DP IB Maths: AI HL

Your notes

3.7 Vector Properties

Contents

3.7.1 Introduction to Vectors

Your notes

Scalars & Vectors

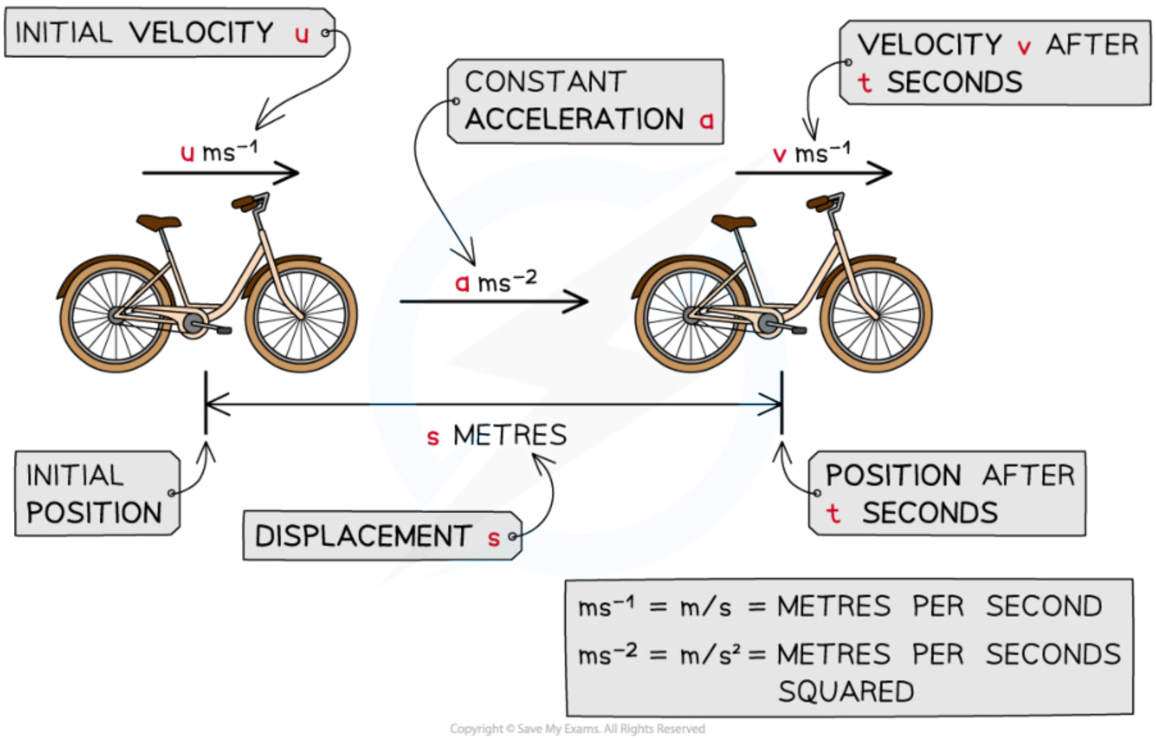
What are scalars?

- **Scalars** are quantities without **direction**
  - They have only a **size (magnitude)**
  - For example: **speed, distance, time, mass**
- **Most scalar quantities** can never be **negative**
  - You cannot have a negative speed or distance

What are vectors?

- **Vectors** are quantities which also have a **direction**, this is what makes them more than just a scalar
  - For example: two objects with **velocities** of 7 m/s and -7 m/s are travelling at the **same speed** but in **opposite directions**
- A **vector quantity** is described by both its **magnitude** and **direction**
- A vector has **components** in the direction of the *x*- , *y*-, and *z*- axes
  - Vector quantities can have **positive** or **negative** components
- Some examples of vector quantities you may come across are **displacement**, **velocity**, **acceleration**, **force/weight**, **momentum**
  - **Displacement** is the position of an object from a starting point
  - **Velocity** is a speed in a given direction (displacement over time)
  - **Acceleration** is the change in velocity over time
- Vectors may be given in either 2– or 3– dimensions

Your notes



Examiner Tip

- Make sure you fully understand the definitions of all the words in this section so that you can be clear about what your exam question is asking of you

Your notes

Worked example

State whether each of the following is a scalar or a vector quantity.

a) A speed boat travels at 3 m/s on a bearing of 052°

Speed with a given direction → velocity

Vector

b) A garden is 1.7 m wide

Length with no direction

Scalar

c) A car accelerates forwards at 5.4 ms^{-2}

Acceleration has direction

Vector

d) A film lasts 2 hours 17 minutes

Time has no direction

Scalar

e) An athlete runs at an average speed of 10.44 ms^{-1}

Speed with no direction is a scalar

Scalar

f) A ball rolls forwards 60 cm before stopping

Displacement has direction

Vector

Your notes

Your notes

Vector Notation

How are vectors represented?

- **Vectors** are usually represented using an arrow in the direction of movement
  - The length of the arrow represents its magnitude
- They are written as lowercase letters either in **bold** or <u>underlined</u>
  - For example a vector from the point O to A will be written **a** or <u>*a*</u>
    - The vector from the point A to O will be written **–a** or -<u>*a*</u>
- If the start and end point of the vector is known, it is written using these points as capital letters with an arrow showing the direction of movement
  - For example: $\overrightarrow{AB}$ or $\overrightarrow{BA}$
- Two vectors are equal only if their corresponding components are equal
- Numerically, vectors are either represented using **column vectors** or **base vectors**
  - Unless otherwise indicated, you may carry out all working and write your answers in either of these two types of vector notation

What are column vectors?

- **Column vectors** are where one number is written above the other enclosed in brackets
- In 2-dimensions the **top number** represents movement in the **horizontal direction** (right/left) and the **bottom number** represents movement in the **vertical direction** (up/down)
- A **positive value** represents movement in the **positive direction** (right/up) and a **negative value** represents movement in the **negative direction** (left/down)
  - For example: The column vector $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ represents **3 units** in the **positive horizontal** (*x*) direction (i.e., **right**) and **2 units** in the **negative vertical** (*y*) direction (i.e., **down**)
- In 3-dimensions the **top number** represents the movement in the ***x* direction** (length), the **middle number** represents movement in the ***y* direction** (width) and the **bottom number** represents the movement in the ***z* direction** (depth)
  - For example: The column vector $\begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix}$ represents **3 units** in the **positive *x* direction**, **4 units** in the **negative *y* direction** and **2 units** in the **positive *z* direction**

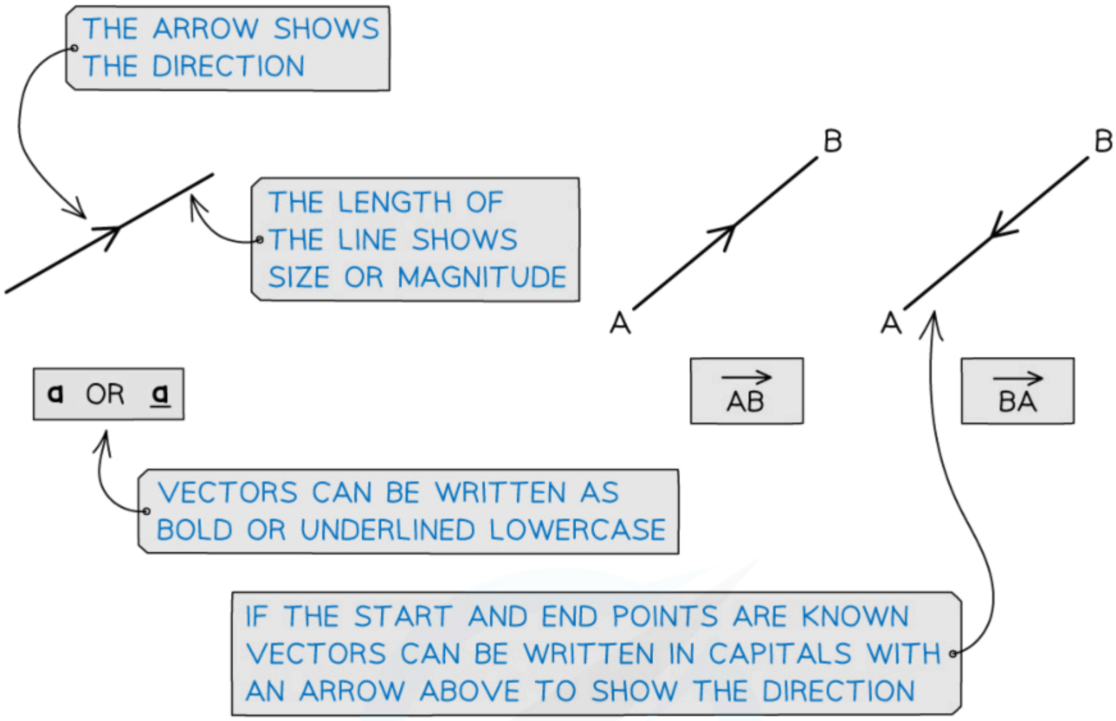
What are base vectors?

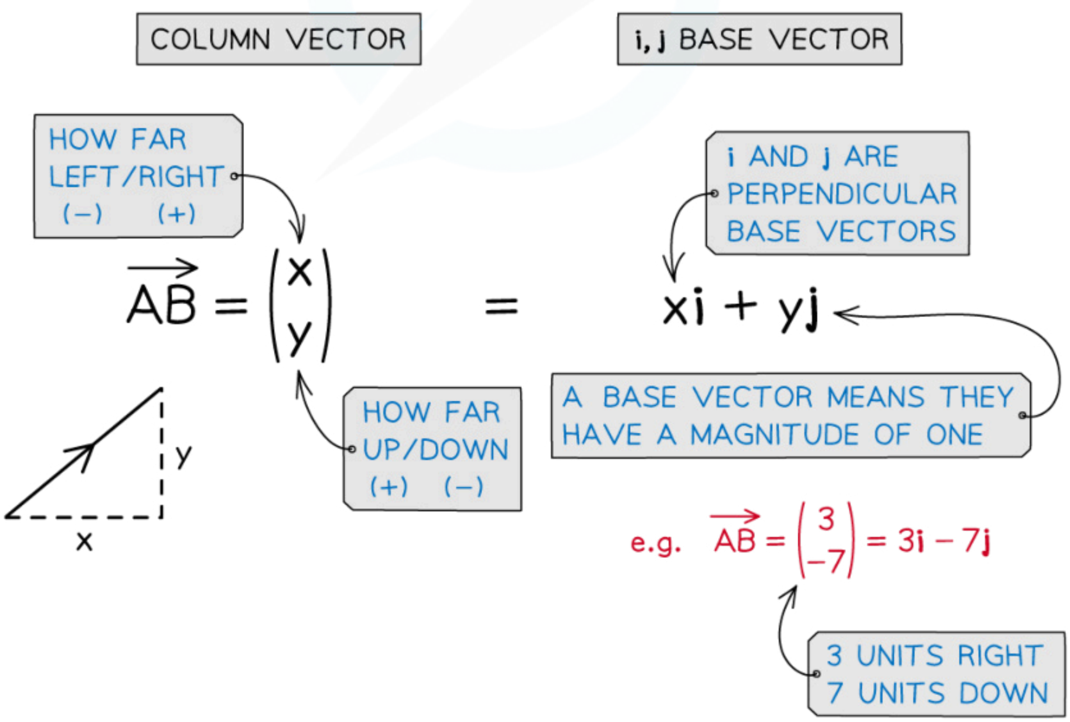
- **Base vectors use i**, **j** and **k** notation **where i**, **j** and **k** are **unit vectors** in the positive *x*, *y*, and *z* directions respectively
  - This is sometimes also called **unit vector notation**
  - A unit vector has a **magnitude of 1**
- In 2-dimensions i represents movement in the horizontal direction (right/left) and j represents the movement in the vertical direction (up/down)

Your notes

- **For example:** The vector (-4**i** + 3**j**) would mean **4 units** in the **negative horizontal** (*x*) direction (i.e., **left**) and **3 units** in the **positive vertical** (*y*) direction (i.e., **up**)
- In 3-dimensions i represents movement in the *x* direction (length), j represents movement in the *y* direction (width) and k represents the movement in the *z* direction (depth)
  - **For example:** The vector (-4**i** + 3**j** – **k**) would mean **4 units** in the **negative *x* direction**, **3 units** in the **positive *y* direction** and **1 unit** in the **negative *z* direction**
- As they are vectors, **i, j and k** are displayed in **bold** in textbooks and online but in handwriting they would be underlined (i, j and k)

Your notes


THE ARROW SHOWS THE DIRECTION
THE LENGTH OF THE LINE SHOWS SIZE OR MAGNITUDE
B
A
B
A
a OR a
AB
BA
VECTORS CAN BE WRITTEN AS BOLD OR UNDERLINED LOWERCASE
IF THE START AND END POINTS ARE KNOWN VECTORS CAN BE WRITTEN IN CAPITALS WITH AN ARROW ABOVE TO SHOW THE DIRECTION


COLUMN VECTOR
i, j BASE VECTOR
HOW FAR LEFT/RIGHT (−) (+)
i AND j ARE PERPENDICULAR BASE VECTORS
AB = (x y) = xi + yj
y
x
HOW FAR UP/DOWN (+) (−)
A BASE VECTOR MEANS THEY HAVE A MAGNITUDE OF ONE
e.g. AB = (3 −7) = 3i − 7j
3 UNITS RIGHT 7 UNITS DOWN

Your notes

Examiner Tip

- Practice working with all types of vector notation so that you are prepared for whatever comes up in the exam
  - Your working and answer in the exam can be in any form unless told otherwise
  - It is generally best to write your final answer in the same form as given in the question, however you will not lose marks for not doing this unless it is specified in the question
- Vectors appear in **bold** (non-italic) font in textbooks and on exam papers, etc (i.e. ***F***, ***a***) but in handwriting should be underlined (i.e. $\underline{F}$, $\underline{a}$)

Your notes

Worked example

a) Write the vector $\begin{pmatrix} -4 \\ 0 \\ 5 \end{pmatrix}$ using **base vector** notation.

$$\begin{pmatrix} -4 \\ 0 \\ 5 \end{pmatrix} = -4\underline{i} + 0\underline{j} + 5\underline{k}$$

$0\underline{j}$ is not needed when giving answer in base vector form.

$$5\underline{k} - 4\underline{i}$$

b) Write the vector $\mathbf{k} - 2\mathbf{j}$ using **column vector** notation.

$$\underline{k} - 2\underline{j} = 0\underline{i} - 2\underline{j} + 1\underline{k}$$

Be careful with negative components and missing terms when working with base vectors

$$\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

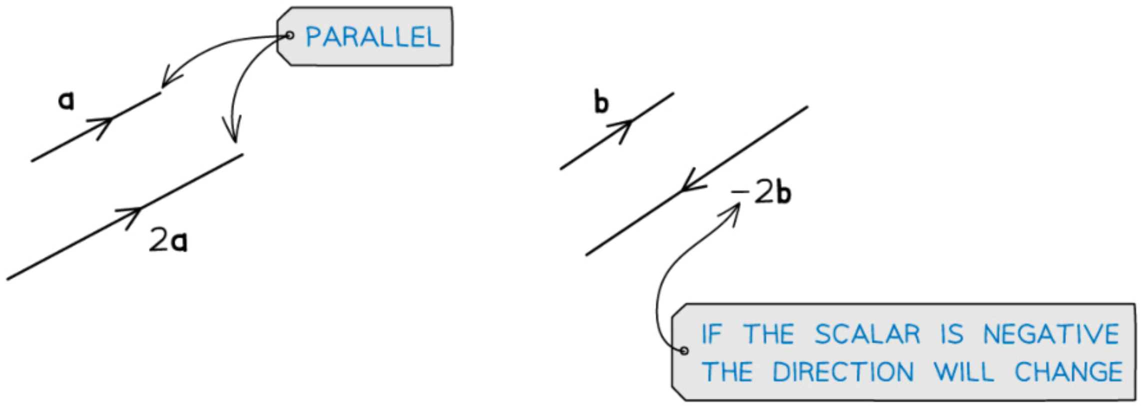
The zero term is needed when using column vector notation

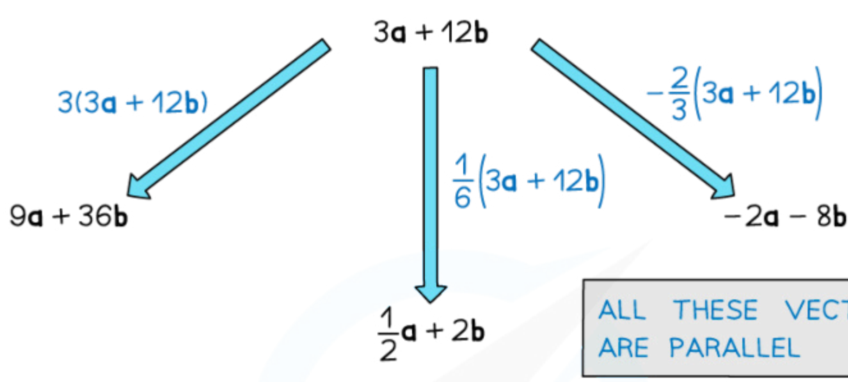
Your notes

Parallel Vectors

How do you know if two vectors are parallel?

- Two vectors are parallel if one is a **scalar multiple** of the other
  - This means that all components of the vector have been multiplied by a **common constant (scalar)**
- Multiplying every component in a vector by a **scalar** will change the **magnitude** of the vector but not the **direction**
  - For example: the vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ and $\mathbf{b} = 2\mathbf{a} = 2\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix}$ will have the **same direction** but the vector **b** will have twice the magnitude of **a**
    - They are **parallel**
- If a vector can be factorised by a **scalar** then it is parallel to any **scalar multiple** of the factorised vector
  - For example: The vector $9\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$ can be factorised by the scalar 3 to $3(3\mathbf{i} + 2\mathbf{j} - \mathbf{k})$ so the vector $9\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$ is parallel to any **scalar multiple** of $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
- If a vector is multiplied by a **negative scalar** its direction will be **reversed**
  - It will still be **parallel** to the original vector
- Two vectors are **parallel** if they have the same or reverse **direction** and **equal** if they have the same **size and direction**





$\mathbf{a} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ $\quad 2\mathbf{a} = 2 \times \begin{pmatrix} 3 \\ -5 \end{pmatrix} = \begin{pmatrix} 6 \\ -10 \end{pmatrix}$

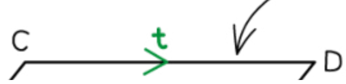
FOR COLUMN VECTORS JUST MULTIPLY TOP AND BOTTOM NUMBERS BY THE SCALAR

$\mathbf{a} = 3\mathbf{i} + 7\mathbf{j}$

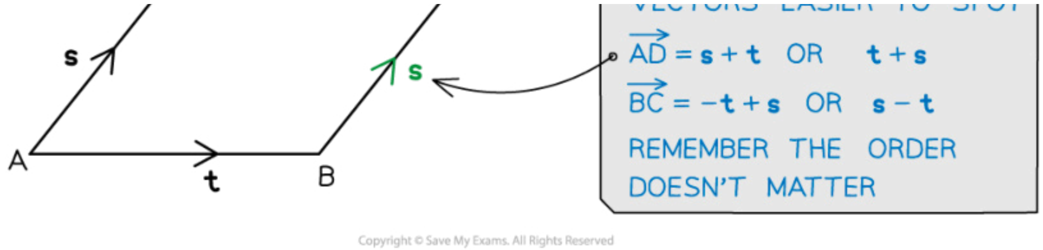
$3\mathbf{a} = 3(3\mathbf{i} + 7\mathbf{j}) = 9\mathbf{i} + 21\mathbf{j}$

FOR **i j** VECTORS JUST MULTIPLY BOTH NUMBERS BY THE SCALAR

ABCD IS A PARALLELOGRAM MEANING $\overrightarrow{AB} = \overrightarrow{CD}$ AND $\overrightarrow{AC} = \overrightarrow{BD}$



LABEL PARALLEL SIDES ON DIAGRAMS, TO MAKE NEW VECTORS EASIER TO SPOT



Your notes

Examiner Tip

- It is easiest to spot that two vectors are parallel when they are in column vector notation
  - in your exam by writing vectors in column vector form and looking for a scalar multiple you will be able to quickly determine whether they are parallel or not

Worked example

Show that the vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$ and $\mathbf{b} = 6\mathbf{k} - 3\mathbf{i}$ are parallel and find the scalar multiple that maps **a** onto **b**.

Convert both vectors into the same form and then look for a value of k such that $\underline{a} = k\underline{b}$, where k is a scalar.

$\underline{a} = \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$

$\underline{b} = 6\underline{k} - 3\underline{i} = -3\underline{i} + 0\underline{j} + 6\underline{k}$

$= \begin{pmatrix} -3 \\ 0 \\ 6 \end{pmatrix} = -\frac{3}{2}\begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$

$= -\frac{3}{2}\underline{a}$

$\underline{b} = -\frac{3}{2}\underline{a}$, $k = -\frac{3}{2}$

Your notes

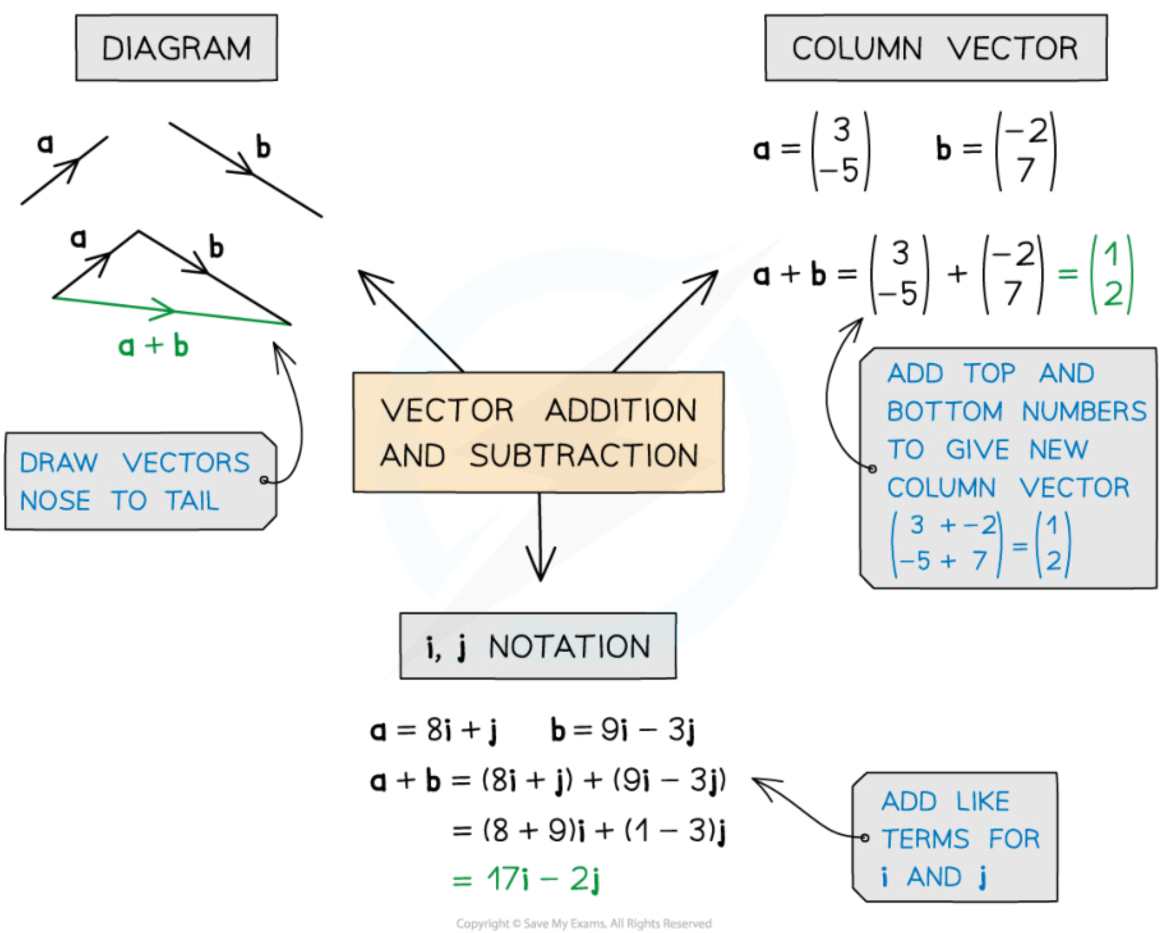
3.7.2 Position & Displacement Vectors

Your notes

Adding & Subtracting Vectors

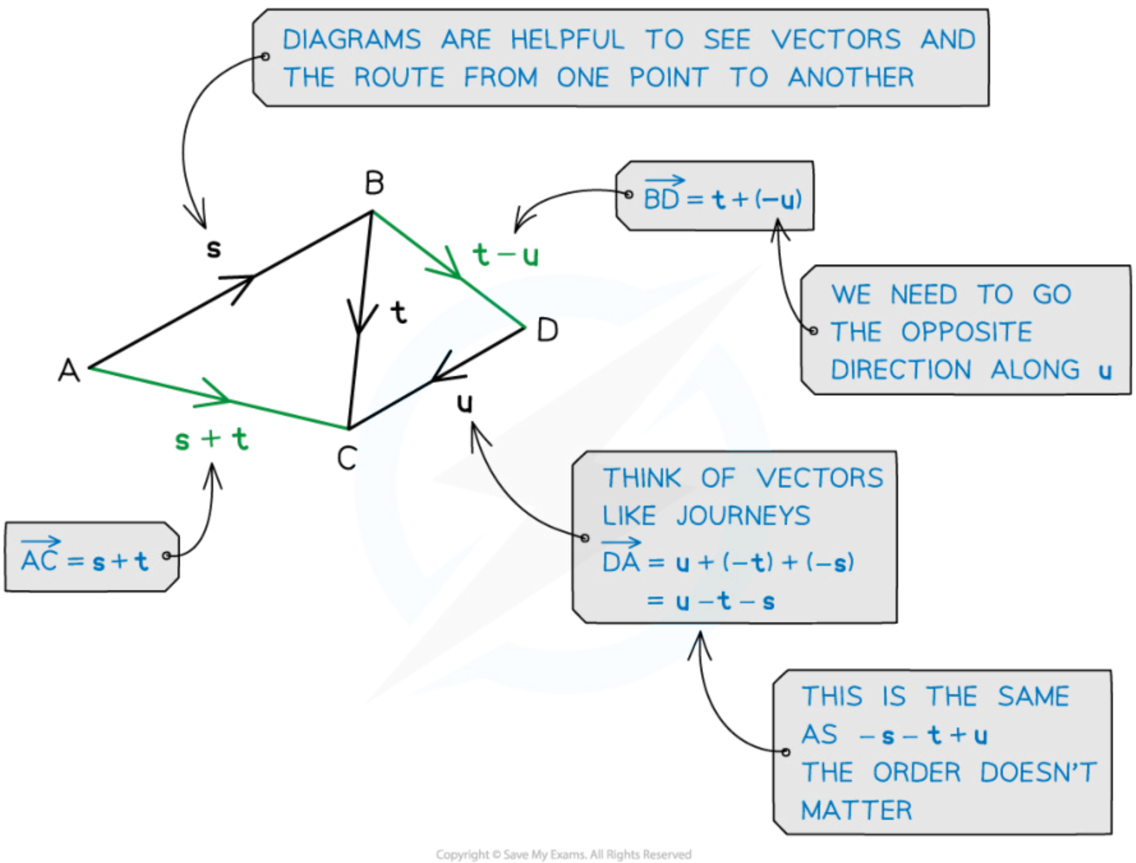
How are vectors added and subtracted numerically?

- To **add** or **subtract** vectors numerically simply add or subtract each of the corresponding components
- In **column vector** notation just add the top, middle and bottom parts together
  - For example: $\begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -8 \end{pmatrix}$
- In **base vector** notation add each of the **i**, **j**, and **k** components together separately
  - For example: (2**i** + **j** – 5**k**) – (**i** + 4**j** + 3**k**) = (**i** – 3**j** – 8**k**)



How are vectors added and subtracted geometrically?

- Vectors can be **added** geometrically by joining the end of one vector to the start of the next one
- The **resultant** vector will be the shortest route from the start of the first vector to the end of the second
  - A **resultant** vector is a vector that results from **adding** or **subtracting** two or more vectors
- If the two vectors have the same **starting position**, the second vector can be **translated** to the end of the first vector to find the resultant vector
  - This results in a **parallelogram** with the resultant vector as the diagonal
- To **subtract** vectors, consider this as **adding on the negative vector**
  - For example: **a** – **b** = **a** + (–**b**)
  - The end of the **resultant vector a – b** will not be anywhere near the end of the vector **b**
    - Instead, it will be at the point where the end of the vector **–b** would be



Examiner Tip

- Working in column vectors tends to be easiest when adding and subtracting
  - in your exam, it can help to convert any vectors into column vectors before carrying out calculations with them
- If there is no diagram, drawing one can be helpful to help you visualise the problem

Your notes

Your notes

Worked example

Find the resultant of the vectors **a** = 5**i** – 2**j** and **b** = $\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$.

$$\underline{a} = 5\underline{i} - 2\underline{j} + 0\underline{k} = \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$$

Writing as a column vector makes adding and subtracting easier.

$$\underline{a} + \underline{b} = \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

Resultant vector = $2\underline{i} - \underline{j} + 2\underline{k}$

Position Vectors

Your notes

What is a position vector?

- A position vector describes the **position** of a point in relation to the **origin**
  - It describes the **direction** and the **distance** from the point O: 0**i** + 0**j** + 0**k** or $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
  - It is different to a **displacement vector** which describes the direction and distance between any two points
- The position vector of point A is written with the notation **a** = $\overrightarrow{OA}$
  - The origin is always denoted O
- The individual components of a position vector are the coordinates of its end point
  - For example the point with coordinates (3, -2, -1) has position vector 3**i** – 2**j** – **k**

Worked example

Determine the position vector of the point with coordinates (4, -1, 8).

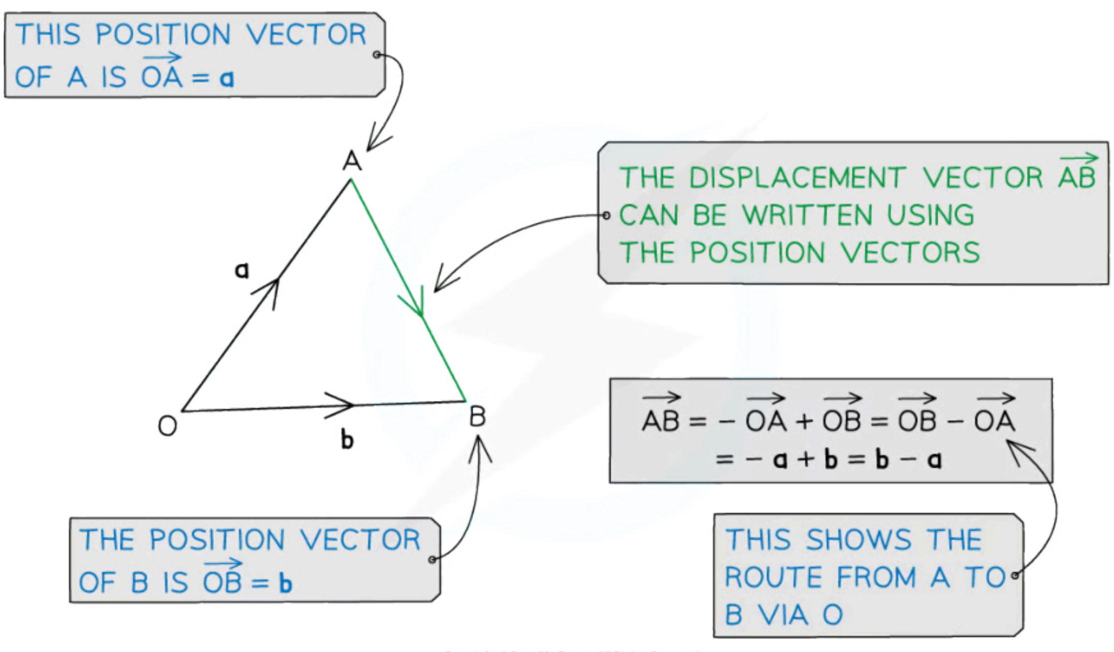
$4\underline{i} - \underline{j} + 8\underline{k}$

Your notes

Displacement Vectors

What is a displacement vector?

- A **displacement vector** describes the shortest route between any two points
  - It describes the **direction** and the **distance** between any two points
  - It is different to a **position vector** which describes the direction and distance from the point O: 0**i** + 0**j** or $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
- The displacement vector of point B from the point A is written with the notation $\overrightarrow{AB}$
- A displacement vector between two points can be written in terms of the displacement vectors of a third point
  - $\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB}$
- A displacement vector can be written in terms of its position vectors
  - For example the displacement vector $\overrightarrow{AB}$ can be written in terms of $\overrightarrow{OA}$ and $\overrightarrow{OB}$
  - $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OB} - \overrightarrow{OA}$
  - For position vector **a** = $\overrightarrow{OA}$ and **b** = $\overrightarrow{OB}$ the displacement vector $\overrightarrow{AB}$ can be written **b** – **a**



Examiner Tip

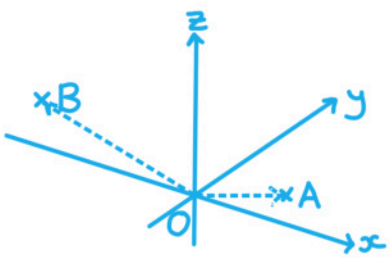
- In an exam, sketching a quick diagram can help to make working out a displacement vector easier

Your notes

Worked example

The point A has coordinates (3, 0, -1) and the point B has coordinates (-2, -5, 7). Find the displacement vector $\overrightarrow{AB}$.

$$\overrightarrow{OA} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \qquad \overrightarrow{OB} = \begin{pmatrix} -2 \\ -5 \\ 7 \end{pmatrix}$$



$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$$

$$= -\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \begin{pmatrix} -2 \\ -5 \\ 7 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$$

$$\overrightarrow{AB} = \begin{pmatrix} -5 \\ -5 \\ 8 \end{pmatrix}$$

3.7.3 Magnitude of a Vector

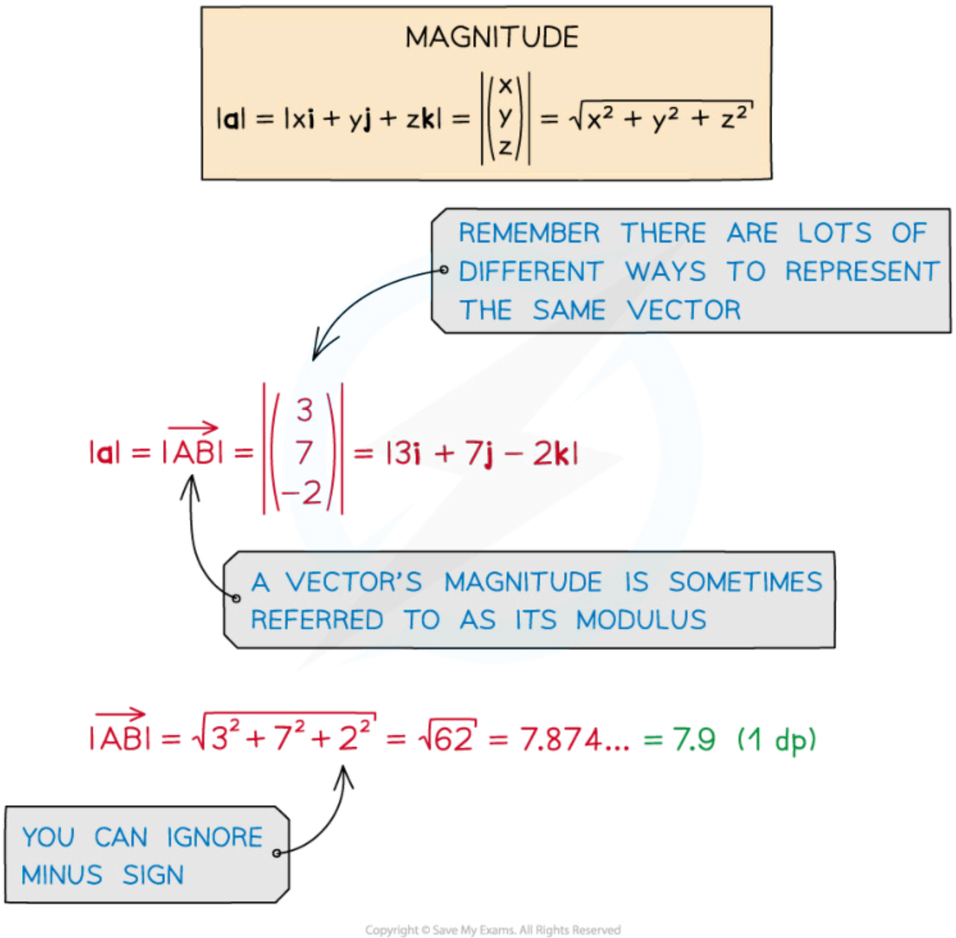
Your notes

Magnitude of a Vector

How do you find the magnitude of a vector?

- The **magnitude** of a vector tells us its **size** or **length**
  - For a **displacement** vector it tells us the **distance** between the two points
  - For a **position** vector it tells us the **distance** of the point from the **origin**
- The magnitude of the vector $\overrightarrow{AB}$ is denoted $\left|\overrightarrow{AB}\right|$
  - The magnitude of the vector **a** is denoted |**a**|
- The magnitude of a vector can be found using **Pythagoras' Theroem**
- The magnitude of a vector $\boldsymbol{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is found using
  - $|\boldsymbol{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$
  - where $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$
  - This is **given in the formula booklet**

Your notes



How do I find the distance between two points?

- **Vectors** can be used to find the distance (or displacement) between two points
  - It is the **magnitude** of the vector between them
- Given the **position vectors** of two points:
  - Find the displacement vector between them
  - Find the magnitude of the displacement vector between them

Examiner Tip

- Finding the magnitude of a vector is the same as finding the distance between two coordinates, it is a useful formula to commit to memory in order to save time in the exam, however it is in your formula booklet if you need it

Worked example

Find the magnitude of the vector AB = 4**i** – **j** + 2**k**.

Magnitude of a vector	$\lvert \boldsymbol{v} \rvert = \sqrt{v_1^2 + v_2^2 + v_3^2}$, where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$$|\overrightarrow{AB}| = \sqrt{4^2 + 1^2 + 2^2} = \sqrt{21}$$

$$|\overrightarrow{AB}| = \sqrt{21}$$

Your notes

Your notes

Unit Vectors

What is a unit vector?

- A **unit vector** has a **magnitude** of 1
- It can be found by dividing a vector by its **magnitude**
  - This will result in a vector with a size of 1 unit in the direction of the original vector
- A unit vector in the direction of **a** is denoted $\frac{\mathbf{a}}{|\mathbf{a}|}$
  - For example a unit vector in the direction 3**i** – 4**j** is $\frac{(3\mathbf{i}-4\mathbf{j})}{\sqrt{3^2+4^2}} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$

Examiner Tip

- Finding the unit vector will not be a question on its own but will be a useful skill for further vectors problems so it is important to be confident with it

Worked example

Find the unit vector in the direction 2**i** – 2**j** + **k**.

Let $\underline{a} = 2\underline{i} - 2\underline{j} + \underline{k}$

Find the magnitude of $\underline{a}$

Magnitude of a vector	$\|v\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$, where $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$|\underline{a}| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3$

Divide $\underline{a}$ by its magnitude:

Unit vector $= \frac{\underline{a}}{|\underline{a}|} = \frac{2\underline{i} - 2\underline{j} + \underline{k}}{3}$

$\frac{2}{3}\underline{i} - \frac{2}{3}\underline{j} + \frac{1}{3}\underline{k}$

Your notes

3.7.4 The Scalar Product

The Scalar ('Dot') Product

What is the scalar product?

- The scalar product (also known as the dot product) is one form in which two vectors can be combined together
- The scalar product between two vectors **a** and **b** is denoted $\mathbf{a} \cdot \mathbf{b}$
- The result of taking the scalar product of two vectors is a **real number**
  - i.e. a scalar
- The scalar product of two vectors gives information about the angle between the two vectors
  - If the scalar product is **positive** then the angle between the two vectors is **acute** (less than 90°)
  - If the scalar product is **negative** then the angle between the two vectors is **obtuse** (between 90° and 180°)
  - If the scalar product is **zero** then the angle between the two vectors is **90°** (the two vectors are **perpendicular**)

How is the scalar product calculated?

- There are **two methods** for calculating the scalar product
- The most common method used to find the scalar product between the two vectors ***v*** and ***w*** is to find the **sum of the product of each component** in the two vectors
  - $\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$
  - Where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$
  - This is **given in the formula booklet**
- The scalar product is also equal to the **product of the magnitudes** of the two vectors and the **cosine of the angle between them**
  - $\boldsymbol{v} \cdot \boldsymbol{w} = |v||w|\cos\theta$
  - Where θ is the angle between ***v*** and ***w***
    - The two vectors ***v*** and ***w*** are joined at the start and pointing away from each other
- The scalar product can be used in the second formula to find the angle between the two vectors

What properties of the scalar product do I need to know?

- If two vectors, ***v*** and ***w***, are **parallel** then the magnitude of the scalar product is equal to the **product** of the magnitudes of the vectors
  - $|\boldsymbol{v} \cdot \boldsymbol{w}| = |\boldsymbol{w}||\boldsymbol{v}|$
  - This is because cos 0° = 1 and cos 180° = -1
- If two vectors are **perpendicular** the scalar product is **zero**
  - This is because cos 90° = 0

Examiner Tip

- Whilst the formulae for the scalar product are given in the formula booklet, the properties of the scalar product are not, however they are important and it is likely that you will need to recall them in your exam so be sure to commit them to memory

Your notes

Your notes

Worked example

Calculate the scalar product between the two vectors $\boldsymbol{v} = \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix}$ and $\boldsymbol{w} = 3\mathbf{j} - 2\mathbf{k} - \mathbf{i}$ using:

i) the formula $\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$,

$\underline{v} = \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix} = 2\underline{i} + 0\underline{j} - 5\underline{k}$

$\underline{w} = 3\underline{j} - 2\underline{k} - \underline{i} = -1\underline{i} + 3\underline{j} - 2\underline{k}$

Be aware of the order of the terms.

Scalar product	$\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$, where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$

$\underline{v} \cdot \underline{w} = (2 \times -1) + (0 \times 3) + (-5 \times -2) = -2 + 10$

$\underline{v} \cdot \underline{w} = 8$

ii) the formula $\boldsymbol{v} \cdot \boldsymbol{w} = |v||w|\cos\theta$, given that the angle between the two vectors is 66.6°.

$\underline{v} = \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix} = 2\underline{i} + 0\underline{j} - 5\underline{k}$ $\quad \underline{w} = -1\underline{i} + 3\underline{j} - 2\underline{k}$

Scalar product	$\boldsymbol{v} \cdot \boldsymbol{w} = \lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert \cos\theta$

Find the magnitude of both vectors:

$|\underline{v}| = \sqrt{2^2 + (-5)^2} = \sqrt{29}$ $\quad |\underline{w}| = \sqrt{1^2 + 3^2 + (-2)^2} = \sqrt{14}$

$v \cdot w = \sqrt{29} \times \sqrt{14} \cos 66.6^\circ$

$\underline{v} \cdot \underline{w} = 8$

Your notes

Angle Between Two Vectors

How do I find the angle between two vectors?

- If two vectors with different directions are placed at the same starting position, they will form an angle between them
- The two formulae for the scalar product can be used together to find this angle
  - $\cos\theta = \dfrac{v_1w_1 + v_2w_2 + v_3w_3}{|\boldsymbol{v}||\boldsymbol{w}|}$
  - This is given in the formula booklet
- To find the angle between two vectors:
  - Calculate the scalar product between them
  - Calculate the magnitude of each vector
  - Use the formula to find $\cos\theta$
  - Use inverse trig to find θ

Examiner Tip

- The formula for this is given in the formula booklet so you do not need to remember it but make sure that you can find it quickly and easily in your exam

Your notes

Worked example

Calculate the angle formed by the two vectors $\boldsymbol{v} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ and $\boldsymbol{w} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$.

$\underline{v} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$, $\underline{w} = 3\underline{i} + 4\underline{j} - \underline{k} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$

Start by finding the scalar product:

$\underline{v} \cdot \underline{w} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$

$= (-1 \times 3) + (3 \times 4) + (2 \times -1) = 7$

Find the magnitude of both vectors:

$|\underline{v}| = \sqrt{(-1)^2 + 3^2 + 2^2} = \sqrt{14}$

$|\underline{w}| = \sqrt{3^2 + 4^2 + (-1)^2} = \sqrt{26}$

Angle between two vectors	$\cos\theta = \frac{v_1 w_1 + v_2 w_2 + v_3 w_3}{\lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert}$

$\cos\theta = \frac{7}{\sqrt{14} \times \sqrt{26}} = 0.3668\ldots$

$\theta = \cos^{-1}(0.3668\ldots)$

$\theta = 68.5^\circ$ (3sf)

Your notes

Perpendicular Vectors

How do I know if two vectors are perpendicular?

- If the **scalar product** of two (non-zero) vectors is **zero** then they are **perpendicular**
  - If $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ then ***v*** and ***w*** must be perpendicular to each other
- Two vectors are **perpendicular** if their **scalar product** is **zero**
  - The value of cos θ = 0 therefore |***v***||***w***|cos θ = 0

Worked example

Find the value of *t* such that the two vectors $\boldsymbol{v} = \begin{pmatrix} 2 \\ t \\ 5 \end{pmatrix}$ and $\boldsymbol{w} = (t-1)\mathbf{i} - \mathbf{j} + \mathbf{k}$ are perpendicular to each other.

The two vectors $\underline{v}$ and $\underline{w}$ are perpendicular if $\underline{v} \cdot \underline{w} = 0$.

$$\underline{v} = \begin{pmatrix} 2 \\ t \\ 5 \end{pmatrix}, \quad \underline{w} = \begin{pmatrix} t-1 \\ -1 \\ 1 \end{pmatrix}$$

$$\underline{v} \cdot \underline{w} = 2(t-1) + t(-1) + 5(1)$$

$$= 2t - 2 - t + 5$$

Therefore $\underline{v}$ and $\underline{w}$ are perpendicular if

$$t + 3 = 0$$

$$\boxed{t = -3}$$

Your notes

3.7.5 The Vector Product

The Vector ('Cross') Product

What is the vector (cross) product?

- The **vector product** (also known as the **cross** product) is a form in which two vectors can be combined together
- The vector product between two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is denoted $\boldsymbol{v} \times \boldsymbol{w}$
- The result of taking the vector product of two vectors is a **vector**
- The **vector product** is a vector **in a plane** that is **perpendicular** to the two vectors from which it was calculated
  - This could be in either direction, depending on the angle between the two vectors
  - The **right-hand** rule helps you see which direction the vector product goes in
    - By pointing your index finger and your middle finger in the direction of the two vectors your thumb will automatically go in the direction of the vector product

How do I find the vector (cross) product?

- There are **two methods** for calculating the vector product
- The **vector product** of the two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ can be written in **component form** as follows:
  - $\boldsymbol{v} \times \boldsymbol{w} = \begin{pmatrix} \boldsymbol{v}_2\boldsymbol{w}_3 - \boldsymbol{v}_3\boldsymbol{w}_2 \\ \boldsymbol{v}_3\boldsymbol{w}_1 - \boldsymbol{v}_1\boldsymbol{w}_3 \\ \boldsymbol{v}_1\boldsymbol{w}_2 - \boldsymbol{v}_2\boldsymbol{w}_1 \end{pmatrix}$
  - Where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$
  - This is **given in the formula booklet**
- The vector product can also be found in terms of its **magnitude** and **direction**
- The **magnitude of the vector product** is equal to the **product of the magnitudes** of the two vectors and the **sine of the angle between them**
  - $|\boldsymbol{v} \times \boldsymbol{w}| = |v||w|\sin\theta$
  - Where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$
    - The two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are joined at the start and pointing away from each other
  - This is **given in the formula booklet**
- The **direction of the vector product** is **perpendicular** to both $\boldsymbol{v}$ and $\boldsymbol{w}$

What properties of the vector product do I need to know?

- If two vectors are **parallel** then the vector product is **zero**

Your notes

- This is because sin 0° = sin 180° = 0
- If $\boldsymbol{v} \times \boldsymbol{w} = 0$ then ***v*** and ***w*** are parallel if they are non-zero
- If two vectors, ***v*** and ***w***, are **perpendicular** then the magnitude of the vector product is equal to the **product** of the magnitudes of the vectors
  - $|\boldsymbol{v} \times \boldsymbol{w}| = |\boldsymbol{w}||\boldsymbol{v}|$
  - This is because sin 90° = 1

Examiner Tip

- The formulae for the vector product are given in the formula booklet, make sure you use them as this is an easy formula to get wrong
- The properties of the vector product are not given in the formula booklet, however they are important and it is likely that you will need to recall them in your exam so be sure to commit them to memory

Your notes

Worked example

Calculate the magnitude of the vector product between the two vectors $\boldsymbol{v} = \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix}$ and

$\boldsymbol{w} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ using

i) the formula $\boldsymbol{v} \times \boldsymbol{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$,

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix} \qquad \underline{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$$

Use the formula to find the cross-product:

$$\underline{v} \times \underline{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix} = \begin{pmatrix} (0)(-1) - (-5)(-2) \\ (-5)(3) - (2)(-1) \\ (2)(-2) - (0)(3) \end{pmatrix} = \begin{pmatrix} -10 \\ -13 \\ -4 \end{pmatrix}$$

Find the magnitude of $\underline{v} \times \underline{w}$:

$$|\underline{v} \times \underline{w}| = \sqrt{(-10)^2 + (-13)^2 + (-4)^2} = \sqrt{285}$$

$$|\underline{v} \times \underline{w}| = 16.9 \text{ (3sf)}$$

ii) the formula , given that the angle between them is 1 radian.

Your notes

Find the magnitude of $\underline{v}$ and $\underline{w}$:

$|\underline{v}| = \sqrt{2^2 + 0^2 + (-5)^2} = \sqrt{29}$

$|\underline{w}| = \sqrt{3^2 + (-2)^2 + (-1)^2} = \sqrt{14}$

$|\underline{v} \times \underline{w}| = |\underline{v}||\underline{w}| \sin\theta$

$= \sqrt{29} \times \sqrt{14} \sin(1^c)$

$|\underline{v} \times \underline{w}| = 17.0$ (3sf)

Areas using Vector Product

Your notes

How do I use the vector product to find the area of a parallelogram?

- The **area of the parallelogram** with two adjacent sides formed by the vectors ***v*** and ***w*** is equal to **the magnitude of the vector product** of two vectors ***v*** and ***w***
  - $A = |\boldsymbol{v} \times \boldsymbol{w}|$ where ***v*** and ***w*** form two **adjacent sides** of the parallelogram
    - This is **given in the formula booklet**

How do I use the vector product to find the area of a triangle?

- The **area of the triangle** with two sides formed by the vectors ***v*** and ***w*** is equal to **half of the magnitude of the vector product** of two vectors ***v*** and ***w***
  - $A = \frac{1}{2}|\boldsymbol{v} \times \boldsymbol{w}|$ where ***v*** and ***w*** form two **sides** of the triangle
    - This is **not** given in the formula booklet

Examiner Tip

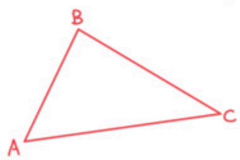
- The formula for the area of the parallelogram is given in the formula booklet but the formula for a triangle is not
  - Remember that the area of a triangle is half the area of a parallelogram

Your notes

Worked example

Find the area of the triangle enclosed by the coordinates (1, 0, 5), (3, -1, 2) and (2, 0, -1).

Let A be (1,0,5), B be (3,-1,2) and C be (2,0,-1)



You can use any two direction vectors moving away from any vertex.

Find the two direction vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$

$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} \qquad \overrightarrow{AC} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -6 \end{pmatrix}$$

Find the cross product of the two direction vectors:

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -6 \end{pmatrix} = \begin{pmatrix} (-1)(-6) - (-3)(0) \\ (-3)(1) - (2)(-6) \\ (2)(0) - (-1)(1) \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \\ 1 \end{pmatrix}$$

Find the magnitude of the cross product

$$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{6^2 + 9^2 + 1^2} = \sqrt{118}$$

Area of the triangle is half the magnitude

$$\text{Area} = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}\sqrt{118}$$

Area = 5.43 u² (3sf)

3.7.6 Components of Vectors

Your notes

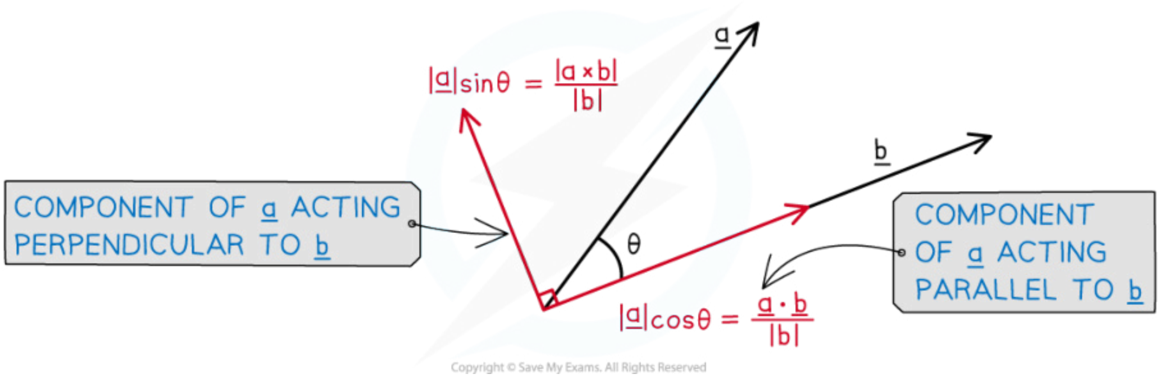
Components of Vectors

Why do we write vectors in component form?

- When working with vectors in context it is often useful to break them down into components acting in a direction that is not one of the base vectors
- The **base vectors** are vectors acting in the directions **i, j** and **k**
- The vector will need to be resolved into components that are acting **perpendicular** to each other
- Usually, one component will be acting **parallel** to the direction of another vector and the other will act **perpendicular** to the direction of the vector
- For example: the components of a **force** parallel and perpendicular to the **line of motion** allows different types of problems to be solved
  - The **parallel** component of a force acting directly on a particle will be the component that causes an effect on the particle
  - The **perpendicular** component of a force acting directly on a particle will be the component that has no effect on the particle
- The two components of the force will have the same combined effect as the original vector

How do we write vectors in component form?

- Use **trigonometry** to resolve a vector acting at an angle
- Given a vector ***a*** acting at an angle θ to another vector ***b***
  - Draw a vector triangle by decomposing the vector ***a*** into its components parallel and perpendicular to the direction of the vector ***b***
- The vector ***a*** will be the **hypotenuse** of the triangle and the two components will make up the **opposite** and **adjacent** sides
- The component of ***a*** acting **parallel** to ***b*** will be equal to the product of the magnitude of ***a*** and the cosine of the angle θ
  - The component of ***a*** acting in the direction of ***b*** equals $|\boldsymbol{a}|\cos\theta$
  - This is equivalent to $\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{b}|}$
- The component of ***a*** acting **perpendicular** to ***b*** will be equal to the product of the magnitude of ***a*** and the sine of the angle θ
  - The component of ***a*** acting perpendicular to the direction of ***b*** equals $|\boldsymbol{a}|\sin\theta$
  - This is equivalent to $\dfrac{|\boldsymbol{a}\times\boldsymbol{b}|}{|\boldsymbol{b}|}$
- The formulae for the components using the **scalar product** and the **vector product** are particularly useful as the angle is not needed
- The question may give you the angle the vector is acting in as a bearing
  - Bearings are always the angle taken from the north



Examiner Tip

- If a question asks you to find a component of a vector it is a good idea to sketch a quick diagram so that you can visualise which vectors are going in which direction
  - This is especially important if the question involves forces

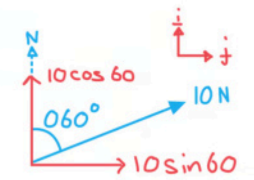
Your notes

Worked example

A force with magnitude 10 N is acting on a bearing of 060° on an object which is moving with velocity vector **v** = 2**i – 3j.**

a) By finding the components of the force in the **i** and **j** direction, write down the force as a vector.

$$F = \begin{pmatrix} 10\sin 60° \\ 10\cos 60° \end{pmatrix} = \begin{pmatrix} 5\sqrt{3} \\ 5 \end{pmatrix}$$



$$F = 5\sqrt{3}\,\mathbf{i} + 5\mathbf{j}$$

b) Find the component of the force acting parallel to the direction of the object.

Your notes

Method 1: Component of $\underline{F}$ acting parallel to $\underline{v} = \frac{\underline{F} \cdot \underline{v}}{|\underline{v}|}$

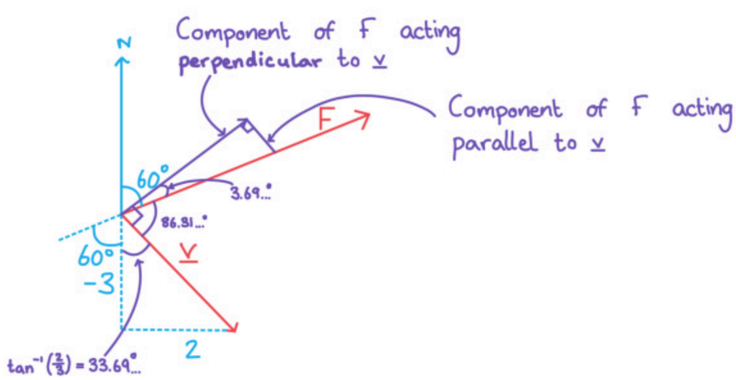
$$\underline{F} \cdot \underline{v} = \begin{pmatrix} 5\sqrt{3} \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \end{pmatrix} = (5\sqrt{3})(2) + (5)(-3)$$

$$= -15 + 10\sqrt{3}$$

$$|\underline{v}| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$$

$$\frac{\underline{F} \cdot \underline{v}}{|\underline{v}|} = \frac{-15 + 10\sqrt{3}}{\sqrt{13}} = 0.644 \text{ N (3 s.f.)}$$

Method 2: Use a diagram:



Component of $\underline{F}$ acting parallel to $\underline{v} = 10 \sin 3.69°$

0.644 N (3 s.f.)

Your notes

3.7.7 Geometric Proof with Vectors

Geometric Proof with Vectors

How can vectors be used to prove geometrical properties?

- If two vectors can be shown to be **parallel** then this can be used to prove parallel lines
  - If two vectors are **scalar multiples** of each other then they are **parallel**
  - To prove that two vectors are parallel simply show that one is a scalar multiple of the other
- If two vectors can be shown to be **perpendicular** then this can be used to prove perpendicular lines
  - If the **scalar product** is zero then the two vectors are **perpendicular**
- If two vectors can be shown to have equal **magnitude** then this can be used to prove two lines are the **same length**
- To prove a 2D shape is a **parallelogram** vectors can be used to
  - Show that there are two pairs of **parallel sides**
  - Show that the **opposite sides** are of **equal length**
    - The vectors opposite each other will be **equal**
  - If the angle between two of the vectors is shown to be 90° then the parallelogram is a **rectangle**
- To prove a 2D shape is a **rhombus** vectors can be used to
  - Show that there are two pairs of **parallel sides**
    - The vectors opposite each other with be **equal**
  - Show that **all four sides** are of **equal length**
  - If the angle between two of the vectors is shown to be 90° then the rhombus is a **square**

How are vectors used to follow paths through a diagram?

- In a geometric diagram the vector $\overrightarrow{\mathrm{AB}}$ forms a path from the point A to the point B
  - This is specific to the path AB
  - If the vector $\overrightarrow{\mathrm{AB}}$ is labelled **a** then any other vector with the same **magnitude** and **direction** as **a** could also be labelled **a**
- The vector $\overrightarrow{\mathrm{BA}}$ would be labelled **–a**
  - It is **parallel** to **a** but pointing in the **opposite direction**
- If the point M is exactly halfway between A and B it is called the midpoint of A and the vector $\overrightarrow{\mathrm{AM}}$ could be labelled $\frac{1}{2}\mathbf{a}$
- If there is a point X on the line AB such that $\overrightarrow{\mathrm{AX}} = 2\overrightarrow{\mathrm{XB}}$ then X is two-thirds of the way along the line $\overrightarrow{\mathrm{AB}}$
  - Other ratios can be found in similar ways
  - A diagram often helps to visualise this
- If a point X divides a line segment AB into the ratio p : q then

Your notes

- $\overrightarrow{AX} = \frac{p}{p+q}\overrightarrow{AB}$
- $\overrightarrow{XB} = \frac{q}{p+q}\overrightarrow{AB}$

How can vectors be used to find the midpoint of two vectors?

- If the point A has position vector **a** and the point B has position vector **b** then the **position vector** of the **midpoint** of $\overrightarrow{AB}$ is $\frac{1}{2}(\mathbf{a}+\mathbf{b})$
  - The **displacement vector** $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$
  - Let **M** be the midpoint of $\overrightarrow{AB}$ then $\overrightarrow{AM} = \frac{1}{2}(\overrightarrow{AB}) = \frac{1}{2}(\mathbf{b}-\mathbf{a})$
  - The **position vector** $\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM} = \mathbf{a} + \frac{1}{2}(\mathbf{b}-\mathbf{a}) = \frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{a} = \frac{1}{2}(\mathbf{a}+\mathbf{b})$

How can vectors be used to prove that three points are collinear?

- Three points are collinear if they all **lie on the same line**
  - The vectors between the three points will be **scalar multiples** of each other
- The points A, B and C are collinear if $\overrightarrow{AB} = k\overrightarrow{BC}$
- If the points A, B and M are collinear and $\overrightarrow{AM} = \overrightarrow{MB}$ then M is the **midpoint** of $\overrightarrow{AB}$

Examiner Tip

- Think of vectors like a journey from one place to another
  - You may have to take a detour e.g. A to B might be A to O then O to B
- Diagrams can help, if there isn't one, draw one
  - If a diagram has been given begin by labelling all known quantities and vectors

Your notes

Worked example

Use vectors to prove that the points *A*, *B*, *C* and *D* with position vectors **a** = (3**i** – 5**j** – 4**k**), **b** = (8**i** - 7**j** - 5**k**), **c** = (3**i** - 2**j** + 4**k**) and **d** = (5**k** – 2**i**) are the vertices of a parallelogram.

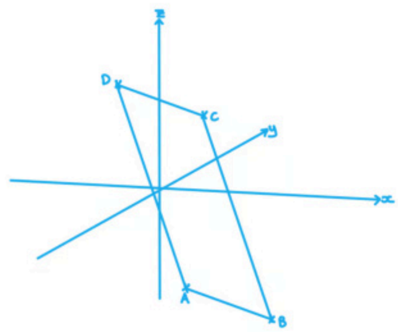
Find the displacement vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$ and $\overrightarrow{DA}$

$$\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 8 \\ -7 \\ -5 \end{pmatrix} - \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ -1 \end{pmatrix}$$

$$\overrightarrow{BC} = \underline{c} - \underline{b} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} - \begin{pmatrix} 8 \\ -7 \\ -5 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \\ 9 \end{pmatrix}$$

$$\overrightarrow{CD} = \underline{d} - \underline{c} = \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix}$$

$$\overrightarrow{DA} = \underline{a} - \underline{d} = \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} - \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ -9 \end{pmatrix}$$



$\overrightarrow{AB} = -\overrightarrow{CD}$ and $\overrightarrow{BC} = -\overrightarrow{DA}$ ∴ ABCD must be a parallelogram