DP IB Maths: AA HL

Your notes

3.11 Vector Planes

Contents

Your notes

3.11.1 Vector Equations of Planes

Equation of a Plane in Vector Form

How do I find the vector equation of a plane?

- A plane is a flat surface which is two-dimensional
  - Imagine a flat piece of paper that continues on forever in both directions
- A plane in often denoted using the capital Greek letter Π
- The vector form of the equation of a plane can be found using **two direction vectors** on the plane
  - The direction vectors must be
    - **parallel** to the plane
    - **not parallel** to each other
  - If **both** direction vectors **lie on the plane** then they will **intersect at a point**
- The formula for finding the **vector equation** of a plane is
  - $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b} + \mu \boldsymbol{c}$
    - Where ***r*** is the **position vector** of any point on the plane
    - ***a*** is the **position vector** of a known point on the plane
    - ***b*** and **c** are two **non-parallel direction** (displacement) **vectors** parallel to the plane
    - λ and μ are scalars
  - The formula is **given in the formula booklet** but you must make sure you know what each part means
- As ***a*** could be the position vector of **any** point on the plane and ***b*** and **c** could be **any non-parallel** direction vectors on the plane there are infinite vector equations for a single plane

How do I determine whether a point lies on a plane?

- Given the equation of a plane $\boldsymbol{r} = \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{pmatrix} + \lambda \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \boldsymbol{b}_3 \end{pmatrix} + \mu \begin{pmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \\ \boldsymbol{c}_3 \end{pmatrix}$ then the point ***r*** with position vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is on the plane if there exists a value of λ and μ such that
  - $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{pmatrix} + \lambda \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \boldsymbol{b}_3 \end{pmatrix} + \mu \begin{pmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \\ \boldsymbol{c}_3 \end{pmatrix}$
  - This means that there exists a single value of λ and μ that satisfy the three **parametric** equations:
    - $x = a_1 + \lambda b_1 + \mu c_1$
    - $y = a_2 + \lambda b_2 + \mu c_2$

- $z = a_3 + \lambda b_3 + \mu c_3$
- Solve two of the equations first to find the values of λ and μ that satisfy the first two equation and then check that this value also satisfies the third equation
- If the values of λ and μ do not satisfy all three equations, then the point ***r*** does not lie on the plane

Examiner Tip

- The formula for the vector equation of a plane is given in the formula booklet, make sure you know what each part means
- Be careful to use different letters, e.g. λ and μ as the scalar multiples of the two direction vectors

Your notes

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Worked example

The points A, B and C have position vectors $\boldsymbol{a} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\boldsymbol{b} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, and $\boldsymbol{c} = 4\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ respectively, relative to the origin O.

(a) Find the vector equation of the plane.

Start by finding the direction vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$

$$\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix}1\\-2\\4\end{pmatrix} - \begin{pmatrix}3\\2\\-1\end{pmatrix} = \begin{pmatrix}-2\\-4\\5\end{pmatrix}$$

$$\overrightarrow{AC} = \underline{c} - \underline{a} = \begin{pmatrix}4\\-1\\3\end{pmatrix} - \begin{pmatrix}3\\2\\-1\end{pmatrix} = \begin{pmatrix}1\\-3\\4\end{pmatrix}$$

All three points lie on the plane, so choose the position vector of one point, e.g. $\overrightarrow{OA}$, to use as 'a' in the vector equation of a plane formula.

Check that $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are not parallel.

$$r = \underline{a} + \lambda\overrightarrow{AB} + \mu\overrightarrow{AC}$$

$$\underline{r} = \begin{pmatrix}3\\2\\-1\end{pmatrix} + \lambda\begin{pmatrix}-2\\-4\\5\end{pmatrix} + \mu\begin{pmatrix}1\\-3\\4\end{pmatrix}$$

(This is one of many correct answers)

(b) Determine whether the point D with coordinates (-2, -3, 5) lies on the plane.

Your notes

Let D have position vector $\underline{d} = \begin{pmatrix} -2 \\ -3 \\ 5 \end{pmatrix}$, then the point D lies on the plane if there exists a value of λ and μ for which: $\begin{pmatrix} -2 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ -4 \\ 5 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$

Find the parametric equations:

$-2 = 3 - 2\lambda + \mu \Rightarrow \mu - 2\lambda = -5$ ①

$-3 = 2 - 4\lambda - 3\mu \Rightarrow 3\mu + 4\lambda = 5$ ②

} solve two equations for λ and μ.

$5 = -1 + 5\lambda + 4\mu \Rightarrow 4\mu + 5\lambda = 6$ ③

Find the value of λ and μ from two equations:

2①: $2\mu - 4\lambda = -10$

+ ②: $3\mu + 4\lambda = 5$

$5\mu = -5$

$\mu = -1$ sub into ①: $(-1) - 2\lambda = -5$

$\lambda = 2$

Check to see if λ and μ satisfy the third equation:

$4(-1) + 5(2) = -4 + 10 = 6$ ✓

The point D lies on the plane.

Your notes

Equation of a Plane in Cartesian Form

How do I find the vector equation of a plane in cartesian form?

- The **cartesian** equation of a plane is given in the form
  - $ax + by + cz = d$
  - This is **given in the formula booklet**
- A **normal vector** to the plane can be used along with a **known point on the plane** to find the cartesian equation of the plane
  - The normal vector will be a vector that is **perpendicular** to the plane
- The **scalar product** of the normal vector and any **direction vector** on the plane will the **zero**
  - The two vectors will be perpendicular to each other
  - The **direction vector** from a fixed-point *A* to any point on the plane, *R* can be written as ***r* – *a***
  - Then $\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{a}) = 0$ and it follows that $(\boldsymbol{n} \cdot \boldsymbol{r}) - (\boldsymbol{n} \cdot \boldsymbol{a}) = 0$
- This gives the **equation of a plane using the normal vector:**
  - $\boldsymbol{n} \cdot \boldsymbol{r} = \boldsymbol{a} \cdot \boldsymbol{n}$
    - Where ***r*** is the **position vector** of any point on the plane
    - ***a*** is the **position vector** of a known point on the plane
    - ***n*** is a vector that is **normal** to the plane
  - This is **given in the formula booklet**
- If the vector ***r*** is given in the form $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and ***a*** and ***n*** are both known vectors given in the form $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ then the Cartesian equation of the plane can be found using:
  - $\boldsymbol{n} \cdot \boldsymbol{r} = ax + by + cz$
  - $\boldsymbol{a} \cdot \boldsymbol{n} = a_1 a + a_2 b + a_3 c$
  - Therefore $ax + by + cz = a_1 a + a_2 b + a_3 c$
  - This simplifies to the form $ax + by + cz = d$

How do I find the equation of a plane in Cartesian form given the vector form?

- The **Cartesian** equation of a plane can be found if you know
  - the **normal vector** and
  - a **point** on the plane
- The **vector equation of a plane** can be used to find the **normal vector** by finding the **vector product** of the two direction vectors
  - A vector product is always perpendicular to the two vectors from which it was calculated
- The vector ***a*** given in the vector equation of a plane is a **known point** on the plane

- Once you have found the normal vector then the point $\boldsymbol{a}$ can be used in the formula $\boldsymbol{n} \cdot \boldsymbol{r} = \boldsymbol{a} \cdot \boldsymbol{n}$ to find the equation in Cartesian form
- To find $ax + by + cz = d$ given $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b} + \mu \boldsymbol{c}$:
  - Let $\boldsymbol{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \boldsymbol{b} \times \boldsymbol{c}$ then $d = \boldsymbol{n} \cdot \boldsymbol{a}$

Your notes

Examiner Tip

- In an exam, using whichever form of the equation of the plane to write down a normal vector to the plane is always a good starting point

Your notes

Worked example

A plane Π contains the point A$(2, 6, -3)$ and has a normal vector $\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$.

a) Find the equation of the plane in its Cartesian form.

The components of the normal vector are the x-, y- and z- coefficients of the Cartesian form:

$3x - y + 4z = d$

The point $(2, 6, -3)$ is on the plane so

$d = 3(2) - (6) + 4(-3) = 6 - 6 - 12 = -12$

Therefore

$$3x - y + 4z = -12$$

b) Determine whether point B with coordinates $(-1, 0, -2)$ lies on the same plane.

Test by putting the coordinates into the equation:

$3(-1) - (0) + 4(-2) = -3 - 8 = -11 \neq -12$

The point with coordinates $(-1, 0, 2)$ does not lie on the plane

3.11.2 Intersections of Lines & Planes

Your notes

Intersection of Line & Plane

How do I tell if a line is parallel to a plane?

- A line is parallel to a plane if its **direction vector** is **perpendicular** to the plane's **normal vector**
- If you know the Cartesian equation of the plane in the form $ax + by + cz = d$ then the values of *a*, *b*, and c are the individual components of a normal vector to the plane
- The **scalar product** can be used to check in the direction vector and the normal vector are perpendicular
  - If two vectors are perpendicular their scalar product will be zero

How do I tell if the line lies inside the plane?

- If the line is parallel to the plane then it will either **never intersect** or it will lie inside the plane
  - Check to see if they have a common point
- If a line is parallel to a plane and they share **any point**, then the line lies inside the plane

How do I find the point of intersection of a line and a plane?

- If a line is **not parallel** to a plane it will **intersect** it at a single point
- If both the **vector equation of the line** and the **Cartesian equation of the plane** is known then this can be found by:
- STEP 1: Set the position vector of the point you are looking for to have the individual components *x*, *y*, and *z* and substitute into the vector equation of the line
  - $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix}$
- STEP 2: Find the parametric equations in terms of *x*, *y*, and *z*
  - $x = x_0 + \lambda l$
  - $y = y_0 + \lambda m$
  - $z = z_0 + \lambda n$
- STEP 3: Substitute these parametric equations into the Cartesian equation of the plane and solve to find λ
  - $a(x_0 + \lambda l) + b(y_0 + \lambda m) + c(z_0 + \lambda n) = d$
- STEP 4: Substitute this value of λ back into the vector equation of the line and use it to find the position vector of the point of intersection

- STEP 5: Check this value in the Cartesian equation of the plane to make sure you have the correct answer

Your notes

Worked example

Find the point of intersection of the line $r = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ with the plane $3x - 4y + z = 8$.

Find the parametric form of the equation of the line:

Let $r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ then $x = 1 + 2\lambda$, $y = -3 - \lambda$, $z = 2 - \lambda$

Substitute into the equation of the plane:

$3(1 + 2\lambda) - 4(-3 - \lambda) + (2 - \lambda) = 8$

Solve to find λ:

$3 + 6\lambda + 12 + 4\lambda + 2 - \lambda = 8$

$\lambda = -1$

Substitute $\lambda = -1$ into the vector equation of the line:

$r = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + (-1)\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1-2 \\ -3+1 \\ 2+1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}$

$(-1, -2, 3)$

Your notes

Intersection of Planes

How do we find the line of intersection of two planes?

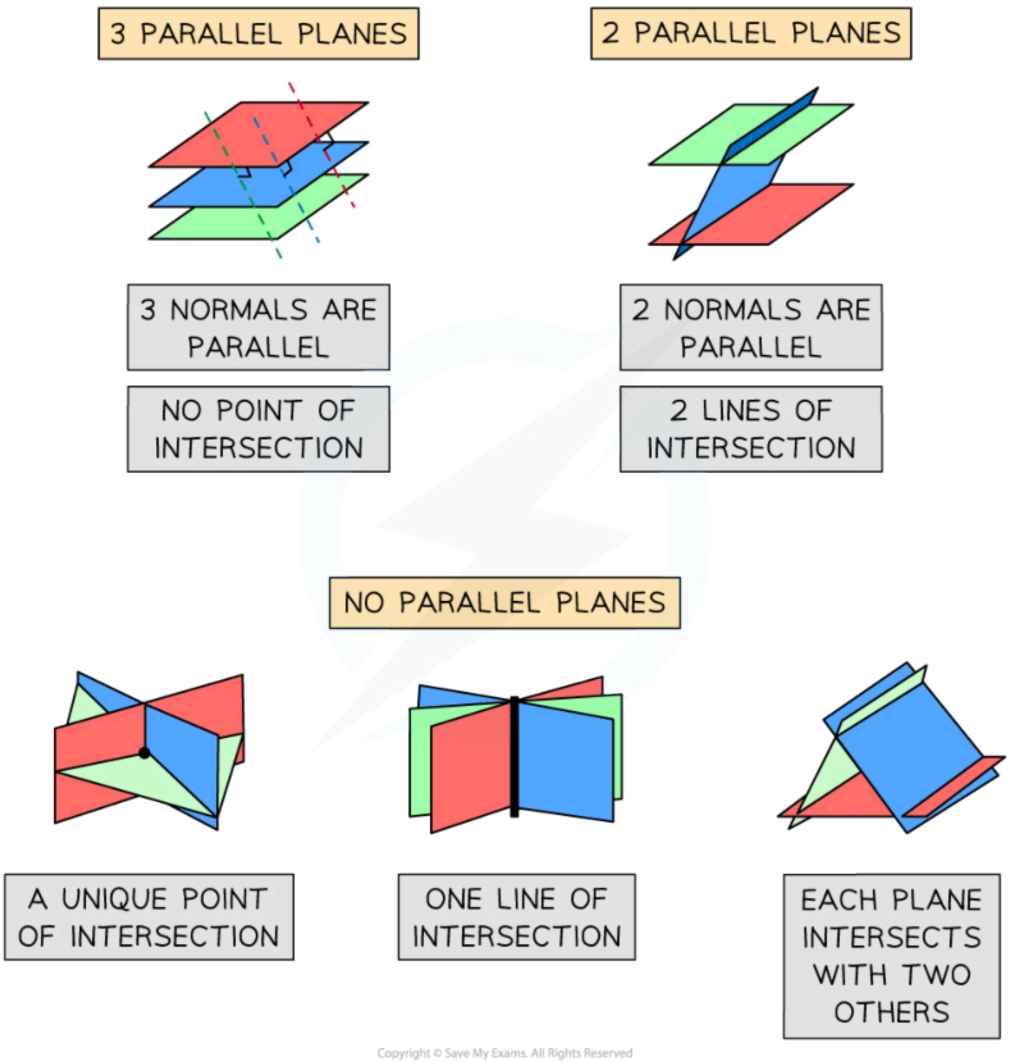
- Two planes will either be **parallel** or they will intersect along a **line**
  - Consider the point where a wall meets a floor or a ceiling
  - You will need to find the **equation of the line** of intersection
- If you have the Cartesian forms of the two planes then the equation of the line of intersection can be found by solving the two equations simultaneously
  - As the solution is a vector equation of a line rather than a unique point you will see below how the equation of the line can be found by part solving the equations
  - For example:
    - $2x - y + 3z = 7$ (1)
    - $x - 3y + 4z = 11$ (2)
- STEP 1: Choose one variable and substitute this variable for λ in both equations
  - For example, letting $x = \lambda$ gives:
    - $2\lambda - y + 3z = 7$ (1)
    - $\lambda - 3y + 4z = 11$ (2)
- STEP 2: Rearrange the two equations to bring λ to one side
  - Equations (1) and (2) become
    - $y - 3z = 2\lambda - 7$ (1)
    - $3y - 4z = \lambda - 11$ (2)
- STEP 3: Solve the equations simultaneously to find the two variables in terms of λ
  - 3(1) – (2) Gives
    - $z = 2 - \lambda$
  - Substituting this into (1) gives
    - $y = -1 - \lambda$
- STEP 4: Write the three parametric equations for *x*, *y*, and *z* in terms of λ and convert into the vector equation of a line in the form $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix}$
  - The parametric equations
    - $x = \lambda$
    - $y = -1 - \lambda$
    - $z = 2 - \lambda$
  - Become
    - $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

Your notes

- If you have fractions in your direction vector you can change its magnitude by multiplying each one by their common denominator
  - The magnitude of the direction vector can be changed without changing the equation of a line
- An alternative method is to find two points on both planes by setting either *x*, *y*, or *z* to zero and solving the system of equations using your GDC or row reduction
  - Repeat this twice to get two points on both planes
  - These two points can then be used to find the vector equation of the line between them
  - This will be the line of intersection of the planes
  - This method relies on the line of intersection having points where the chosen variables are equal to zero

How do we find the relationship between three planes?

- Three planes could either be **parallel**, intersect at one **point**, or intersect along a **line**
- If the three planes have a **unique point of intersection** this point can be found by using your GDC (or row reduction) to solve the three equations in their Cartesian form
  - Make sure you know how to use your GDC to solve a **system of linear equations**
  - Enter all three equations in for the three variables *x*, *y*, and *z*
  - Your GDC will give you the unique solution which will be the coordinates of the point of intersection
- If the three planes do not intersect at a unique point you will not be able to use your GDC to solve the equations
  - If there are no solutions to the system of Cartesian equations then there is no unique point of intersection
- If the three planes are all **parallel** their **normal vectors** will be parallel to each other
  - Show that the normal vectors all have equivalent **direction vectors**
  - These direction vectors may be **scalar multiples** of each other
- If the three planes have **no point of intersection** and are **not all parallel** they may have a relationship such as:
  - Each plane intersects two other planes such that they form a **prism** (none are parallel)
  - Two planes are parallel with the third plane intersecting each of them
  - Check the normal vectors to see if any two of the planes are parallel to decide which relationship they have
- If the three planes intersect along a line there will not be a unique solution to the three equations but there will be a **vector equation of a line** that will satisfy the three equations
- The system of equations will need to be solved by **elimination** or **row reduction**
  - Choose one variable to substitute for λ
  - Solve two of the equations simultaneously to find the other two variables in terms of λ
  - Write *x*, *y*, and *z* in terms of λ in the parametric form of the equation of the line and convert into the vector form of the equation of a line



Examiner Tip

- In an exam you may need to decide the relationship between three planes by using row reduction to determine the number of solutions
  - Make sure you are confident using row reduction to solve systems of linear equations
  - Make sure you remember the different forms three planes can take

Your notes

Worked example

Two planes Π_1 and Π_2 are defined by the equations:

$$\Pi_1 : 3x + 4y + 2z = 7$$

$$\Pi_2 : x - 2y + 3z = 5$$

Find the vector equation of the line of intersection of the two planes.

STEP 1: Let $z = \lambda$, then $3x + 4y + 2\lambda = 7$ ①

$x - 2y + 3\lambda = 5$ ②

You can substitute any variable here, look at the equations to see which is easiest.

STEP 2: ① : $3x + 4y = 7 - 2\lambda$

② : $x - 2y = 5 - 3\lambda$

Write the two equations as simultaneous equations for the two remaining constants.

STEP 3: Find x and y in terms of λ:

① − 2②: $(3x + 4y = 7 - 2\lambda)$

$+ (2x - 4y = 10 - 6\lambda)$

$5x = 17 - 8\lambda$

$x = \frac{17}{5} - \frac{8\lambda}{5}$

sub into ② $\frac{17}{5} - \frac{8\lambda}{5} - 2y + 3\lambda = 5$

$y = \frac{7\lambda}{10} - \frac{8}{10}$

STEP 4: $x = \frac{17}{5} - \frac{8\lambda}{5}$, $y = \frac{7\lambda}{10} - \frac{4}{5}$, $z = \lambda$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{17}{5} \\ -\frac{4}{5} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -\frac{8}{5} \\ \frac{7}{10} \\ 1 \end{pmatrix}$$

The components of the direction vector can be multiplied by a scalar without changing the direction.

$$\mathbf{r} = \begin{pmatrix} 17/5 \\ -4/5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 16 \\ 7 \\ 10 \end{pmatrix}$$

Your notes

3.11.3 Angles Between Lines & Planes

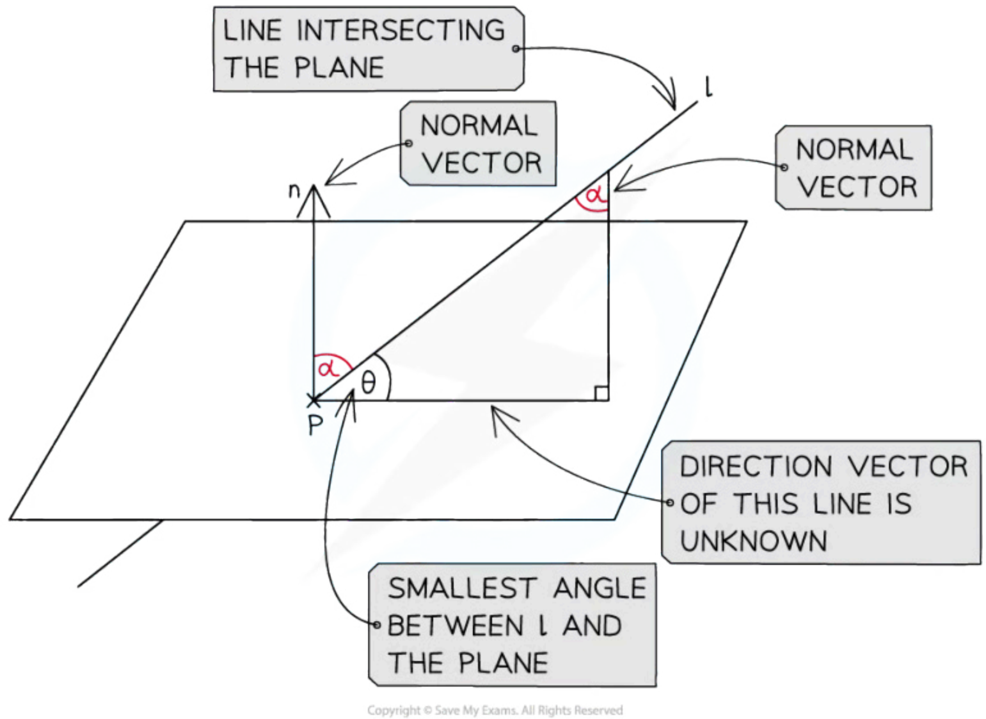
Angle Between Line & Plane

What is meant by the angle between a line and a plane?

- When you find the angle between a line and a plane you will be finding the angle between the line itself and the line on the plane that creates the smallest angle with it
  - This means the line on the plane directly under the line as it joins the plane
- It is easiest to think of these two lines making a right-triangle with the normal vector to the plane
  - The line joining the plane will be the **hypotenuse**
  - The line on the plane will be **adjacent** to the angle
  - The normal will the **opposite** the angle

How do I find the angle between a line and a plane?

- You need to know:
  - A **direction vector** for the **line** (***b***)
    - This can easily be identified if the equation of the line is in the form $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$
  - A **normal vector** to the **plane** (***n***)
    - This can easily be identified if the equation of the plane is in the form $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$
- Find the **acute angle** between the **direction of the line** and the **normal to the plane**
  - Use the formula $\cos\alpha = \dfrac{|\boldsymbol{b} \cdot \boldsymbol{n}|}{|\boldsymbol{b}||\boldsymbol{n}|}$
    - The **absolute value** of the **scalar product** ensures that the angle is **acute**
- **Subtract** this angle from **90°** to find the **acute angle** between the line and the plane
  - Subtract the angle from $\dfrac{\pi}{2}$ if working in **radians**



Your notes

Examiner Tip

- Remember that if the scalar product is negative your answer will result in an obtuse angle
  - Taking the absolute value of the scalar product will ensure that you get the acute angle as your answer

Your notes

Worked example

Find the angle in radians between the line L with vector equation $\boldsymbol{r} = (2-\lambda)\boldsymbol{i} + (\lambda+1)\boldsymbol{j} + (1-2\lambda)\boldsymbol{k}$ and the plane Π with Cartesian equation $x - 3y + 2z = 5$.

Rewrite line equation in standard vector form:

$$r = \begin{pmatrix} 2-\lambda \\ 1+\lambda \\ 1-2\lambda \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$$

↖ direction vector of the line

Find the normal vector of the plane:

$$x - 3y + 2z = 5 \Rightarrow \text{normal vector} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$$

(components of the normal vector)

Find the angle between the direction vector and the normal vector, α:

Angle between two vectors	$\cos\theta = \frac{v_1w_1 + v_2w_2 + v_3w_3}{\lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert}$

$$\cos\alpha = \frac{\left| \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \right|}{\sqrt{(-1)^2 + (1)^2 + (-2)^2} \times \sqrt{1^2 + (-3)^2 + 2^2}} = \frac{\left|(-1)(1) + (1)(-3) + (-2)(2)\right|}{\sqrt{6}\sqrt{14}}$$

$$\theta = \frac{\pi}{2} - \cos^{-1}\alpha$$

$$\theta = \frac{\pi}{2} - \cos^{-1}\left(\frac{|-8|}{\sqrt{6}\sqrt{14}}\right)$$

Using the absolute value ensures we find the acute angle.

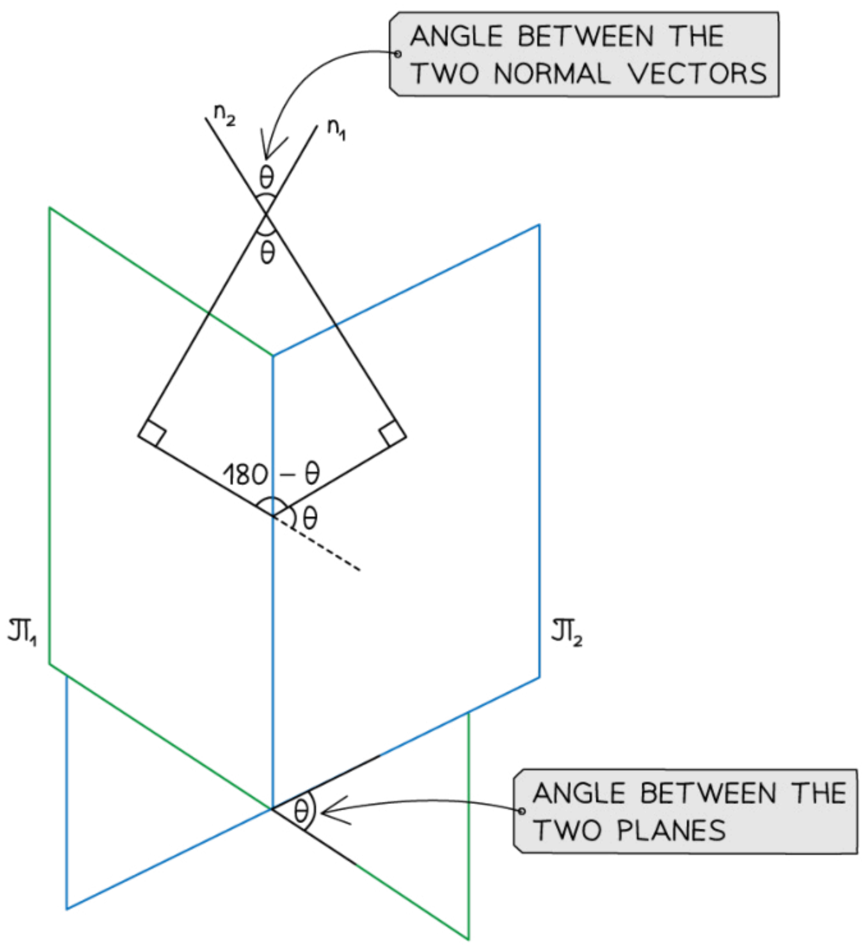
$\theta = 1.06$ radians (3 s.f.)

Your notes

Angle Between Two Planes

How do I find the angle between two planes?

- The angle between two planes is equal to the angle between their **normal vectors**
  - It can be found using the **scalar product** of their normal vectors
  - $\cos\theta = \frac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{|\boldsymbol{n}_1||\boldsymbol{n}_2|}$
- If two planes Π_1 and Π_2 with normal vectors n_1 and n_2 meet at an angle then the two planes and the two normal vectors will form a quadrilateral
  - The angles between the planes and the normal will both be 90°
  - The angle between the two planes and the angle opposite it (between the two normal vectors) will add up to 180°



Your notes

Examiner Tip

- In your exam read the question carefully to see if you need to find the acute or obtuse angle
  - When revising, get into the practice of double checking at the end of a question whether your angle is acute or obtuse and whether this fits the question

Worked example

Find the acute angle between the two planes which can be defined by equations Π_1: $2x - y + 3z = 7$ and Π_2: $x + 2y - z = 20$.

Find the normal vectors of each of the planes:

Π_1 : $2x - y + 3z = 7 \Rightarrow$ normal vector, $n_1 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

Π_2 : $x + 2y - z = 20 \Rightarrow$ normal vector, $n_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

Find the angle between the two normal vectors:

Angle between two vectors	$\cos\theta = \dfrac{v_1w_1 + v_2w_2 + v_3w_3}{\lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert}$

$$\cos\theta = \frac{n_1 \cdot n_2}{|n_1||n_2|} = \frac{|(2)(1) + (-1)(2) + (3)(-1)|}{\sqrt{2^2 + (-1)^2 + 3^2} \times \sqrt{1^2 + 2^2 + (-1)^2}} = \frac{|-3|}{\sqrt{14} \times \sqrt{6}}$$

Using the absolute value ensures we find the acute angle.

$$\theta = \cos^{-1}\left(\frac{3}{2\sqrt{21}}\right)$$

$$\theta = 1.24 \text{ radians } (3 \text{ s.f.})$$

Your notes

3.11.4 Shortest Distances with Planes

Shortest Distance Between a Line and a Plane

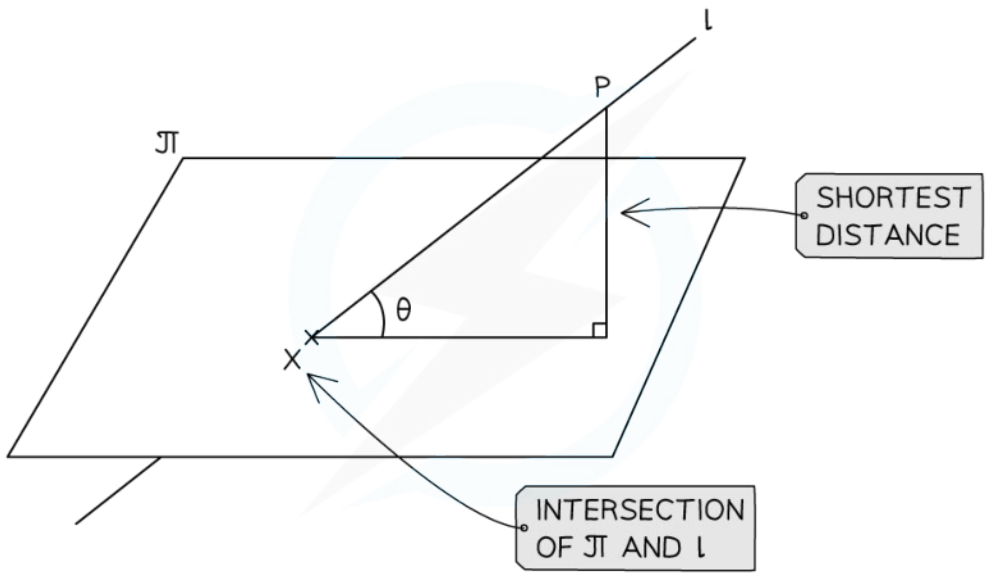
How do I find the shortest distance between a point and a plane?

- The shortest distance from any point to a plane will always be the **perpendicular** distance from the point to the plane
- Given a point, P with position vector **p** and a plane Π with equation $\mathbf{r}\cdot\mathbf{n} = d$
  - STEP 1: Find the **vector equation of the line** perpendicular to the plane that goes through the point, *P*
    - This will have the position vector of the point, *P*, and the direction vector **n**
    - $\mathbf{r} = \mathbf{p} + \lambda\mathbf{n}$
  - STEP 2: Find the value of λ at the **point of intersection** of this line with Π by substituting the equation of the line into the equation of the plane
  - STEP 3: Find the **distance** between the point and the point of intersection
    - Substitute λ into the equation of the line to find the coordinates of the point on the plane closest to point P
    - Find the distance between this point and point P
    - As a shortcut, this distance will be equal to $|\lambda\mathbf{n}|$

How do I find the shortest distance between a given point on a line and a plane?

- The shortest distance from any point on a line to a plane will always be the **perpendicular** distance from the point to the plane
- You can follow the same **steps above**
- A question may provide the acute angle between the line and the plane
  - Use right-angled trigonometry to find the perpendicular distance between the point on the line and the plane
    - Drawing a clear diagram will help

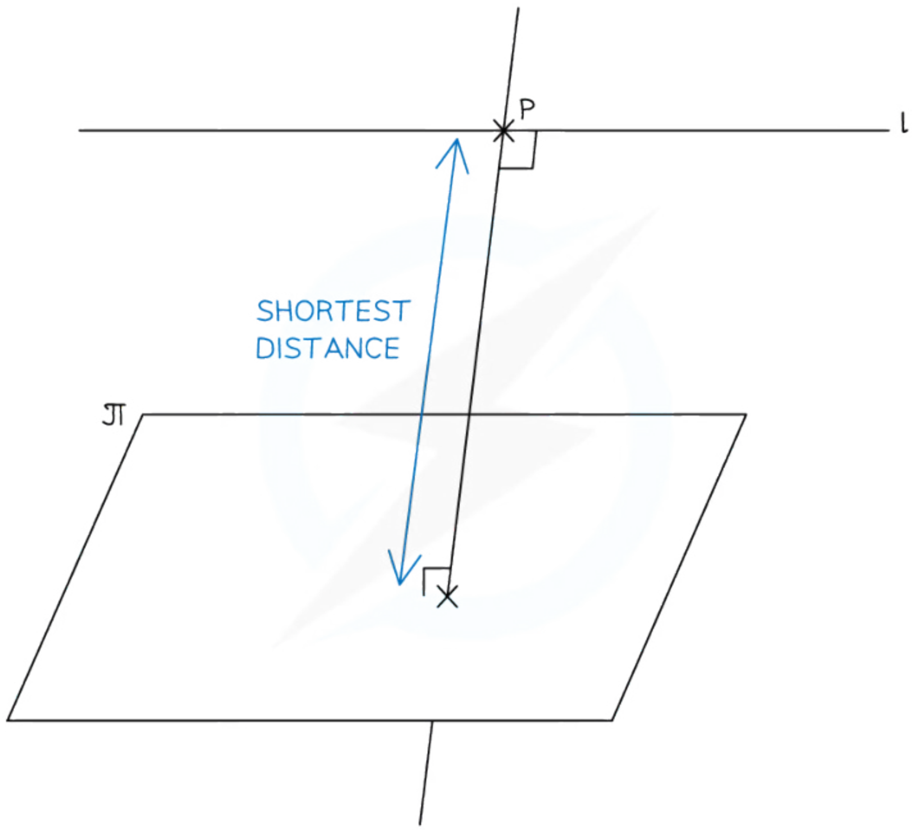
Your notes



How do I find the shortest distance between a plane and a line parallel to the plane?

- The shortest distance between a line and a plane that are parallel to each other will be the **perpendicular** distance from the line to the plane
- Given a line l_1 with equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and a plane Π parallel to l_1 with equation $\mathbf{r}\cdot\mathbf{n} = d$
  - Where **n** is the **normal vector** to the plane
  - STEP 1: Find the equation of the line l_2 perpendicular to l_1 and Π going through the point **a** in the form $\mathbf{r} = \mathbf{a} + \mu\mathbf{n}$
  - STEP 2: Find the point of intersection of the line l_2 and Π
  - STEP 3: Find the distance between the point of intersection and the point,

Your notes



Examiner Tip

- Vector planes questions can be tricky to visualise, read the question carefully and sketch a very simple diagram to help you get started

Your notes

Worked example

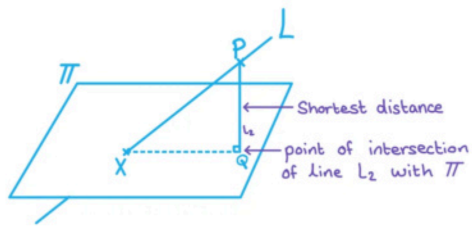
The plane Π has equation $\mathbf{r}\cdot\begin{pmatrix}2\\-1\\1\end{pmatrix}=6$.

The line L has equation $\mathbf{r}=\begin{pmatrix}2\\3\\1\end{pmatrix}+s\begin{pmatrix}1\\-2\\4\end{pmatrix}$.

The point $P\ (-2,\ 11,\ -15)$ lies on the line L.

Find the shortest distance between the point P and the plane Π.

Your notes



STEP 1: Use the given point, P and the known normal to the plane, n to write an equation for the line perpendicular to Π, L_2.

$$\mathbf{r} = \begin{pmatrix} -2 \\ 11 \\ -15 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

P ↗ ↖ n

STEP 2: Find the point of intersection, Q, of the new line, L_2, with Π.

$$\left(\begin{pmatrix} -2 \\ 11 \\ -15 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\right) \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 6$$

$$2(-2+2\lambda) - (11-\lambda) + (\lambda - 15) = 6$$

$$-4 + 4\lambda - 11 + \lambda + \lambda - 15 = 6$$

$$6\lambda - 30 = 6$$

$$\lambda = 6 \Rightarrow \overrightarrow{OQ} = \begin{pmatrix} -2 \\ 11 \\ -15 \end{pmatrix} + 6\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \\ -9 \end{pmatrix}$$

STEP 3: Find the distance between P and Q.

$$|\overrightarrow{PQ}| = \sqrt{(10--2)^2 + (5-11)^2 + (-9--15)^2} = 6\sqrt{6} \text{ units}$$

Shortest distance $= 6\sqrt{6}$ units

Your notes

Shortest Distance Between Two Planes

How do I find the shortest distance between two parallel planes?

- Two **parallel** planes will never intersect
- The shortest distance between two **parallel planes** will be the **perpendicular distance** between them
- Given a plane Π_1 with equation $\mathbf{r} \cdot \mathbf{n} = d$ and a plane Π_2 with equation $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b} + \mu \mathbf{c}$ then the shortest distance between them can be found
  - STEP 1: The equation of the line perpendicular to both planes and through the point **a** can be written in the form **r** = **a** + s**n**
  - STEP 2: Substitute the equation of the line into $\mathbf{r} \cdot \mathbf{n} = d$ to find the coordinates of the point where the line meets Π_1
  - STEP 3: Find the distance between the two points of intersection of the line with the two planes

How do I find the shortest distance from a given point on a plane to another plane?

- The shortest distance from any point, *P* on a plane, Π_1, to another plane, Π_2 will be the **perpendicular** distance from the point to Π_2
  - STEP 1: Use the given coordinates of the point *P* on Π_1 and the normal to the plane Π_2 to find the vector equation of the line through *P* that is perpendicular to Π_1
  - STEP 2: Find the point of intersection of this line with the plane Π_2
  - STEP 3: Find the distance between the two points of intersection

Examiner Tip

- There are a lot of steps when answering these questions so set your methods out clearly in the exam

Your notes

Worked example

Consider the parallel planes defined by the equations:

$$\Pi_1 : \mathbf{r} \cdot \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = 44,$$

$$\Pi_2 : \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Find the shortest distance between the two planes Π_1 and Π_2.

Your notes

Find the equation of the line perpendicular to the planes through the point (0,0,3)

$$L: \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + s\begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}$$

position vector of Π_2 ; Normal vector of Π_1

Substitute the equation of L into the equation of Π_1:

$$\begin{pmatrix} 3s \\ -5s \\ 3+2s \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = 44$$

$$3(3s) - 5(-5s) + 2(3+2s) = 44$$

$$38s + 6 = 44$$

$$s = 1$$

Substitute $s = 1$ back into the equation of L:

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ 5 \end{pmatrix}$$

Find the distance between (0,0,3) and (3,−5,5)

$$d = \sqrt{3^2 + (-5)^2 + (5-3)^2}$$

$$= \sqrt{38}$$

Shortest distance $= \sqrt{38}$ units