

DP IB Maths: AA HL



Your notes

5.6 Kinematics

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Your notes

5.6.1 Kinematics Toolkit

Displacement, Velocity & Acceleration

What is kinematics?

- **Kinematics** is the branch of mathematics that models and analyses the **motion** of objects
- Common words such as **distance**, **speed** and **acceleration** are used in kinematics but are used according to their technical definition

What definitions do I need to be aware of?

- Firstly, only motion of an object in a **straight line** is considered
 - this could be a **horizontal** straight line
 - the **positive** direction would be to the **right**
 - or this could be a **vertical** straight line
 - the **positive** direction would be **upwards**

Particle

- A **particle** is the general term for an **object**
 - some questions may use a **specific** object such as a **car** or a **ball**

Time t seconds

- **Displacement**, **velocity** and **acceleration** are all **functions** of time t
- **Initially** time is zero $t = 0$

Displacement S m

- The **displacement** of a particle is its **distance relative** to a **fixed point**
 - the fixed point is often (but not always) the particle's **initial position**
- **Displacement** will be **zero** $S = 0$ if the object is at or has returned to its initial position
- **Displacement** will be negative if its **position relative** to the **fixed point** is in the **negative direction** (left or down)

Distance d m

- Use of the word **distance** needs to be considered carefully and could refer to
 - the distance **travelled** by a particle
 - the **(straight line)** distance the particle is from a **particular point**
- Be careful not to confuse **displacement** with **distance**
 - if a bus route starts and ends at a bus depot, when the bus has returned to the depot, its **displacement** will be **zero** but the distance the bus has travelled will be the length of the route
- **Distance** is always **positive**

Velocity V m s⁻¹

- The **velocity** of a particle is the **rate of change** of its **displacement** at time t



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- **Velocity** will be **negative** if the **particle** is moving in the **negative direction**
- A **velocity of zero** means the particle is **stationary** $v = 0$

Speed $|v| \text{ m s}^{-1}$

- **Speed** is the **magnitude** (a.k.a. absolute value or modulus) of **velocity**
 - as the particle is **moving** in a **straight line**, **speed** is the **velocity ignoring** the **direction**
 - if $v = 4$, $|v| = 4$
 - if $v = -6$, $|v| = 6$

Acceleration $a \text{ m s}^{-2}$

- The **acceleration** of a particle is the **rate of change** of its **velocity** at time t
- Acceleration can be **negative** but this alone cannot fully describe the particle's motion
 - if **velocity** and **acceleration** have the **same** sign the particle is **accelerating** (speeding up)
 - if **velocity** and **acceleration** have **different** signs then the particle is **decelerating** (slowing down)
 - if **acceleration** is **zero** $a = 0$ the particle is moving with **constant** velocity
 - in all cases the **direction** of **motion** is determined by the **sign** of **velocity**

Are there any other words or phrases in kinematics I should know?

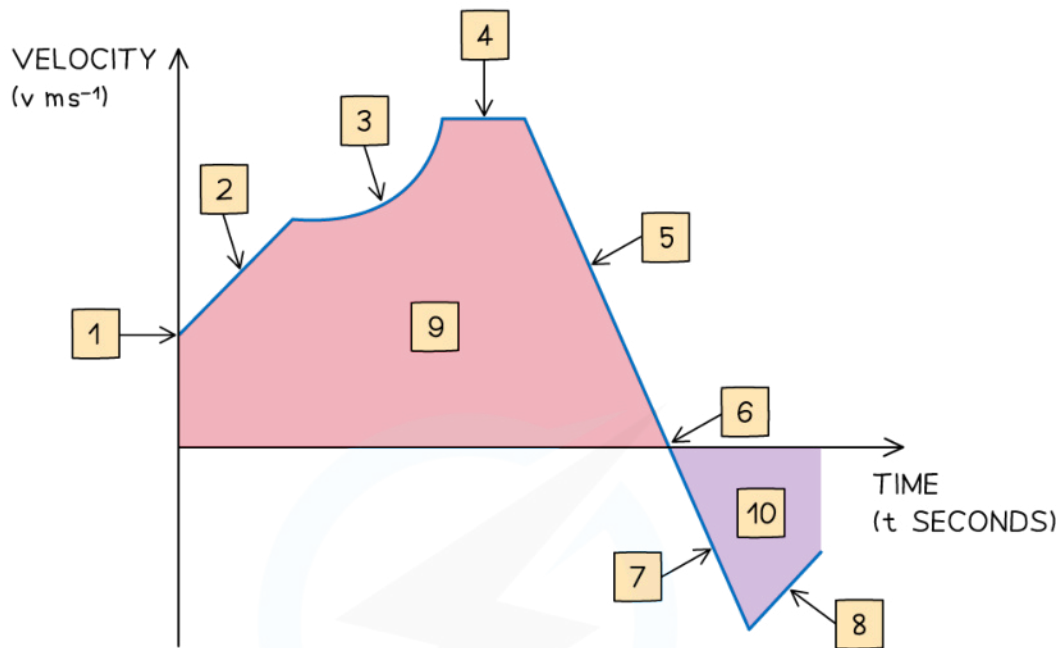
- Certain words and phrases can imply values or directions in kinematics
 - a particle described as “at **rest**” means that its velocity is zero, $v = 0$
 - a particle described as moving “**due east**” or “**right**” or would be moving in the **positive horizontal** direction
 - this also means that $v > 0$
 - a particle “**dropped from the top of a cliff**” or “**down**” would be moving in the **negative vertical** direction
 - this also means that $v < 0$

What are the key features of a velocity–time graph?

- The **gradient** of the graph equals the **acceleration** of an object
- A **straight line** shows that the object is **accelerating** at a **constant rate**
- A **horizontal** line shows that the object is moving at a **constant velocity**
- The **area** between graph and the x-axis tells us the **change in displacement** of the object
 - Graph **above** the x-axis means the object is moving **forwards**
 - Graph **below** the x-axis means the object is moving **backwards**
- The **total displacement** of the object from its starting point is the sum of the **areas above** the x-axis **minus** the sum of the **areas below** the x-axis
- The **total distance travelled** by the object is the sum of **all** the **areas**
- If the graph **touches** the **x-axis** then the object is **stationary** at that time
- If the graph is **above** the **x-axis** then the object has positive velocity and is **travelling forwards**
- If the graph is **below** the **x-axis** then the object has negative velocity and is **travelling backwards**



Your notes



1 INITIAL VELOCITY

2 CONSTANT ACCELERATION

3 VARIABLE ACCELERATION

4 CONSTANT VELOCITY

5 DECELERATING (SLOWING DOWN BUT STILL MOVING FORWARDS)

6 INSTANTANEOUSLY AT REST (STATIONARY FOR AN INSTANT)

7 SPEEDING UP BUT MOVING BACKWARDS

8 SLOWING DOWN BUT STILL MOVING BACKWARDS

9 DISTANCE TRAVELLED FORWARDS

10 DISTANCE TRAVELLED BACKWARDS

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 **Examiner Tip**

- In an exam if you are given an expression for the velocity then sketching a velocity-time graph can help visualise the problem

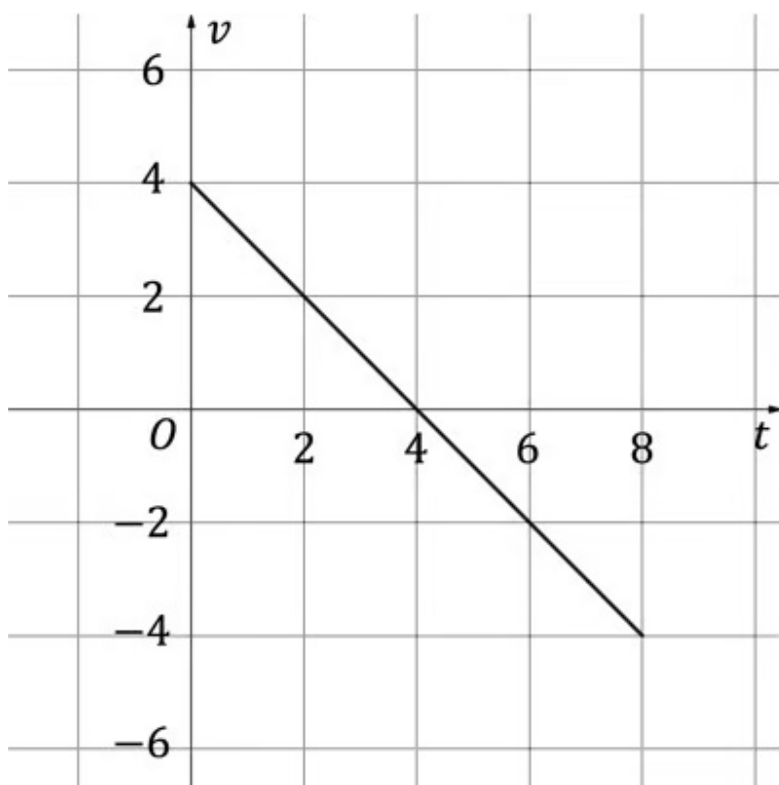


Your notes

Worked example

A particle is projected vertically upwards from ground level, taking 8 seconds to return to the ground.

The velocity-time graph below illustrates the motion of the particle for these 8 seconds.



- i) How many seconds does the particle take to reach its maximum height?
Give a reason for your answer.
- ii) State, with a reason, whether the particle is accelerating or decelerating at time $t = 3$.



Your notes

i. At maximum height, velocity is zero

$$v = 0 \text{ at } t = 4$$

∴ The particle takes 4 seconds to reach its maximum height. This is because its velocity is 0 m s^{-1} at 4 seconds.

ii. At $t = 3$, velocity is POSITIVE

Acceleration is the gradient of velocity

At $t = 3$, acceleration is NEGATIVE

∴ At 3 seconds the particle is decelerating as its velocity and acceleration have different signs.



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5.6.2 Calculus for Kinematics

Differentiation for Kinematics

How is differentiation used in kinematics?

- **Displacement, velocity and acceleration** are related by calculus
- In terms of differentiation and derivatives
 - **velocity** is the **rate of change** of **displacement**
 - $v = \frac{ds}{dt}$ or $v(t) = s'(t)$
 - **acceleration** is the **rate of change** of **velocity**
 - $a = \frac{dv}{dt}$ or $a(t) = v'(t)$
 - so **acceleration** is also the **second derivative** of **displacement**
 - $a = \frac{d^2s}{dt^2}$ or $a(t) = s''(t)$
- If a graph is not given you can use your GDC to draw one
 - you can then use your GDC's graphing features to find **gradients**
 - **velocity** is the **gradient** on a **displacement** (-time) graph
 - **acceleration** is the **gradient** on a **velocity** (-time) graph



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Worked example

The displacement, s m, of a particle at t seconds, is modelled by $s(t) = 2t^3 - 27t^2 + 84t$

- i. Find $v(t)$ and $a(t)$.
- ii. Find the times at which the particle is at rest.

$$\text{i. } v(t) = s'(t) = 6t^2 - 54t + 84 = 6(t^2 - 9t + 14)$$

$$a(t) = v'(t) = 12t - 54 = 6(2t - 9)$$

$$\therefore v(t) = 6(t-7)(t-2)$$

$$a(t) = 6(2t-9)$$

It's not essential to factorise the final answers

- ii. The particle is at rest when $v(t) = 0$
- $$6(t-7)(t-2) = 0$$
- $$t = 7, t = 2$$

\therefore The particle is at rest at 2 seconds and 7 seconds



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Integration for Kinematics

How is integration used in kinematics?

- Since **velocity** is the **derivative** of **displacement** ($v = \frac{ds}{dt}$) it follows that

$$s = \int v \, dt$$

- Similarly, **velocity** will be an **antiderivative** of **acceleration**

$$v = \int a \, dt$$

How would I find the constant of integration in kinematics problems?

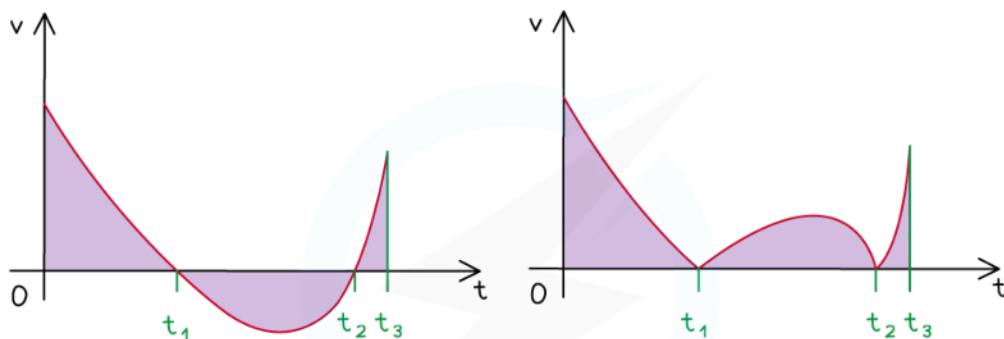
- A **boundary** or **initial** condition would need to be known
 - phrases involving the word “**initial**”, or “**initially**” are referring to **time** being **zero**, i.e. $t = 0$
 - you might also be given information about the object at some other time (this is called a **boundary condition**)
 - substituting** the values in from the **initial or boundary condition** would allow the **constant of integration** to be found

How are definite integrals used in kinematics?

- Definite integrals can be used to find the displacement of a particle between two points in time
 - $\int_{t_1}^{t_2} v(t) \, dt$ would give the **displacement** of the particle **between** the times $t = t_1$ and $t = t_2$
 - This can be found using a velocity-time graph by **subtracting** the **total area below** the horizontal axis from the **total area above**
 - $\int_{t_1}^{t_2} |v(t)| \, dt$ gives the **distance** a particle has **travelled** between the times $t = t_1$ and $t = t_2$
 - This can be found using a velocity-time graph by **adding** the **total area below** the horizontal axis to the **total area above**
 - Use a GDC to plot the modulus graph $y = |v(t)|$



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$\int_0^{t_3} v(t) dt$ IS THE
DISPLACEMENT OF THE
PARTICLE FROM ITS INITIAL
POSITION AT TIME t_3

$\int_0^{t_3} |v(t)| dt$ IS THE
DISTANCE THE PARTICLE
HAS TRAVELLED AT TIME t_3

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Examiner Tip

- Sketching the velocity-time graph can help you visualise the distances travelled using areas between the graph and the horizontal axis



Your notes

Worked example

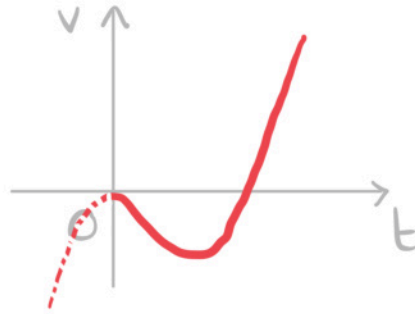
A particle moving in a straight horizontal line has velocity ($v \text{ m s}^{-1}$) at time t seconds modelled by $v(t) = 8t^3 - 12t^2 - 2t$.

- i. Given that the initial position of the particle is at the origin, find an expression for its displacement from the origin at time t seconds.
- ii. Find the displacement of the particle from the origin in the first five seconds of its motion.
- iii. Find the distance travelled by the particle in the first five seconds of its motion.



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Use your GDC to sketch a velocity(-time) graph and use it to check to see if your answers are sensible.



i. "initial" - $t=0$, "origin" - $s=0$

$$s(t) = \int v(t) dt = \int (8t^3 - 12t^2 - 2t) dt$$

$$s(t) = 2t^4 - 4t^3 - t^2 + c$$

where c is a constant

$$\text{at } t=0, s=0, \therefore c=0$$

$$\therefore s(t) = 2t^4 - 4t^3 - t^2$$

ii. "first five seconds" - $t_1=0$, $t_2=5$

Using a GDC this would be

$$s = \int_0^5 (8t^3 - 12t^2 - 2t) dt$$

$$s = 725 \text{ m}$$

iii. Using a GDC this would be

$$d = \int_0^5 |8t^3 - 12t^2 - 2t| dt$$

d for distance

$$d = 736.734020\dots$$

$$\therefore d = 737 \text{ m (3 s.f.)}$$