

# DP IB Maths: AI SL



Your notes

## 4.4 Probability Distributions

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## 4.4.1 Discrete Probability Distributions

### Discrete Probability Distributions

#### What is a discrete random variable?

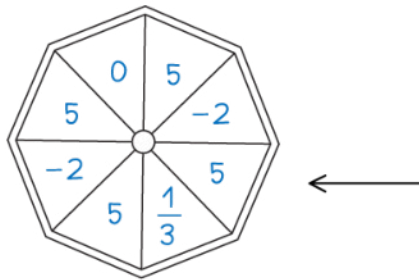
- A **random variable** is a variable whose value depends on the outcome of a **random event**
  - The value of the random variable is not known until the event is carried out (this is what is meant by 'random' in this case)
- **Random variables** are denoted using **upper case letters** ( $X$ ,  $Y$ , etc)
- **Particular outcomes** of the event are denoted using **lower case letters** ( $x$ ,  $y$ , etc)
- $P(X = x)$  means "the probability of the random variable  $X$  taking the value  $x$ "
- A **discrete** random variable (often abbreviated to DRV) can only take **certain values** within a set
  - Discrete random variables **usually count** something
  - Discrete random variables usually can only take a finite number of values but it is possible that it can take an infinite number of values (see the examples below)
- **Examples** of discrete random variables include:
  - The number of times a coin lands on heads when flipped 20 times
    - this has a finite number of outcomes:  $\{0, 1, 2, \dots, 20\}$
  - The number of emails a manager receives within an hour
    - this has an infinite number of outcomes:  $\{1, 2, 3, \dots\}$
  - The number of times a dice is rolled until it lands on a 6
    - this has an infinite number of outcomes:  $\{1, 2, 3, \dots\}$
  - The number that a dice lands on when rolled once
    - this has a finite number of outcomes:  $\{1, 2, 3, 4, 5, 6\}$

#### What is a probability distribution of a discrete random variable?

- A **discrete probability distribution** fully describes **all the values** that a discrete random variable can take along with their **associated probabilities**
  - This can be given in a **table**
  - Or it can be given as a **function** (called a discrete probability distribution function or "pdf")
  - They can be represented by **vertical line graphs** (the possible values for along the horizontal axis and the probability on the vertical axis)
- The **sum of the probabilities** of **all the values** of a discrete random variable is **1**
  - This is usually written  $\sum P(X = x) = 1$
- A **discrete uniform distribution** is one where the random variable takes a finite number of values each with an **equal probability**
  - If there are  $n$  values then the probability of each one is  $\frac{1}{n}$

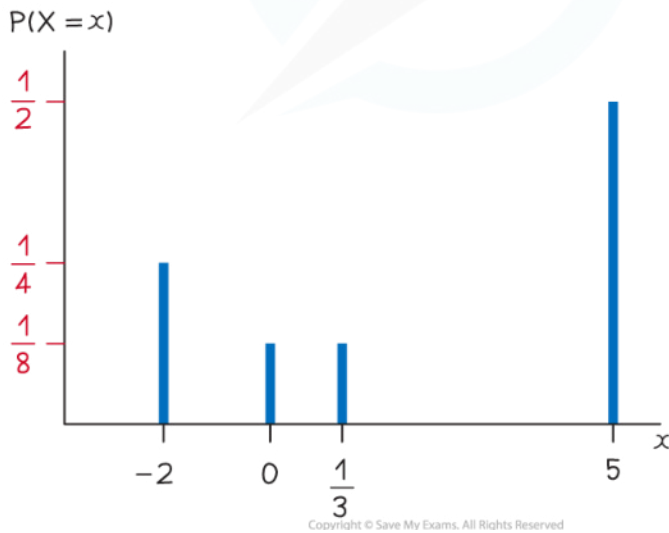


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LET  $x$  BE THE NUMBER THAT THE SPINNER LANDS ON

$x$	-2	0	$\frac{1}{3}$	5
$P(X=x)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{2}$

$$P(X=x) = \begin{cases} \frac{1}{8} & x = 0, \frac{1}{3} \\ \frac{1}{4} & x = -2 \\ \frac{1}{2} & x = 5 \\ 0 & \text{OTHERWISE} \end{cases}$$


### How do I calculate probabilities using a discrete probability distribution?

- First **draw a table** to represent the probability distribution
  - If it is given as a function then find each probability
  - If any probabilities are unknown then use algebra to represent them
- **Form an equation** using  $\sum P(X=x) = 1$ 
  - Add together all the probabilities and make the sum equal to 1
- To find  $P(X=k)$

- If  $k$  is a possible value of the random variable  $X$  then  $P(X = k)$  will be given in the table
- If  $k$  is not a possible value then  $P(X = k) = 0$
- To find  $P(X \leq k)$ 
  - Identify all possible values,  $x_i$ , that  $X$  can take which satisfy  $x_i \leq k$
  - Add together all their corresponding probabilities
  - $P(X \leq k) = \sum_{x_i \leq k} P(X = x_i)$
  - Some mathematicians use the notation  $F(x)$  to represent the cumulative distribution
    - $F(x) = P(X \leq x)$
- Using a similar method you can find  $P(X < k)$ ,  $P(X > k)$  and  $P(X \geq k)$
- As all the probabilities add up to 1 you can form the following equivalent equations:
  - $P(X < k) + P(X = k) + P(X > k) = 1$
  - $P(X > k) = 1 - P(X \leq k)$
  - $P(X \geq k) = 1 - P(X < k)$

### How do I know which inequality to use?

- $P(X \leq k)$  would be used for phrases such as:
  - At most, no greater than, etc
- $P(X < k)$  would be used for phrases such as:
  - Fewer than
- $P(X \geq k)$  would be used for phrases such as:
  - At least, no fewer than, etc
- $P(X > k)$  would be used for phrases such as:
  - Greater than, etc



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### Worked example

The probability distribution of the discrete random variable  $X$  is given by the function

$$P(X=x) = \begin{cases} kx^2 & x = -3, -1, 2, 4 \\ 0 & \text{otherwise.} \end{cases}$$

a) Show that  $k = \frac{1}{30}$ .

Construct a table

$x$	-3	-1	2	4
$P(X=x)$	$9k$	$k$	$4k$	$16k$

Substitute in the values of  $x$   
e.g.  $P(X=-3) = k(-3)^2 = 9k$

The probabilities add up to 1

$$9k + k + 4k + 16k = 1$$

$$30k = 1$$

$$k = \frac{1}{30}$$

b) Calculate  $P(X \leq 3)$ .

Substitute  $k$  into the probabilities

$x$	-3	-1	2	4
$P(X=x)$	$\frac{3}{10}$	$\frac{1}{30}$	$\frac{2}{15}$	$\frac{8}{15}$

$$X \leq 3 : X = -3, -1, 2$$

$$P(X \leq 3) = P(X=-3) + P(X=-1) + P(X=2)$$

$$= \frac{3}{10} + \frac{1}{30} + \frac{2}{15}$$

$$P(X \leq 3) = \frac{7}{15}$$



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## 4.4.2 Expected Values

### Expected Values $E(X)$

#### What does $E(X)$ mean and how do I calculate $E(X)$ ?

- $E(X)$  means the **expected value** or the **mean** of a **random variable  $X$** 
  - The expected value does not need to be an obtainable value of  $X$
  - For example: the expected value number of times a coin will land on tails when flipped 5 times is 2.5
- For a **discrete** random variable, it is calculated by:
  - **Multiplying each value** of  $X$  with its corresponding **probability**
  - **Adding** all these terms together

$$E(X) = \sum xP(X = x)$$

- This is given in the **formula booklet**
- Look out for **symmetrical** distributions (where the values of  $X$  are symmetrical and their probabilities are symmetrical) as the mean of these is the same as the median
  - For example: if  $X$  can take the values 1, 5, 9 with probabilities 0.3, 0.4, 0.3 respectively then by symmetry the mean would be 5

#### How can I decide if a game is fair?

- Let  $X$  be the random variable that represents the **gain/loss** of a player in a game
  - $X$  will be **negative** if there is a **loss**
- Normally the expected gain or loss is calculated by **subtracting** the **cost to play** the game from the **expected value** of the **prize**
- If  $E(X)$  is **positive** then it means the player can **expect to make a gain**
- If  $E(X)$  is **negative** then it means the player can **expect to make a loss**
- The game is called **fair** if the **expected gain is 0**
  - $E(X) = 0$



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### Worked example

Daphne pays \$15 to play a game where she wins a prize of \$1, \$5, \$10 or \$100. The random variable  $W$  represents the amount she wins and has the probability distribution shown in the following table:

$w$	1	5	10	100
$P(W = w)$	0.35	0.5	0.05	0.1

- a) Calculate the expected value of Daphne's prize.

Formula booklet

Expected value of a discrete random variable $X$	$E(X) = \sum x P(X = x)$
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$$E(W) = \sum \omega P(W = \omega)$$

$$= 1 \times 0.35 + 5 \times 0.5 + 10 \times 0.05 + 100 \times 0.1$$

$$\text{Expected value} = \$13.35$$

- b) Determine whether the game is fair.

A game is fair if expected gain/loss is 0

Prize - cost

$$13.35 - 15 = -1.65$$

Expected loss is \$1.65 so game is not fair