

DP IB Maths: AA HL



4.3 Probability

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Your notes

4.3.1 Probability & Types of Events

Probability Basics

What key words and terminology are used with probability?

- An **experiment** is a repeatable activity that has a result that can be observed or recorded
 - Trials** are what we call the repeats of the experiment
- An **outcome** is a possible result of a trial
- An **event** is an outcome or a collection of outcomes
 - Events are usually denoted with capital letters: A , B , etc
 - $n(A)$ is the number of outcomes that are included in event A
 - An event can have one or more than one outcome
- A **sample space** is the set of all possible outcomes of an experiment
 - This is denoted by U
 - $n(U)$ is the total number of outcomes
 - It can be represented as a **list** or a **table**

How do I calculate basic probabilities?

- If all outcomes are **equally likely** then probability for each outcome is the same
 - Probability for each outcome is $\frac{1}{n(U)}$
- Theoretical probability** of an event can be calculated without using an experiment by dividing the number of outcomes of that event by the total number of outcomes

$$P(A) = \frac{n(A)}{n(U)}$$

- This is given in the **formula booklet**
 - Identifying all possible outcomes either as a list or a table can help
- Experimental probability** (also known as **relative frequency**) of an outcome can be calculated using results from an experiment by dividing its frequency by the number of trials

- Relative frequency** of an outcome is $\frac{\text{Frequency of that outcome from the trials}}{\text{Total number of trials } (n)}$

How do I calculate the expected number of occurrences of an outcome?

- Theoretical probability** can be used to calculate the **expected number of occurrences** of an outcome from n trials
- If the probability of an outcome is p and there are n trials then:
 - The expected number of occurrences is **np**
 - This **does not mean** that there will **exactly np occurrences**

- If the experiment is repeated multiple times then we expect the number of occurrences to average out to be np



Your notes

What is the complement of an event?

- The probabilities of all the outcomes **add up to 1**
- Complementary events are when there are **two events** and **exactly one** of them will occur
 - One event has to occur but both events can not occur at the same time
- The **complement of event A** is the event where event **A does not happen**
 - This can be thought of as **not A**
 - This is denoted A'

$$P(A) + P(A') = 1$$

- This is in the **formula booklet**
- It is commonly written as $P(A') = 1 - P(A)$

What are different types of combined events?

- The **intersection** of two events (A and B) is the event where **both A and B** occur
 - This can be thought of as **A and B**
 - This is denoted as $A \cap B$
- The **union** of two events (A and B) is the event where **A or B or both occur**
 - This can be thought of as **A or B**
 - This is denoted $A \cup B$
- The event where A occurs given that event B has occurred is called **conditional probability**
 - This can be thought as **A given B**
 - This is denoted $A|B$

How do I find the probability of combined events?

- The probability of A or B (or both) occurring can be found using the formula

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- This is given in the **formula booklet**
- You subtract the probability of A and B both occurring because it has been included twice (once in $P(A)$ and once in $P(B)$)
- The probability of A and B occurring can be found using the formula

$$P(A \cap B) = P(A)P(B|A)$$

- A rearranged version is given in the **formula booklet**
- Basically you multiply the probability of A by the probability of B then happening

 **Examiner Tip**

- In an exam drawing a Venn diagram or tree diagram can help even if the question does not ask you to



Your notes



Your notes

Worked example

Dave has two fair spinners, A and B. Spinner A has three sides numbered 1, 4, 9 and spinner B has four sides numbered 2, 3, 5, 7. Dave spins both spinners and forms a two-digit number by using the spinner A for the first digit and spinner B for the second digit.

T is the event that the two-digit number is a multiple of 3.

- a) List all the possible two-digit numbers.

A two-way table would be a systematic way to list all the outcomes

	2	3	5	7
1	12	13	15	17
4	42	43	45	47
9	92	93	95	97

- b) Find $P(T)$.

$$P(T) = \frac{n(T)}{n(U)}$$

← Number of multiples of 3
← Total number of outcomes

{12, 15, 42, 45, 93} are the multiples of 3

$$P(T) = \frac{5}{12}$$

- c) Find $P(T')$.

$$P(T) + P(T') = 1 \Rightarrow P(T') = 1 - P(T)$$

$$P(T') = 1 - \frac{5}{12}$$

$$P(T') = \frac{7}{12}$$



Your notes

Independent & Mutually Exclusive Events

What are mutually exclusive events?

- Two events are **mutually exclusive** if they **cannot both occur**
 - For example: when rolling a dice the events "getting a prime number" and "getting a 6" are mutually exclusive
- If A and B are mutually exclusive events then:
 - $P(A \cap B) = 0$

What are independent events?

- Two events are **independent** if **one occurring does not affect the probability of the other occurring**
 - For example: when flipping a coin twice the events "getting a tails on the first flip" and "getting a tails on the second flip" are independent
- If A and B are independent events then:
 - $P(A|B) = P(A)$ and $P(B|A) = P(B)$
- If A and B are independent events then:
 - $P(A \cap B) = P(A)P(B)$
 - This is given in the **formula booklet**
 - This is a useful formula to test whether two events are statistically independent

How do I find the probability of combined mutually exclusive events?

- If A and B are **mutually exclusive** events then
$$P(A \cup B) = P(A) + P(B)$$
 - This is given in the **formula booklet**
 - This occurs because $P(A \cap B) = 0$
- For any two events A and B the events $A \cap B$ and $A \cap B'$ are **mutually exclusive** and A is the **union** of these two events
 - $P(A) = P(A \cap B) + P(A \cap B')$
 - This works for any two events A and B



Your notes

Worked example

- a) A student is chosen at random from a class. The probability that they have a dog is 0.8, the probability they have a cat is 0.6 and the probability that they have a cat or a dog is 0.9. Find the probability that the student has both a dog and a cat.

Let D be event "has a dog" and C be "has a cat"

$$P(D \cup C) = P(D) + P(C) - P(D \cap C)$$

$$0.9 = 0.8 + 0.6 - P(D \cap C)$$

$$P(D \cap C) = 0.5$$

- b) Two events, Q and R , are such that $P(Q) = 0.8$ and $P(Q \cap R) = 0.1$. Given that Q and R are independent, find $P(R)$.

Q and R independent $\Rightarrow P(Q \cap R) = P(Q)P(R)$

$$0.1 = 0.8 \times P(R) \quad \therefore P(R) = \frac{0.1}{0.8}$$

$$P(R) = 0.125 \quad \text{or} \quad \frac{1}{8}$$

- c) Two events, S and T , are such that $P(S) = 2P(T)$. Given that S and T are mutually exclusive and that $P(S \cup T) = 0.6$ find $P(S)$ and $P(T)$.

S and T mutually exclusive $\Rightarrow P(S \cup T) = P(S) + P(T)$

$$0.6 = P(S) + P(T)$$

$$0.6 = 2P(T) + P(T) \quad P(S) = 2P(T)$$

$$0.6 = 3P(T)$$

$$P(T) = 0.2 \quad \text{and} \quad P(S) = 0.4$$



Your notes

4.3.2 Conditional Probability

Conditional Probability

What is conditional probability?

- **Conditional probability** is where the probability of an **event** happening can vary depending on the outcome of a prior event
- The event A happening **given that** event B has happened is denoted $A|B$
- A common example of conditional probability involves selecting multiple objects from a bag **without replacement**
 - The probability of selecting a certain item changes depending on what was selected before
 - This is because the total number of items will change as they are not replaced once they have been selected

How do I calculate conditional probabilities?

- Some conditional probabilities can be calculated by using counting outcomes
 - Probabilities without replacement can be calculated like this
 - For example: There are 10 balls in a bag, 6 of them are red, two of them are selected without replacement
 - To find the probability that the second ball selected is red given that the first one is red count how many balls are left:
 - A red one has already been selected so there are 9 balls left and 5 are red so the probability is $\frac{5}{9}$
- You can use sample space diagrams to find the probability of A given B :
 - reduce your sample space to just include outcomes for event B
 - find the proportion that also contains outcomes for event A
- There is a formula for conditional probability that you should use
 - $$P(A|B) = \frac{P(A \cap B)}{P(B)}$$
 - This is given in the **formula booklet**
 - This can be rearranged to give $P(A \cap B) = P(B)P(A|B)$
 - By symmetry you can also write $P(A \cap B) = P(A)P(B|A)$

How do I tell if two events are independent using conditional probabilities?

- If A and B are two events then they are independent if:
 - $P(A|B) = P(A) = P(A|B')$
- Equally you can still use $P(A \cap B) = P(A)P(B)$ to test for independence
 - This is given in the **formula booklet**



Your notes

Worked example

Let R be the event that it is raining in Weatherville and T be the event that there is a thunderstorm in Weatherville.

It is known that $P(T) = 0.035$, $P(T \cap R) = 0.03$ and $P(T|R) = 0.15$.

- a) Find the probability that it is raining in Weatherville.

Formula booklet

Conditional probability	$P(A B) = \frac{P(A \cap B)}{P(B)}$
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$$P(T|R) = \frac{P(T \cap R)}{P(R)}$$

Substitute the values in

$$0.15 = \frac{0.03}{P(R)}$$

$$P(R) = \frac{0.03}{0.15}$$

$$P(R) = 0.2$$

- b) State whether the events R and T are independent. Give a reason for your answer.

If R and T are independent then $P(T|R) = P(T)$

$$P(T|R) = 0.15 \text{ and } P(T) = 0.035$$

R and T are not independent as $P(T|R) \neq P(T)$



Your notes

4.3.3 Bayes' Theorem

Bayes' Theorem

What is Bayes' Theorem

- **Bayes' Theorem** allows you **switch the order** of conditional probabilities
 - If you know $P(B)$, $P(B')$ and $P(A|B)$ then Bayes' Theorem allows you to find $P(B|A)$
- Essentially if you have a **tree diagram** you will already know the conditional probabilities of the **second branches**
 - Bayes' Theorem allows you to find the **conditional probabilities** if you **switch the order** of the events
- For any two events A and B Bayes' Theorem states:

$$P(B|A) = \frac{P(B)P(A|B)}{P(B)P(A|B) + P(B')P(A|B')}$$

- This is given in the **formula booklet**
- This formula is derived using the formulae:
 - $P(B|A) = \frac{P(B \cap A)}{P(A)}$
 - $P(A) = P(B \cap A) + P(B' \cap A)$
 - $P(B \cap A) = P(B)P(A|B)$ and $P(B' \cap A) = P(B')P(A|B')$
- Bayes' Theorem can be **extended** to **mutually exclusive events** B_1, B_2, \dots, B_n and any other event A
 - In your exam you will have a **maximum of three** mutually exclusive events

$$P(B_i|A) = \frac{P(B_i)P(A|B_i)}{P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3)}$$

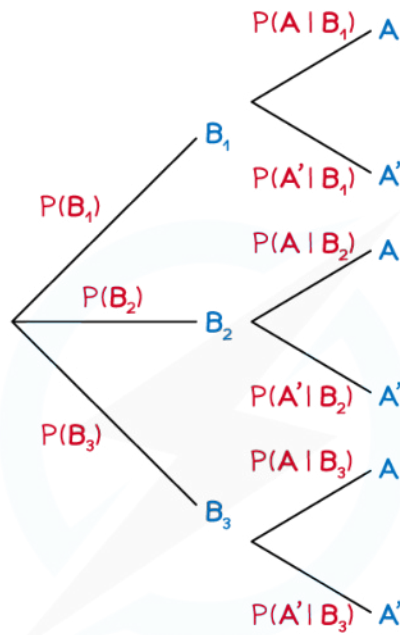
- This is given in the **formula booklet**

How do I calculate conditional probabilities using Bayes' Theorem?

- Start by **drawing a tree diagram**
 - Label B_1 & B_2 (& B_3 if necessary) on the **first set** of branches
 - Label A & A' on the **second set** of branches
- The questions will give you enough information to **label the probabilities** on this tree
- **Identify the probabilities needed** to use Bayes' Theorem
 - The probabilities will come in pairs: $P(B_i)$ and $P(A|B_i)$



Your notes



$$P(B_1 | A) = \frac{P(B_1) P(A | B_1)}{P(B_1) P(A | B_1) + P(B_2) P(A | B_2) + P(B_3) P(A | B_3)}$$

SWAP B_i for B_1, B_2 or B_3 DEPENDING ON THE PROBABILITY NEEDED
THE DENOMINATOR DOES NOT CHANGE

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 **Examiner Tip**

- In an exam you are less likely to make a mistake when using the formula if you draw a tree diagram first



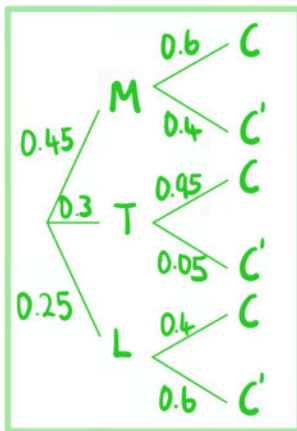
Your notes

Worked example

Lucy is doing a quiz. For each question there's a 45% chance that it is about music, 30% chance that it is about TV and 25% chance that it is about literature. The probability that Lucy answers a question correctly is 0.6 for music, 0.95 for TV and 0.4 for literature.

a) Draw a tree diagram to represent this information.

Let M be the event that the question is on music
 T be the event that the question is on TV
 L be the event that the question is on literature
 C be the event that Lucy answers correctly



$$P(C'|M) = 1 - P(C|M) \\ = 1 - 0.6$$

$$P(C'|T) = 1 - P(C|T) \\ = 1 - 0.95$$

$$P(C'|L) = 1 - P(C|L) \\ = 1 - 0.4$$

b) Given that Lucy answered a question correctly, find the probability that it was about TV.

Formula booklet

Bayes' theorem	$P(B_i A) = \frac{P(B_i)P(A B_i)}{P(B_1)P(A B_1) + P(B_2)P(A B_2) + P(B_3)P(A B_3)}$
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Let $A=C$ $B_1=M$ $B_2=T$ $B_3=L$

$$P(T|C) = \frac{P(T)P(C|T)}{P(M)P(C|M) + P(T)P(C|T) + P(L)P(C|L)} \\ = \frac{0.3 \times 0.95}{0.45 \times 0.6 + 0.3 \times 0.95 + 0.25 \times 0.4}$$

$$P(T|C) = \frac{57}{131} = 0.435 \text{ (3sf)}$$



Your notes



Your notes

4.3.4 Sample Space Diagrams

Venn Diagrams

What is a Venn diagram?

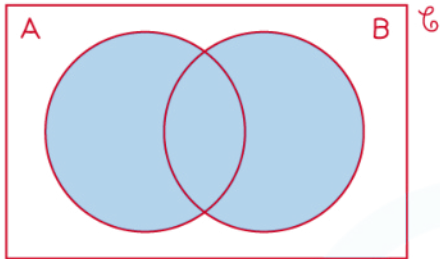
- A Venn diagram is a way to illustrate **events** from an **experiment** and are particularly useful when there is an overlap between possible **outcomes**
- A Venn diagram consists of
 - a **rectangle** representing the **sample space (U)**
 - The rectangle is labelled U
 - Some mathematicians instead use S or ξ
 - a **circle** for each **event**
 - Circles may or may not overlap depending on which **outcomes** are shared between **events**
- The numbers in the circles represent either the **frequency** of that event or the **probability** of that event
 - If the **frequencies** are used then they should **add up to the total frequency**
 - If the **probabilities** are used then they should **add up to 1**

What do the different regions mean on a Venn diagram?

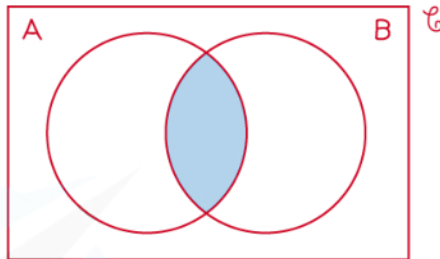
- A' is represented by the regions that are **not in** the A circle
- $A \cap B$ is represented by the region where the A and B circles **overlap**
- $A \cup B$ is represented by the regions that **are in** A or B or both
- Venn diagrams show '**AND**' and '**OR**' statements easily
- Venn diagrams also instantly show **mutually exclusive** events as these circles will **not overlap**
- **Independent** events can not be instantly seen
 - You need to use probabilities to deduce if two events are independent



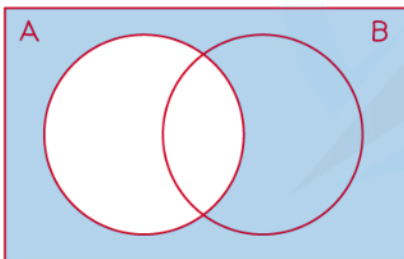
Your notes



$A \cup B$ (UNION)
"A OR B OR BOTH"

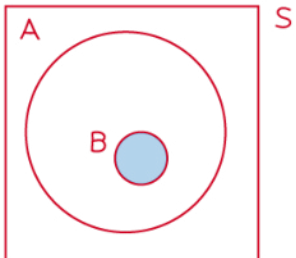


$A \cap B$ (INTERSECTION)
"A AND B"

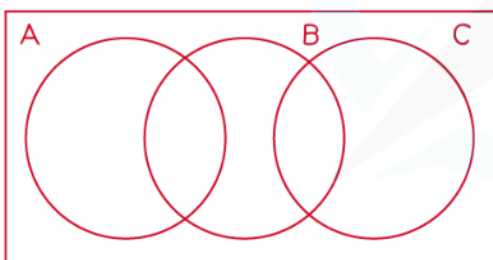


A' (COMPLEMENT)
"NOT A"

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THE BUBBLE FOR EVENT B LIES ENTIRELY IN THE BUBBLE FOR EVENT A IF EVENT B OCCURS, SO DOES EVENT A (BUT NOT NECESSARILY VICE VERSA)



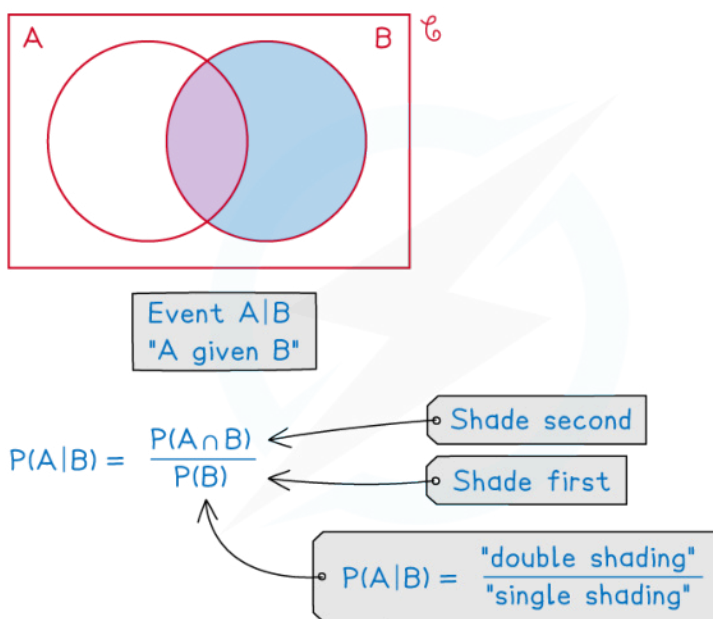
THE BUBBLES FOR EVENTS A AND C DO NOT OVERLAP: THEY ARE MUTUALLY EXCLUSIVE

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How do I solve probability problems involving Venn diagrams?

- Draw, or add to a given Venn diagram, filling in as many values as possible from the information provided in the question
- It is usually helpful to work from the centre outwards
 - Fill in **intersections** (overlaps) first
- If two events are independent you can use the formula
 - $P(A \cap B) = P(A)P(B)$
- To find the conditional probability $P(A|B)$
 - Add together the frequencies/probabilities in the B circle
 - This is your denominator
 - Out of those frequencies/probabilities add together the ones that are also in the A circle
 - This is your numerator
 - Evaluate the fraction



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Examiner Tip

- If you struggle to fill in a Venn diagram in an exam:
 - Label the missing parts using algebra
 - Form equations using known facts such as:
 - the sum of the probabilities should be 1
 - $P(A \cap B) = P(A)P(B)$ if A and B are independent events



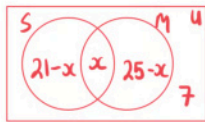
Your notes

Worked example

40 people are asked if they have sugar and/or milk in their coffee. 21 people have sugar, 25 people have milk and 7 people have neither.

- a) Draw a Venn diagram to represent the information.

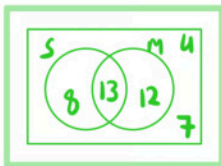
Find the centre first



Total should be 40

$$(21-x) + x + (25-x) + 7 = 40$$

$$53 - x = 40 \quad \therefore x = 13$$



- b) One of the 40 people are randomly selected, find the probability that they have sugar but not milk with their coffee.

S and not M is the part of S circle that does not include M

$$P(S \cap M') = \frac{8}{40}$$

Remember to write as a fraction of the total

$$P(S \cap M') = \frac{1}{5}$$

- c) Given that a person who has sugar is selected at random, find the probability that they have milk with their coffee.

Given that sugar has been selected we only want the S circle as our total.

Out of the S circle 13 also have milk

$$P(M|S) = \frac{13}{21}$$



Your notes

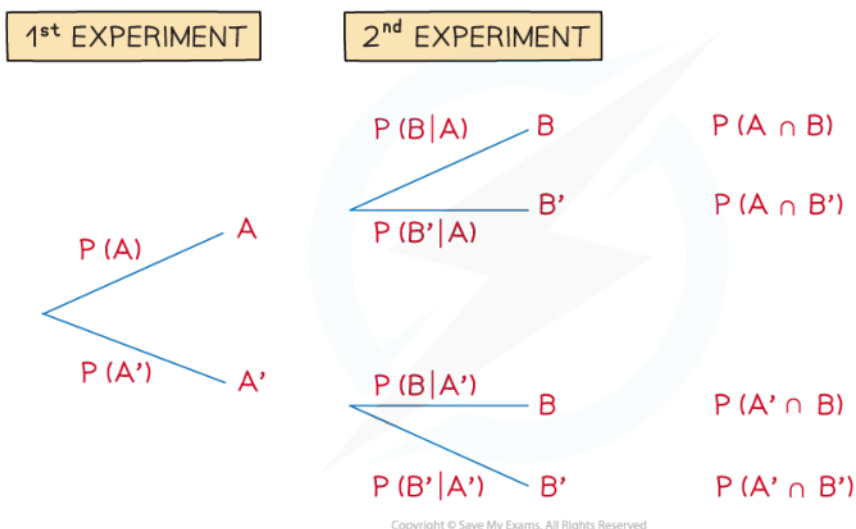
Tree Diagrams

What is a tree diagram?

- A **tree diagram** is another way to show the outcomes of combined events
 - They are very useful for intersections of events
- The events on the branches must be **mutually exclusive**
 - Usually they are an event and its complement
- The probabilities on the second sets of branches **can depend** on the outcome of the first event
 - These are **conditional probabilities**
- When selecting the items from a bag:
 - The second set of branches will be the **same** as the first if the items **are replaced**
 - The second set of branches will be the **different** to the first if the items **are not replaced**

How are probabilities calculated using a tree diagram?

- To find the probability that two events happen together you **multiply** the corresponding probabilities on their branches
 - It is helpful to find the probability of all combined outcomes once you have drawn the tree
- To find the probability of an event you can:
 - add together** the probabilities of the **combined outcomes** that are part of that event
 - For example: $P(A \cup B) = P(A \cap B) + P(A \cap B') + P(A' \cap B)$
 - subtract** the probabilities of the combined outcomes that are not part of that event from 1
 - For example: $P(A \cup B) = 1 - P(A' \cap B')$



Do I have to use a tree diagram?

- If there are **multiple events** or trials then a tree diagram can get big
- You can break down the problem by using the words **AND/OR/NOT** to help you find probabilities without a tree

- You can speed up the process by only drawing parts of the tree that you are interested in

Which events do I put on the first branch?

- If the events A and B are **independent** then the **order does not matter**
- If the events A and B are **not independent** then the **order does matter**
 - If you have the probability of **A given B** then put **B on the first set** of branches
 - If you have the probability of **B given A** then put **A on the first set** of branches

Examiner Tip

- In an exam do not waste time drawing a full tree diagram for scenarios with lots of events unless the question asks you to
 - Only draw the parts that you are interested in



Your notes

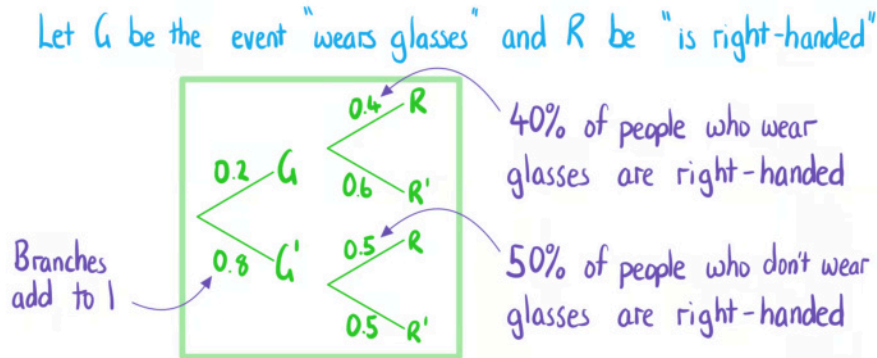


Your notes

Worked example

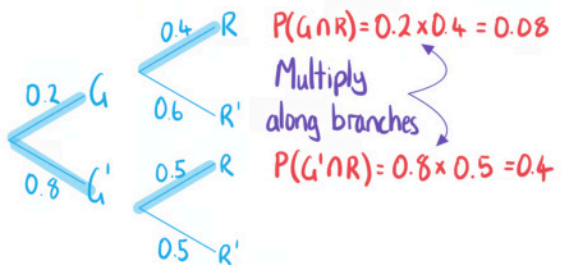
20% of people in a company wear glasses. 40% of people in the company who wear glasses are right-handed. 50% of people in the company who don't wear glasses are right-handed.

- a) Draw a tree diagram to represent the information.



- b) One of the people in the company are randomly selected, find the probability that they are right-handed.

Find options that contain R



$$P(R) = P(G \cap R) + P(G' \cap R) = 0.08 + 0.4$$

$$P(R) = 0.48$$

- c) Given that a person who is right-handed is selected at random, find the probability that they wear glasses.

$$P(G|R) = \frac{P(G \cap R)}{P(R)} = \frac{0.08}{0.48}$$

$$P(G|R) = \frac{1}{6}$$



Your notes