


DP IB Maths: AI HL



Your notes

3.9 Modelling with Vectors

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Your notes

3.9.1 Kinematics with Vectors

Kinematics using Vectors

How are vectors related to kinematics?

- Kinematics is the use of mathematics to model motion in objects
- If an object is moving in **one dimension** then its velocity, displacement and time are related using the formula $s = vt$
 - where s is **displacement**, v is **velocity** and t is the **time taken**
- If an object is moving in **more than one dimension** then **vectors** are needed to represent its **velocity** and **displacement**
 - Whilst **time** is a **scalar quantity**, **displacement** and **velocity** are both **vector quantities**
- Vectors are often used in questions in the context of **forces**, **acceleration** or **velocity**
- The position of an object at a particular time can be modelled using a vector equation

How do I find the direction of a vector?

- Vectors have opposite directions if they are the same size but opposite signs
- The direction of a vector is what makes it more than just a scalar
 - E.g. two objects with velocities of 7 m/s and -7 m/s are travelling at the **same speed** but in **opposite directions**
- Two vectors are **parallel** if and only if one is a **scalar multiple** of the other
- For real-life contexts such as mechanics, direction can be calculated from a given vector using **trigonometry**
 - **Given the i and j components a right-triangle can be created and the angle found using SOHCAHTOA**
- It is usually given as a **bearing** or as an angle calculated **anticlockwise** from the positive x -axis

How do I find the distance between two moving objects?

- If two objects are moving with constant velocity in non-parallel directions the distance between them will change
- The distance between them can be found by finding the magnitude of their position vectors at any point in time
- The **shortest distance** between the two objects at a particular time can be found by finding the value of the time at which the magnitude is at its minimum value
 - Let the time when the objects are at the shortest distance be t
 - Find the distance, d , in terms of t by substituting into the equation for the magnitude of their position vectors
 - d^2 will be an expression in terms of t which can be differentiated and set to 0
 - Solving this will give the time at which the distance is at a minimum
 - Substitute this back into the expression for d to find the shortest distance

 **Examiner Tip**

- Kinematics questions can have a lot of information in, read them carefully and pick out the parts that are essential to the question
- Look out for where variables used are the same and/or different within vector equations, you will need to use different techniques to find these



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Worked example

Two objects, A and B, are moving so that their position relative to a fixed point, O at time t , in minutes can be defined by the position vectors $\mathbf{r}_A = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + t \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ and $\mathbf{r}_B = \begin{pmatrix} 2 \\ 5 \end{pmatrix} + t \begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

The unit vectors \mathbf{i} and \mathbf{j} are a displacement of 1 metre due East and North of O respectively.

- a) Find the coordinates of the initial position of the two objects.

The initial position is when $t = 0$

$$\mathbf{r}_A = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + 0 \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$\mathbf{r}_B = \begin{pmatrix} 2 \\ 5 \end{pmatrix} + 0 \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

A (3, -1) and B (2, 5)

- b) Find the shortest distance between the two objects and the time at which this will occur.



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Let the shortest distance occur at time, t , then:

$$A: (3-2t, -1+4t) \quad B: (2+3t, 5-t)$$

Find the distance between A and B in terms of t :

$$d = \sqrt{[(2+3t)-(3-2t)]^2 + [(5-t)-(-1+4t)]^2}$$

$$= \sqrt{(-1+5t)^2 + (6-5t)^2}$$

$$= \sqrt{(1-10t+25t^2) + (36-60t+25t^2)}$$

$$d^2 = 37 - 70t + 50t^2$$

Find the minimum point of d^2 :

$$\frac{dd^2}{dt} = -70 + 100t \quad \therefore -70 + 100t = 0$$

$$t = \frac{70}{100} = 0.7$$

$$\text{When } t = 0.7, \quad d = \sqrt{37 - 70(0.7) + 50(0.7)^2} = \sqrt{12.5}$$

$$d = 3.54 \text{ m (3 s.f.)}$$

3.9.2 Constant & Variable Velocity



Your notes

Vectors & Constant Velocity

How are vectors used to model linear motion?

- If an object is moving with **constant velocity** it will travel in a **straight line**
- For an object moving in a **straight line** in two or three dimensions its velocity, displacement and time can be related using the vector equation of a line
 - $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$
 - Letting
 - \mathbf{r} be the position of the object at the time, t
 - \mathbf{a} be the position vector, \mathbf{r}_0 at the start ($t = 0$)
 - λ represent the time, t
 - \mathbf{b} be the **velocity** vector, \mathbf{v}
 - Then the position of the object at the time, t can be given by
 - $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$
- The **velocity vector** is the direction vector in the equation of the line
- The speed of the object will be the magnitude of the velocity $|\mathbf{v}|$



Your notes

Worked example

A car, moving at constant speed, takes 2 minutes to drive in a straight line from point A $(-4, 3)$ to point B $(6, -5)$.

At time t , in minutes, the position vector (\mathbf{p}) of the car relative to the origin can be given in the form

$$\mathbf{p} = \mathbf{a} + t\mathbf{b}.$$

Find the vectors \mathbf{a} and \mathbf{b} .

Vector \mathbf{a} represents the initial position and vector \mathbf{b} represents the direction vector per minute.

Position vector $\vec{OA} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$

At $t = 0$ minutes, $\mathbf{p} = \mathbf{a}$ so $\mathbf{a} = \vec{OA} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$

Position vector $\vec{OB} = \begin{pmatrix} 6 \\ -5 \end{pmatrix}$

At $t = 2$ minutes, the car is at the point B and so $\vec{OB} = \mathbf{a} + 2\mathbf{b}$

$$\begin{pmatrix} 6 \\ -5 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} + 2\mathbf{b}$$

Direction vector $2\mathbf{b} = \begin{pmatrix} 6 \\ -5 \end{pmatrix} - \begin{pmatrix} -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 10 \\ -8 \end{pmatrix}$

$$\mathbf{a} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 5 \\ -4 \end{pmatrix}$$



Your notes

Vectors & Variable Velocity

How are vectors used to model motion with variable velocity?

- The **velocity** of a particle is the rate of change of its **displacement** over time
- In one dimension velocity, v , is found by taking the derivative of the displacement, s , with respect to time, t
 - $v = \frac{ds}{dt}$
- In more than one dimension **vectors** are used to represent motion
- For displacement given as a function of time in the form
 - $\mathbf{r}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$
- The velocity vector can be found by differentiating each component of the vector individually
 - $\mathbf{v} = \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}$
 - $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \begin{pmatrix} f_1'(t) \\ f_2'(t) \end{pmatrix}$
 - The velocity should be left as a **vector**
 - The speed is the magnitude of the velocity
- If the velocity vector is known, displacement can be found by **integrating** each component of the vector individually
 - The constant of integration for each component will need to be found
- The **acceleration** of a particle is the rate of change of its **velocity** over time
- In one dimension acceleration, a , is found by taking the derivative of the velocity, v , with respect to time, t
 - $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$
- In two dimensions acceleration can be found by differentiating each component of the velocity vector individually
 - $\mathbf{a} = \begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix}$
 - $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \begin{pmatrix} v_1'(t) \\ v_2'(t) \end{pmatrix}$
 - $\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \begin{pmatrix} f_1''(t) \\ f_2''(t) \end{pmatrix}$



Your notes

- If the acceleration vector is known, the velocity vector can be found by **integrating** each component of the acceleration vector individually
 - The constant of integration for each component will need to be found

Examiner Tip

- Look out for clues in the question as to whether you should treat the question as a constant or variable velocity problem
 - 'moving at a constant speed' will imply using a linear model
 - an object falling or rolling would imply variable velocity

Worked example

A ball is rolling down a hill with velocity $\underline{v} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} + t \begin{pmatrix} 0 \\ -0.8 \end{pmatrix}$. At the time $t = 0$ the position vector of the ball is $3\mathbf{i} - 2\mathbf{j}$.

- a) Find the acceleration vector of the ball's motion.

$$\underline{v} = \begin{pmatrix} 5 \\ 3 - 0.8t \end{pmatrix} \Rightarrow \underline{a} = \frac{d\underline{v}}{dt} = \begin{pmatrix} 0 \\ -0.8 \end{pmatrix}$$

$$\underline{a} = -0.8\mathbf{j}$$

- b) Find the position vector of the ball at the time, t .

$$\underline{r} = \int \underline{v} dt = \int \begin{pmatrix} 5 \\ 3 - 0.8t \end{pmatrix} dt = \begin{pmatrix} 5t + c \\ 3t - \frac{0.8t^2}{2} + d \end{pmatrix}$$

$$\text{at } t=0, \underline{r} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

A constant of integration is needed for both components.

$$\begin{pmatrix} 5(0) + c \\ 3(0) - \frac{0.8(0)^2}{2} + d \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \therefore c = 3, d = -2$$

$$\underline{r} = (5t + 3)\mathbf{i} + (3t - 0.4t^2 - 2)\mathbf{j}$$