x^2)} \phantom{+ 0x - 1} \\
 20x^2 + 0x \phantom{- 1} \\
 \underline{-(20x^2 + 60x)} \phantom{- 1} \\
 -60x - 1 \\
 \underline{-(-60x - 180)} \\
 179
 \end{array}$$

$$\begin{aligned}
 &x^4 + 11x^2 - 1 \\
 &= (x^3 - 3x^2 + 20x - 60)(x + 3) + 179
 \end{aligned}$$

- b) Find the quotient and remainder for  $\frac{x^4 + 4x^3 - x + 1}{x^2 - 2x}$ . Hence write  $x^4 + 4x^3 - x + 1$  in the form  $Q(x) \times (x^2 - 2x) + R(x)$ .



Your notes

When dividing by quadratics use the same steps as above.

$$\begin{array}{r}
 x^2 + 6x + 12 \\
 x^2 - 2x \overline{) x^4 + 4x^3 + 0x^2 - x + 1} \\
 \underline{-(x^4 - 2x^3)} \phantom{+ 1} \\
 6x^3 + 0x^2 - x + 1 \\
 \underline{-(6x^3 - 12x^2)} \phantom{+ 1} \\
 12x^2 - x + 1 \\
 \underline{-(12x^2 - 24x)} \\
 23x + 1
 \end{array}$$

$$\begin{aligned}
 &x^4 + 4x^3 - x + 1 \\
 &= (x^2 + 6x + 12)(x^2 - 2x) + 23x + 1
 \end{aligned}$$



Your notes

## 2.7.3 Polynomial Functions

### Sketching Polynomial Graphs

In exams you'll commonly be asked to sketch the graphs of different polynomial functions with and without the use of your GDC.

#### What's the relationship between a polynomial's degree and its zeros?

- If a **real polynomial**  $P(x)$  has **degree**  $n$ , it will have  **$n$  zeros** which can be written in the form  $a + bi$ , where  $a, b \in \mathbb{R}$ 
  - For example:
    - A quadratic will have 2 zeros
    - A cubic function will have 3 zeros
    - A quartic will have 4 zeros
  - Some of the zeros may be **repeated**
- Every **real polynomial** of **odd degree** has **at least one real zero**

#### How do I sketch the graph of a polynomial function without a GDC?

- Suppose  $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  is a **real polynomial** with **degree**  $n$
- To sketch the graph of a polynomial you need to know three things:
  - The **y-intercept**
    - Find this by **substituting**  $x = 0$  to get  $y = a_0$
  - The **roots**
    - You can find these by **factorising** or solving  $y = 0$
  - The **shape**
    - This is determined by the **degree** ( $n$ ) and the sign of the **leading coefficient** ( $a_n$ )

#### How does the multiplicity of a real root affect the graph of the polynomial?

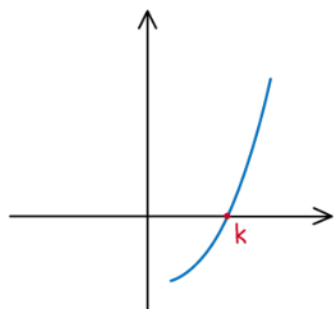
- The **multiplicity** of a root is the number of times it is **repeated** when the polynomial is factorised
  - If  $x = k$  is a root with **multiplicity**  $m$  then  $(x - k)^m$  is a **factor** of the polynomial
- The graph either **crosses** the x-axis or **touches** the x-axis at a **root**  $x = k$  where  $k$  is a real number
  - If  $x = k$  has **multiplicity 1** then the graph **crosses** the x-axis at  $(k, 0)$
  - If  $x = k$  has **multiplicity 2** then the graph has a **turning point** at  $(k, 0)$  so **touches** the x-axis
    - If  $x = k$  has **odd multiplicity**  $m \geq 3$  then the graph has a **stationary point of inflection** at  $(k, 0)$  so **crosses** the x-axis
    - If  $x = k$  has **even multiplicity**  $m \geq 4$  then the graph has a **turning point** at  $(k, 0)$  so **touches** the x-axis





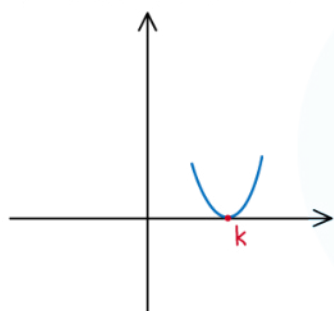
Your notes

$(x - k)$  IS A FACTOR



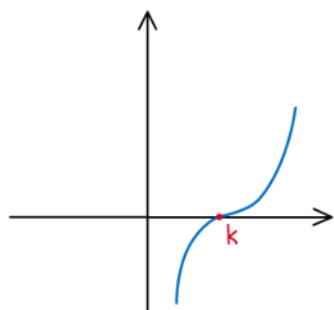
CURVE CROSSES THE  $x -$  AXIS

$(x - k)^2$  IS A FACTOR



CURVE TOUCHES THE  $x -$  AXIS  
AT THE TURNING POINT

$(x - k)^3$  IS A FACTOR



CURVE CROSSES THE  $x -$  AXIS  
AT THE STATIONARY POINT OF INFLECTION

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### How do I determine the shape of the graph of the polynomial?

- Consider what happens as  $x$  tends to  $\pm \infty$ 
  - If  $a_n$  is **positive** and  $n$  is **even** then the graph **approaches from the top left** and **tends to the top right**
    - $\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow +\infty} f(x) = +\infty$



Your notes

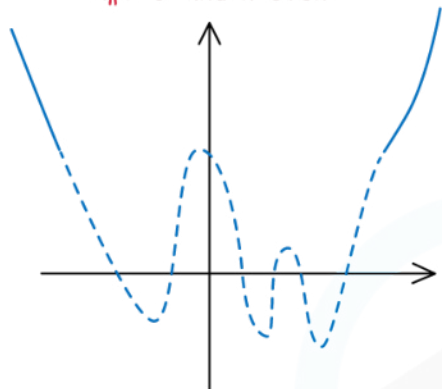
- If  $a_n$  is **negative** and  $n$  is **even** then the graph **approaches from the bottom left** and **tends to the bottom right**
  - $\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow +\infty} f(x) = +\infty$
- If  $a_n$  is **positive** and  $n$  is **odd** then the graph **approaches from the bottom left** and **tends to the top right**
  - $\lim_{x \rightarrow -\infty} f(x) = -\infty$  and  $\lim_{x \rightarrow +\infty} f(x) = +\infty$
- If  $a_n$  is **negative** and  $n$  is **odd** then the graph **approaches from the top left** and **tends to the bottom right**
  - $\lim_{x \rightarrow -\infty} f(x) = +\infty$  and  $\lim_{x \rightarrow +\infty} f(x) = -\infty$
- Once you know the **shape**, the **real roots** and the **y-intercept** then you simply connect the points using a **smooth curve**
- There will be **at least one turning point** in-between each pair of roots
  - If the degree is  $n$  then there is **at most  $n - 1$  stationary points** (some will be **turning points**)
    - Every real polynomial of **even degree** has **at least one turning point**
    - Every real polynomial of **odd degree bigger than 1** has **at least one point of inflection**
  - If it is a calculator paper then you can use your GDC to find the coordinates of the turning points
  - You won't need to find their location without a GDC unless the question asks you to



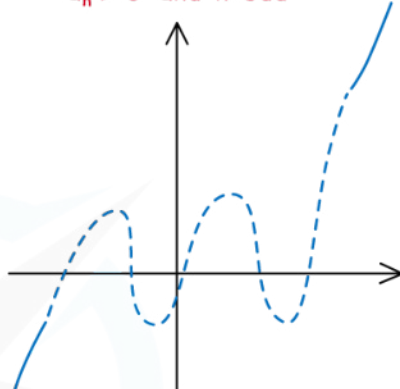
Your notes

$$y = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

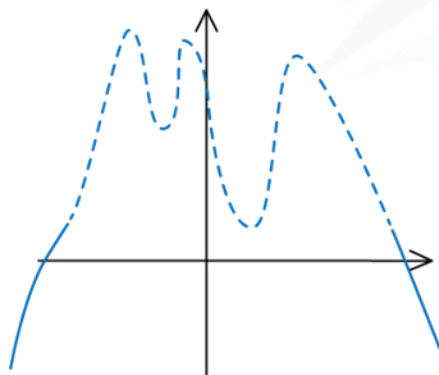
$a_n > 0$  and  $n$  even



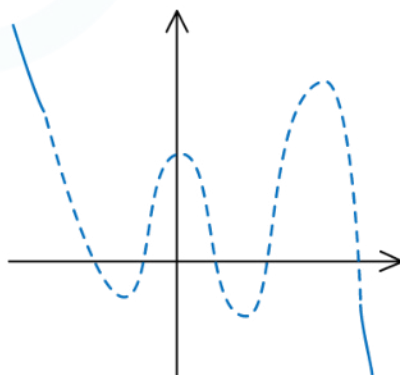
$a_n > 0$  and  $n$  odd



$a_n < 0$  and  $n$  even



$a_n < 0$  and  $n$  odd



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### Examiner Tip

- If it is a calculator paper then you can use your GDC to find the coordinates of any turning points
- If it is the non-calculator paper then you will not be required to find the turning points when sketching unless specifically asked to



Your notes

### Worked example

- a) The function  $f$  is defined by  $f(x) = (x + 1)(2x - 1)(x - 2)^2$ . Sketch the graph of  $y = f(x)$ .

Find the y-intercept

$$x = 0: y = (1)(-1)(-2)^2 = -4$$

Find the roots and determine if graphs crosses or touches the x-axis

$$(x + 1)(2x - 1)(x - 2)^2$$

$$(-1, 0) \quad \left(\frac{1}{2}, 0\right) \quad (2, 0)$$

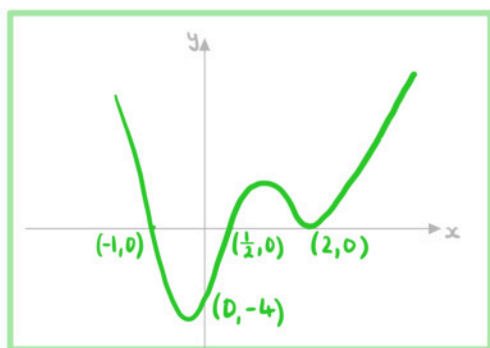
cross cross touch

Determine the shape by looking at the leading term

$$\text{Leading term is } (x)(2x)(x)^2 = 2x^4$$

$$\text{As } x \rightarrow -\infty \quad y \rightarrow +\infty$$

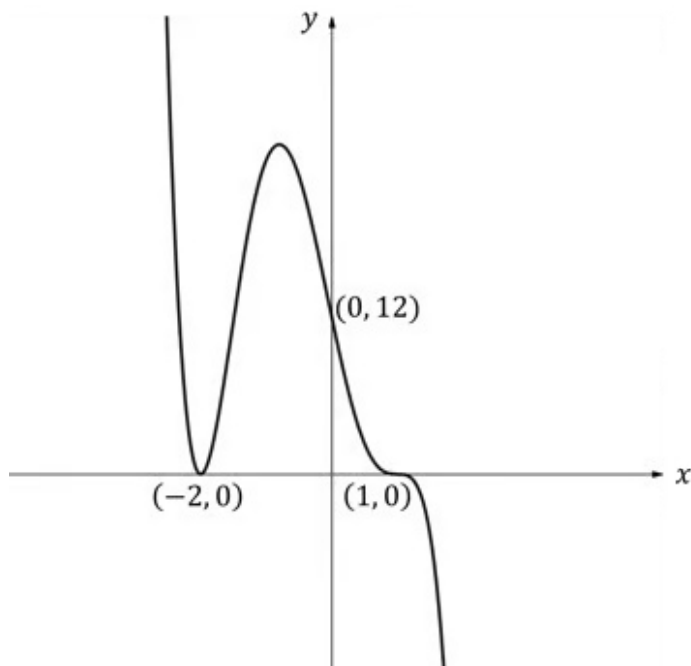
$$\text{As } x \rightarrow +\infty \quad y \rightarrow +\infty$$



- b) The graph below shows a polynomial function. Find a possible equation of the polynomial.



Your notes



Touches at  $(-2, 0)$   $(x+2)^2$  is a factor

Point of inflection at  $(1, 0)$   $(x-1)^3$  is a factor

Write in the form of:  $y = a(x+2)^2(x-1)^3$

Use the y-intercept to find a

$$12 = a(2)^2(-1)^3 \Rightarrow -4a = 12 \quad \therefore a = -3$$

$$y = -3(x+2)^2(x-1)^3$$



Your notes

## Solving Polynomial Equations

### What is “The Fundamental Theorem of Algebra”?

- Every **real polynomial** with degree  $n$  can be factorised into  $n$  **complex linear factors**
  - Some of which may be **repeated**
  - This means the polynomial will have  $n$  zeros (some may be repeats)
- Every **real polynomial** can be expressed as a product of **real linear factors** and **real irreducible quadratic factors**
  - An irreducible quadratic is where it **does not have real roots**
    - The **discriminant** will be negative:  $b^2 - 4ac < 0$
- If  $a + bi$  ( $b \neq 0$ ) is a **zero** of a **real polynomial** then its **complex conjugate**  $a - bi$  is also a **zero**
- Every **real polynomial** of **odd degree** will have **at least one real zero**

### How do I solve polynomial equations?

- Suppose you have an equation  $P(x) = 0$  where  $P(x)$  is a **real polynomial of degree  $n$** 
  - $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$
- You may be given one zero or you might have to find a zero  $x = k$  by substituting values into  $P(x)$  until it equals 0
- If you know a **root** then you know a **factor**
  - If you know  $x = k$  is a **root** then  $(x - k)$  is a **factor**
  - If you know  $x = a + bi$  is a **root** then you know a **quadratic factor**  $(x - (a + bi))(x - (a - bi))$ 
    - Which can be written as  $((x - a) - bi)((x - a) + bi)$  and **expanded quickly using difference of two squares**
- You can then **divide**  $P(x)$  by this factor to get **another factor**
  - For example: dividing a cubic by a linear factor will give you a quadratic factor
- You then may be able to factorise this new factor

#### Examiner Tip

- If a polynomial has three or less terms check whether a substitution can turn it into a quadratic
  - For example:  $x^6 + 3x^3 + 2$  can be written as  $(x^3)^2 + 3(x^3) + 2$



Your notes

### Worked example

Given that  $x = \frac{1}{2}$  is a zero of the polynomial defined by  $f(x) = 2x^3 - 3x^2 + 5x - 2$ , find all three zeros of  $f$ .

$x = \frac{1}{2}$  is a root  $\therefore (2x-1)$  is a factor

Find the quadratic factor  $(2x^3 - 3x^2 + 5x - 2) = (2x-1)(ax^2 + bx + c)$

Compare coefficients :  $2x^3 = 2ax^3 \quad \therefore a=1$

$-2 = -c \quad \therefore c=2$

$5x = 2cx - bx \Rightarrow 5 = 4 - b \quad \therefore b=-1$

Solve the quadratic :  $x^2 - x + 2 = 0$

Formula booklet

Solutions of a quadratic equation	$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
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$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(2)}}{2(1)}$$

$$x = \frac{1 \pm \sqrt{-7}}{2} = \frac{1}{2} \pm \frac{\sqrt{7}}{2}i$$

Roots :  $\frac{1}{2}, \frac{1}{2} + \frac{\sqrt{7}}{2}i, \frac{1}{2} - \frac{\sqrt{7}}{2}i$



Your notes

## 2.7.4 Roots of Polynomials

### Sum & Product of Roots

#### How do I find the sum & product of roots of polynomials?

- Suppose  $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  is a **polynomial** of **degree**  $n$  with  $n$  roots

$$\alpha_1, \alpha_2, \dots, \alpha_n$$

- The polynomial is written as  $\sum_{r=0}^n a_r x^r = 0$ ,  $a_n \neq 0$  in the **formula booklet**
  - $a_n$  is the coefficient of the **leading term**
  - $a_{n-1}$  is the coefficient of the  **$x^{n-1}$  term**
    - Be careful: this could be equal to zero
  - $a_0$  is the **constant term**
    - Be careful: this could be equal to zero
- In factorised form:  $P(x) = a_n(x - \alpha_1)(x - \alpha_2)\dots(x - \alpha_n)$ 
  - Comparing coefficients of the  **$x^{n-1}$  term** and the **constant term** gives
    - $a_{n-1} = a_n(-\alpha_1 - \alpha_2 - \dots - \alpha_n)$
    - $a_0 = a_n(-\alpha_1) \times (-\alpha_2) \times \dots \times (-\alpha_n)$
- The **sum** of the roots is given by:
  - $\alpha_1 + \alpha_2 + \dots + \alpha_n = -\frac{a_{n-1}}{a_n}$
- The **product** of the roots is given by:
  - $\alpha_1 \times \alpha_2 \times \dots \times \alpha_n = \frac{(-1)^n a_0}{a_n}$ 
    - both of these formulae are in your **formula booklet**

#### How can I find unknowns if I am given the sum and/or product of the roots of a polynomial?

- If you know a complex root of a real polynomial then its **complex conjugate** is **another root**
- Form **two equations** using the roots
  - One using the **sum of the roots formula**
  - One using the **product of the roots formula**
- Solve** for any unknowns



### Examiner Tip

- Examiners might trick you by not having an  $x^{n-1}$  term or a constant term
- To make sure you do not get tricked you can write out the full polynomial using 0 as a coefficient where needed
  - For example: Write  $x^4 + 2x^2 - 5x$  as  $x^4 + 0x^3 + 2x^2 - 5x + 0$



Your notes



Your notes

 **Worked example**

$2 - 3i$ ,  $\frac{5}{3}i$  and  $\alpha$  are three roots of the equation  $18x^5 - 9x^4 + 32x^3 + 794x^2 - 50x + k = 0$

a) Use the sum of all the roots to find the value of  $\alpha$ .

It is a real polynomial so if  $a+bi$  is a root then  $a-bi$  is also a root

Roots:  $2-3i$ ,  $2+3i$ ,  $\frac{5}{3}i$ ,  $-\frac{5}{3}i$ ,  $\alpha$

Formula booklet

Sum & product of the roots of polynomial equations of the form $\sum_{i=1}^n a_i x^i = 0$	Sum is $-\frac{a_{n-1}}{a_n}$
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$18x^5 - 9x^4 + 32x^3 + 794x^2 - 50x + k$   
 $a_n = 18$   $a_{n-1} = -9$

$$(2-3i) + (2+3i) + \left(\frac{5}{3}i\right) + \left(-\frac{5}{3}i\right) + \alpha = \frac{-(-9)}{18}$$

$$4 + \alpha = \frac{1}{2}$$

$$\alpha = -\frac{7}{2}$$

b) Use the product of all the roots to find the value of  $k$ .

Formula booklet

Sum & product of the roots of polynomial equations of the form $\sum_{i=1}^n a_i x^i = 0$	product is $\frac{(-1)^n a_0}{a_n}$
-------------------------------------------------------------------------------------------	-------------------------------------

$18x^5 - 9x^4 + 32x^3 + 794x^2 - 50x + k$   
 $a_n = 18$   $n = 5$   $a_0 = k$

$$(2-3i)(2+3i)\left(\frac{5}{3}i\right)\left(-\frac{5}{3}i\right)\left(-\frac{7}{2}\right) = \frac{(-1)^5 k}{18}$$

$$(13)\left(\frac{25}{9}\right)\left(-\frac{7}{2}\right) = \frac{-k}{18}$$

$$-\frac{2275}{18} = -\frac{k}{18}$$

$$k = 2275$$